Homology, Homotopy and Applications, vol.6(1), 2004, pp.363–411

DIAGONALS ON THE PERMUTAHEDRA, MULTIPLIHEDRA AND ASSOCIAHEDRA

SAMSON SANEBLIDZE AND RONALD UMBLE

(*communicated by Ross Street*)

Abstract

We construct an explicit diagonal Δ_P on the permutahedra P. Related diagonals on the multiplihedra J and the associahedra K are induced by Tonks' projection $P \to K$ [**19**] and its factorization through J. We introduce the notion of a permutahedral set $\mathcal{Z}$ and lift Δ_P to a diagonal on $\mathcal{Z}$. We show that the double cobar construction $\Omega^2 C_*(X)$ is a permutahedral set; consequently Δ_P lifts to a diagonal on $\Omega^2 C_*(X)$. Finally, we apply the diagonal on K to define the tensor product of A_∞-(co)algebras in maximal generality.

1. Introduction

A permutahedral set is a combinatorial object generated by permutahedra P and equipped with appropriate face and degeneracy operators. Permutahedral sets are distinguished from cubical or simplicial set by higher order (non-quadratic) relations among face and degeneracy operators. In this paper we define the notion of a permutahedral set and observe that the double cobar construction $\Omega^2 C_*(X)$ is a naturally occurring example. We construct an explicit diagonal $\Delta_P : C_*(P) \to C_*(P) \otimes C_*(P)$ on the cellular chains of permutahedra and show how to lift Δ_P to a diagonal on any permutahedral set. We factor Tonks' projection $\theta : P \to K$ through the multiplihedron J and obtain diagonals Δ_J on $C_*(J)$ and Δ_K on $C_*(K)$. We apply Δ_K to define the tensor product of A_∞-(co)algebras in maximal generality; this resolves a long-standing problem in the theory of operads. Gaberdiel and Zwiebach's open string field theory [**5**] provides a setting in which this tensor product can be applied.

The paper is organized as follows: Sections 2 and 5 review the families of polytopes we consider. The diagonal Δ_P is defined in Section 3 and lifted to general permutahedral sets in Section 4. The related diagonals Δ_J and Δ_K are obtained in Section 6 and applied in Section 7 to define the tensor product of A_∞-(co)algebras in maximal generality. Sections 5 through 7 do not depend on Section 4.

This research was funded in part by Award No. GM1-2083 of the U.S. Civilian Research and Development Foundation for the Independent States of the Former Soviet Union (CRDF) and by Award No. 99-00817 of INTAS
This research was funded in part by a Millersville University faculty research grant.
Received November 22, 2002, revised June 7, 2004; published on September 29, 2004.
2000 Mathematics Subject Classification: Primary 55U05, 52B05, 05A18, 05A19; Secondary 55P35.
Key words and phrases: Diagonal, permutahedron, multiplihedron, associahedron.

The first author wishes to acknowledge conversations with Jean-Louis Loday from which our representation of the permutahedron as a subdivision of the cube emerged. The second author wishes to thank Millersville University for its generous financial support and the University of North Carolina at Chapel Hill for its kind hospitality during work on parts of this project.

2. The Permutahedra

Let S_n be the symmetric group on $\underline{n} = \{1, 2, \ldots, n\}$. Recall that the permutahedron P_n is the convex hull of $n!$ vertices $(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n$, $\sigma \in S_n$ [**4**], [**14**], [**20**]. As a cellular complex, P_n is an $(n-1)$-dimensional convex polytope whose $(n-p)$-faces are indexed by (ordered) partitions $U_1|\cdots|U_p$ of $\underline{n}$. We shall define the permutahedra inductively as subdivisions of the standard n-cube I^n. With this representation the combinatorial connection between faces and partitions is immediately clear.

Assign the label $\underline{1}$ to the single point P_1. If P_{n-1} has been constructed and $u = U_1|\cdots|U_p$ is one of its faces, form the sequence $u_* = \{u_0 = 0, u_1, \ldots, u_{p-1}, u_p = \infty\}$ where $u_j = \#\left(U_{p-j+1} \cup \cdots \cup U_p\right)$, $1 \leqslant j \leqslant p-1$ and $\#$ denotes cardinality. Define the *subdivision of* I *relative to* u to be

$$I/u_* = I_1 \cup I_2 \cup \cdots \cup I_p,$$

where $I_j = \left[1 - \frac{1}{2^{u_{j-1}}}, 1 - \frac{1}{2^{u_j}}\right]$ and $\frac{1}{2^\infty} = 0$. Then

$$P_n = \bigcup_{u \in P_{n-1}} u \times I/u_*$$

with faces labeled as follows (see Figures 1 and 2):

Face of $u \times I/u_*$	**Partition of** $\underline{n}$
$u \times 0$	$U_1\vert\cdots\vert U_p\vert n$
$u \times (I_j \cap I_{j+1})$	$U_1\vert\cdots\vert U_{p-j}\vert n\vert U_{p-j+1}\vert\cdots\vert U_p, \quad 1 \leqslant j \leqslant p-1$
$u \times 1$	$n\vert U_1\vert\cdots\vert U_p$
$u \times I_j$	$U_1\vert\cdots\vert U_{p-j+1} \cup n\vert\cdots\vert U_p.$

A *cubical* vertex of P_n is a vertex common to both P_n and I^{n-1}. Note that u is a cubical vertex of P_{n-1} if and only if $u|n$ and $n|u$ are cubical vertices of P_n. Thus the cubical vertices of P_3 are $1|2|3$, $2|1|3$, $3|1|2$ and $3|2|1$ since $1|2$ and $2|1$ are cubical vertices of P_2.

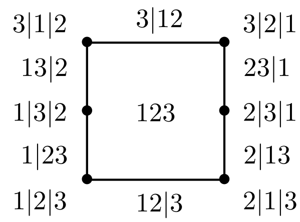


Figure 1: P_3 as a subdivision of $P_2 \times I$.

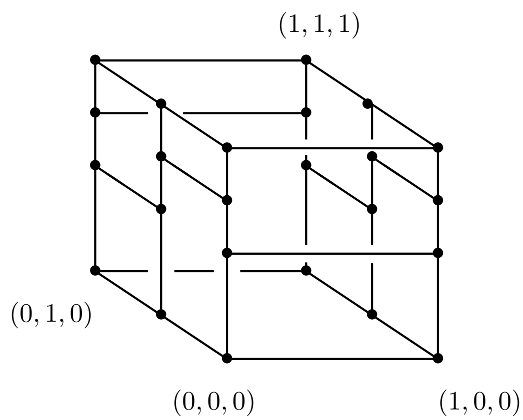


Figure 2a: P_4 as a subdivision of $P_3 \times I$.

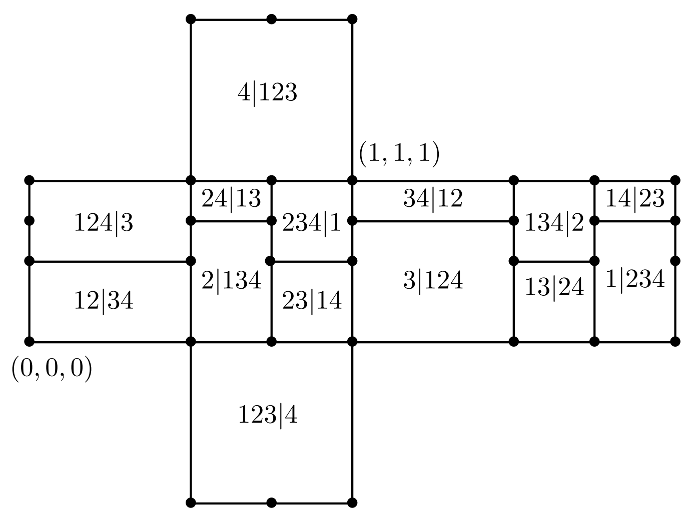


Figure 2b: The 2-faces of P_4.

3. A Diagonal on the Permutahedra

In this section we construct a combinatorial diagonal on the cellular chains of the permutahedron P_{n+1}. Given a polytope X, let $(C_*(X), \partial)$ denote the cellular chains on X with boundary ∂.

Definition 1. *A map $\Delta_X : C_*(X) \to C_*(X) \otimes C_*(X)$ is a* diagonal *on $C_*(X)$ if*

1. $\Delta_X(C_*(e)) \subseteq C_*(e) \otimes C_*(e)$ *for each cell $e \subseteq X$ and*
2. $(C_*(X), \Delta_X, \partial)$ *is a DG coalgebra.*

In general, the DG coalgebra $(C_*(X), \Delta_X, \partial)$ is non-coassociative, non-cocommutative and non-counital; thus the statement (2) in Definition 1 is equivalent to stating that Δ_X is a chain map. We remark that a diagonal Δ_P on $C_*(P_{n+1})$ is unique if the following two additional properties hold:

1. *The canonical cellular projection $\rho_{n+1} : P_{n+1} \to I^n$ induces a DG coalgebra map $C_*(P_{n+1}) \to C_*(I^n)$* (see Section 4, Figures 3 and 4) *and*

2. *There is a minimal number of components $a \otimes b$ in $\Delta_P(C_k(P_{n+1}))$ for $0 \leqslant k \leqslant n$.*

Since the uniqueness of Δ_P is not used in our work, verification of these facts is left to the interested reader.

Definition 2. *A partition $A_1|\cdots|A_p$ is* step increasing *iff $A_p|\cdots|A_1$ is* step decreasing *iff $\min A_j < \max A_{j+1}$ for all $j \leqslant p-1$. A* step *partition is either step increasing or step decreasing.*

Think of $\sigma \in S_{p+q-1}$ as an ordered sequence of positive integers; let $\overleftarrow{\sigma}_j$ and $\overrightarrow{\sigma}_{q-i+1}$ denote its j^{th} decreasing and i^{th} increasing subsequence of maximal length. Then $\overleftarrow{\sigma}_1|\cdots|\overleftarrow{\sigma}_p$ and $\overrightarrow{\sigma}_q|\cdots|\overrightarrow{\sigma}_1$ are step increasing and step decreasing partitions of $\underline{p+q-1}$, respectively (see Example 1 below).

Definition 3. *A pairing of partitions $A_1|\cdots|A_p \otimes B_q|\cdots|B_1$ is a* strong complementary pair *(SCP) if there exists $\sigma \in S_{p+q-1}$ such that $A_j = \overleftarrow{\sigma}_j$ and $B_i = \overrightarrow{\sigma}_i$ as unordered sets for all i, j.*

SCP's have a natural matrix representation.

Definition 4. *A $q \times p$ matrix $O = (o_{ij})$ is* ordered *if:*

1. $\{o_{i,j}\} = \{0, 1, \ldots, p+q-1\}$;
2. *Each row and column of O is non-zero;*
3. *Non-zero entries in O are distinct and increase in each row and column.*

Let $\mathcal{O}$ denote the set of ordered matrices. Note that the rows and columns of an ordered matrix $O^{q \times p}$ form a partition of $\underline{p+q-1}$.

Definition 5. *Given $O \in \mathcal{O}^{q \times p}$, let $V_i = row_i(O) \cap \mathbb{Z}^+$ for $i \leqslant q$ and $U_j = col_j(O) \cap \mathbb{Z}^+$ for $j \leqslant p$. The* row face of O *is the face $r(O) = V_q|\cdots|V_1 \subset P_{p+q-1}$; the* column face of O *is the face $c(O) = U_1|\cdots|U_p \subset P_{p+q-1}$.*

Definition 6. *An ordered matrix E is a* <u>*step matrix*</u> *if:*

1. *Non-zero entries in each row of E appear in consecutive columns;*
2. *Non-zero entries in each column of E appear in consecutive rows;*
3. *The sub, main and super diagonals of E contain a single non-zero entry.*

Let $\mathcal{E}$ denote the set of step matrices. If $E = (e_{i,j}) \in \mathcal{E}^{q\times p}$, condition (1) in Definition 6 groups the non-zero entries in each row together in a horizontal block, condition (2) groups the non-zero entries in each column together in a vertical block and condition (3) links horizontal and vertical blocks to produce a "staircase path" of non-zero entries connecting the lower-left and upper-right entries $e_{q,1}$ and $e_{1,p}$ (see Example 1 below). Clearly, $c(E) \otimes r(E) = \overleftarrow{\sigma}_1|\cdots|\overleftarrow{\sigma}_p \otimes \overrightarrow{\sigma}_q|\cdots|\overrightarrow{\sigma}_1$ for some $\sigma \in S_{p+q-1}$, so $c(E) \otimes r(E)$ is an SCP. Furthermore, one can recover E from $\sigma = (x_1\ x_2\ \cdots\ x_{n+1})$ in the following way: Set $e_{q,1} = x_1$. Inductively, assume $e_{i,j} = x_{k-1}$; if $x_{k-1} < x_k$, set $e_{i,j+1} = x_k$; otherwise, set $e_{i-1,j} = x_k$. Let E_σ denote the step matrix given by $\sigma \in S = \lim\limits_{\to} S_{n+1}$. We have proved:

Proposition 1. *There exist one-to-one correspondences*

$$\mathcal{E} \leftrightarrow S \leftrightarrow \{\textit{Step increasing partitions}\} \leftrightarrow \{\textit{Step decreasing partitions}\} \leftrightarrow \{\textit{SCP's}\}$$

$$E_\sigma \quad \leftrightarrow \quad \sigma \quad \leftrightarrow \quad \overleftarrow{\sigma}_1|\cdots|\overleftarrow{\sigma}_p \quad \leftrightarrow \quad \overrightarrow{\sigma}_q|\cdots|\overrightarrow{\sigma}_1 \quad \leftrightarrow \quad \overleftarrow{\sigma}_1|\cdots|\overleftarrow{\sigma}_p \otimes \overrightarrow{\sigma}_q|\cdots|\overrightarrow{\sigma}_1.$$

Example 1. *The permutation*

$$\sigma = (9\ 7\ 1\ 3\ 8\ 4\ 6\ 5\ 2)$$

corresponds to step matrix

$$E_\sigma = \begin{array}{|c|c|c|c|}\hline & & & 2 \\ \hline & & & 5 \\ \hline & & 4 & 6 \\ \hline 1 & 3 & 8 & \\ \hline 7 & & & \\ \hline 9 & & & \\ \hline \end{array}\ .$$

and the SCP

$$c(E_\sigma) \otimes r(E_\sigma) = 971|3|84|652 \otimes 9|7|138|46|5|2.$$

We now introduce matrix transformations that operate like the vertical and horizontal shifts one performs in a tableau puzzle. For $(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, define the *down-shift* and *right-shift* operators $D_{i,j}, R_{i,j} : \mathcal{O} \to \mathcal{O}$ on $O^{q\times p} = (o_{i,j})$ by

1. $D_{i,j}O = O$ unless $i \leqslant q-1$, $o_{i+1,j} = 0$, $o_{i,j}o_{i,k} > 0$ for some $k \neq j$, $o_{i,j} > o_{i+1,\ell}$ for $1 \leqslant \ell < j$, and $o_{i+1,\ell} > o_{i,j}$ whenever $o_{i+1,\ell} > 0$ and $j < \ell \leqslant p$, in which case $D_{i,j}O$ is obtained from O by transposing $o_{i,j}$ and $o_{i+1,j}$;
2. $R_{i,j}O = O$ unless $j \leqslant p-1$, $o_{i,j+1} = 0$, $o_{i,j}o_{k,j} > 0$ for some $k \neq i$, $o_{i,j} > o_{\ell,j+1}$ for $1 \leqslant \ell < i$, and $o_{\ell,j+1} > o_{i,j}$ whenever $o_{\ell,j+1} > 0$ and $j < \ell \leqslant q$, in which case $R_{i,j}O$ is obtained from O by transposing $o_{i,j}$ and $o_{i,j+1}$.

Definition 7. *A matrix $F \in \mathcal{O}$ is a* <u>*configuration matrix*</u> *if there is a step matrix E and a sequence of shift operators $G_1, \ldots, G_m$ such that*

1. $F = G_m \cdots G_1 E$;
2. *If* $G_m \cdots G_1 = \cdots D_{i_2,j_2} \cdots D_{i_1,j_1} \cdots$, *then* $i_1 \leqslant i_2$;
3. *If* $G_m \cdots G_1 = \cdots R_{k_2,\ell_2} \cdots R_{k_1,\ell_1} \cdots$, *then* $\ell_1 \leqslant \ell_2$.

When this occurs, we say that F is <u>*derived from*</u> *E and refer to the pairing $c(F) \otimes r(F)$ as a* <u>*complementary pair (CP) related to*</u> *$c(E) \otimes r(E)$.*

Let $\mathcal{C}$ denote the set of configuration matrices. For $F = (f_{i,j}) \in \mathcal{C}$ with column face $U_1|\cdots|U_p$ and row face $V_q|\cdots|V_1$, choose proper subsets $N_i = \{f_{i,n_1} < \cdots < f_{i,n_k} |\max V_{i+1} < f_{i,n_1}\} \subset V_i$ and $M_j = \{f_{m_1,j} < \cdots < f_{m_\ell,j} |\max U_{j+1} < f_{m_1,j}\} \subset U_j$ and define

$$D^i_{N_i} F = D_{i,n_k} \cdots D_{i,n_1} F \quad \text{and} \quad R^j_{M_j} F = R_{m_\ell,j} \cdots R_{m_1,j} F.$$

We often suppress the superscript when it is clear from context. The fact that $D_{i,j+1} R_{i,j} F = R_{i+1,j} D_{i,j} F$ wherever both maps in the composition act non-trivially, gives the following useful reformulation of Definition 7:

Proposition 2. *A matrix $F \in \mathcal{O}$ with $c(F) = U_1|\cdots|U_p$ and $r(F) = V_q|\cdots|V_1$ is a configuration matrix if and only if there exists $E \in \mathcal{E}$ and proper subsets $M_j \subset U_j$ and $N_i \subset V_i$ such that*

$$F = D_{N_{q-1}} \cdots D_{N_1} R_{M_{p-1}} \cdots R_{M_1} E.$$

Example 2. *Four configuration matrices F can be derived from the step matrix*

$E =$

	2	3
1	5	
4		

:

$D_\varnothing D_\varnothing R_\varnothing R_\varnothing E =$

	2	3
1	5	
4		

$\leftrightarrow$ 14|25|3 $\otimes$ 4|15|23,

$D_\varnothing D_\varnothing R_5 R_\varnothing E =$

	2	3
1		5
4		

$\leftrightarrow$ 14|2|35 $\otimes$ 4|15|23,

$D_5 D_\varnothing R_\varnothing R_\varnothing E =$

	2	3
1		
4	5	

$\leftrightarrow$ 14|25|3 $\otimes$ 45|1|23,

$D_5 D_\varnothing R_5 R_\varnothing E =$

	2	3
1		
4		5

$\leftrightarrow$ 14|2|35 $\otimes$ 45|1|23.

Up to sign, the CP's

$$c(F) \otimes r(F) = (14|2|35 + 14|25|3) \otimes (4|15|23 + 45|1|23)$$

are components of $\Delta_P(\underline{5})$.

Let us associate formal "configuration signs" to configuration matrices. The signs we introduce here can be derived by induction on dimension given that $P_2 = I$ and Δ_P is a chain map. *Henceforth we assume that all blocks in a partition are increasingly ordered.* First note that a face $u = U_1|\cdots|U_p \subset P_{n+1}$ is an $(n-p+1)$-face of $p-1$ faces in dimension $n-p+2$. Thus there are $(p-1)!$ ways to produce u by successively inserting bars into $\underline{n+1}$, each of which has an associated sign. Of these, we need the right-most and left-most insertion procedures.

When each $x \in \underline{n+1}$ has degree 1, the sign of a permutation $\sigma \in S_{n+1}$ is the Koszul sign that arises from the action of σ. Thus, if σ transposes adjacent subsets $U, V \subset \underline{n+1}$ for example, then $sgn(\sigma) = (-1)^{\#U\#V}$. For $u = U_1|\cdots|U_p \subset P_{n+1}$, denote the sign of the permutation $\underline{n+1} \to U_1 \cup \cdots \cup U_p$ by $psgn(u)$; note that σ is an unshuffle of $\underline{n}$ when $p = 2$, in which case we denote $psgn(u) = shuff(U_1; U_2)$. Let $m_i = \#U_i - 1$ and identify u with the Cartesian product $P_{m_1+1} \times \cdots \times P_{m_p+1}$; then

$$C_{n-p+1}(u) = C_{m_1}(U_1) \otimes \cdots \otimes C_{m_p}(U_p).$$

Finally, think of the symbol $|$ as an operator with degree -1 that acts by sliding in from the left; then

$$|(U \otimes V) = (-1)^{\#U}\, U|V.$$

Definition 8. *Given a partition $M|N$ of $\underline{n+1}$, define* <u>*face operators with respect to M and N*</u>*, $d_M, d^N : C_n(P_{n+1}) \to C_{n-1}(P_{n+1})$ by*

$$d_M(\underline{n+1}) = d^N(\underline{n+1}) = (-1)^{\#M}\ shuff(M;N)\ \ M|N.$$

For $u = U_1|\cdots|U_p \subset P_{n+1}$ and non-empty $M \subset U_k$, define the <u>*face operator with respect to M*</u>*, $d^k_M : C_{n-p+1}(u) \to C_{n-p}(u)$, by*

$$d^k_M(u) = (1^{\otimes k-1} \otimes d_M \otimes 1^{\otimes p-k})(u);$$

for $v = V_q|\cdots|V_1 \subset P_{n+1}$ and non-empty $N \subset V_k$, define the <u>*face operator with respect to N*</u>*, $d^N_k : C_{n-q+1}(v) \to C_{n-q}(v)$, by*

$$d^N_k(v) = (1^{\otimes q-k} \otimes d^N \otimes 1^{\otimes k-1})(v).$$

Then

$$d^k_M(u) = \epsilon(M)\ \ U_1|\cdots|M|U_k \setminus M|\cdots|U_p,$$

$$d^N_k(v) = \epsilon(N)\ \ V_q|\cdots|V_k \setminus N|N|\cdots|V_1,$$

where

$$\epsilon(M) = (-1)^{m_1+\cdots+m_{k-1}+\#M}\, shuff(M; U_k \setminus M) \quad \text{and} \quad m_i = \#U_i - 1,$$

$$\epsilon\,(N) = (-1)^{n_q+\cdots+n_{k+1}+\#(V_k\setminus N)}\, shuff\,(V_k \setminus N; N) \quad \text{and} \quad n_i = \#V_i - 1.$$

Face operators give rise to boundary operators $\partial : C_{n-p+1}\,(u) \to C_{n-p}\,(u)$ and $\partial : C_{n-q+1}\,(v) \to C_{n-q}\,(v)$ in the standard way:

$$\partial\,(u) = \sum_{\substack{1\leqslant k\leqslant p\\ M\subset U_k}} \epsilon(M)\, d^k_M\,(U_1|\cdots|U_p)$$

and similarly for $\partial\,(v)$; in either case,

$$\partial\,(\underline{n+1}) = \sum_{\substack{M,N\subset\underline{n+1}\\ N=\underline{n+1}\setminus M}} (-1)^{\#M}\ \ shuff\,(M;N)\ M|N. \tag{3.1}$$

The sign coefficients in 3.1 were given by R. J. Milgram in [**14**]. Thus, two types of signs appear when d^k_M is applied to $U_1|\cdots|U_p$: First, Koszul's sign appears when d_M passes $U_1\otimes\cdots\otimes U_{k-1}$ and second, Milgram's sign appears when d_M is applied to U_k.

A *partitioning procedure* is a composition of the form

$$d^{k_{p-1}}_{M_{p-1}}\cdots d^{k_2}_{M_2} d_{M_1}.$$

For example, a partition $u = U_1|\cdots|U_p$ of $\underline{n+1}$ can be obtained from the *right-most* partitioning procedure by setting $M_0 = \underline{n+1}$, $M_i = M_{i-1} \setminus U_{p-i+1}$ and $k_i = 1$ for $1 \leqslant i \leqslant p-1$; then

$$d^1_{M_{p-1}}\cdots d^1_{M_2} d_{M_1}\,(\underline{n+1}) = sgn_1\,(u)\ \ U_1|\cdots|U_p,$$

where

$$sgn_1\,(u) = (-1)^{\epsilon_1}\, psgn\,(u) \quad \text{and} \quad \epsilon_1 = \sum\nolimits_{i=1}^{p-1} i\cdot\#U_{p-i}.$$

Note that when $v = V_q|\cdots|V_1$ we have $\epsilon_1 = \sum_{i=1}^{q-1} i\cdot\#V_{i+1}$. Alternatively, u can be obtained from the *left-most* partitioning procedure by setting $M_i = U_i$ and $k_i = i$ for $1 \leqslant i \leqslant p-1$; then

$$d^{p-1}_{U_{p-1}}\cdots d^2_{U_2} d_{U_1}\,(\underline{n+1}) = sgn_2\,(u)\ \ U_1|\cdots|U_p,$$

where

$$sgn_2\,(u) = (-1)^{\epsilon_2}\, psgn\,(u) \quad \text{and} \quad \epsilon_2 = \epsilon_1 + \tbinom{p-1}{2}.$$

Let $rsgn(U_i)$ denote the sign of the order-reversing permutation on U_i, then

$$rsgn(U_i) = (-1)^{\frac{1}{2}(\#U_i)(\#U_i-1)};$$

define

$$rsgn(u) = \prod_{i=1}^{p} rsgn(U_i) = (-1)^{\frac{1}{2}\left[(\#U_1)^2+\cdots+(\#U_p)^2-(n+1)\right]}.$$

Definition 9. *If $F \in \mathcal{C}^{q\times p}$ is derived from $E \in \mathcal{E}$, the configuration sign of F is defined to be*

$$csgn(F) = (-1)^{\binom{q}{2}}\ rsgn(c(E)) \cdot sgn_1 r(F) \cdot sgn_2 c(E) \cdot sgn_2 c(F).$$

In particular, for $F = E \in \mathcal{E}^{q\times p}$ we have

$$csgn(E) = (-1)^{\binom{q}{2}}\ rsgn(c(E)) \cdot sgn_1 r(E).$$

Signs that arise from the action of shift operators are now determined. For $x \in \mathbb{Z}$ and $Y \subseteq \mathbb{Z}$, denote the lower and upper cuts of Y at x by $[Y, x) = \{y \in Y \mid y < x\}$ and $(x, Y] = \{y \in Y \mid y > x\}$, respectively.

Proposition 3. *If $F = (f_{i,j}) \in \mathcal{C}$, $c(F) = U_1|\cdots|U_p$ and $r(F) = V_q|\cdots|V_1$, then*

$$csgn(D_{i,j}F) \cdot csgn(F) = -(-1)^{\#(f'_{i+1,j}, V'_{i+1}] \cup [V_i, f_{i,j})},$$

$$csgn(R_{i,j}F) \cdot csgn(F) = -(-1)^{\#(f_{i,j}, U_j] \cup [U'_{j+1}, f'_{i,j+1})},$$

where $F' = \left(f'_{i,j}\right)$ is the image of F, $U'_1|\cdots|U'_p = c(F')$ and $V'_q|\cdots|V'_1 = r(F')$.

Proof. Note that $c(F) = c(D_{i,j}F)$ and $r(F) = r(R_{i,j}F)$. Then for example,

$$\begin{aligned} csgn(D_{i,j}F) \cdot csgn(F) &= (-1)^{\binom{q}{2}}\, rsgn(c(E)) \cdot sgn_1 r(D_{i,j}F) \cdot sgn_2 c(E) \cdot sgn_2 c(D_{i,j}F) \\ &\quad \cdot\ (-1)^{\binom{q}{2}}\ rsgn(c(E)) \cdot sgn_1 r(F) \cdot sgn_2 c(E) \cdot sgn_2 c(F) \\ &= sgn_1 r(F) \cdot sgn_1 r(D_{i,j}F) \cdot sgn_2 c(F) \cdot sgn_2 c(D_{i,j}F) \\ &= psgn(r(F)) \cdot psgn(r(D_{i,j}F)) = -sgn(\sigma), \end{aligned}$$

where σ is the permutation $V_q \cup \cdots \cup V_1 \mapsto V_q \cup \cdots V'_{i+1} \cup V'_i \cdots \cup V_1$. □

The configuration signs of "edge matrices," which appear in our subsequent discussion of permutahedral sets, have a particularly nice form.

Definition 10. *$E \in \mathcal{E}$ is an edge matrix if $e_{1,1} = 1$.*

Let Γ denote the set of all edge matrices. With one possible exception, all blocks in the column and row face of an edge matrix consist of singleton sets. Thus if $E \in \Gamma^{q\times p}$,

$$c(E) \otimes r(E) = A|a_2|\cdots|a_p \otimes b_q|\cdots|b_2|B,$$

where $A = \{1 < b_2 < \cdots < b_q\}$ and $B = \{1 < a_2 < \cdots < a_p\}$. Since $c(E)$ and $r(E)$ meet at the cubical vertex $b_q|\cdots|b_2|1|a_2|\cdots|a_p$ of P_{p+q-1}, there is a canonical bijection

$$\Gamma \leftrightarrow \{\textit{cubical vertices of } P = \sqcup P_{n+1}\}.$$

The proof of the following proposition is now immediate:

Proposition 4. *If E is an edge matrix and $b_q|\cdots|b_2|1|a_2|\cdots|a_p$ is the corresponding cubical vertex, then*

$$csgn(E) = \textit{shuff}(b_2, \ldots, b_q; a_2, \ldots, a_p).$$

We are ready to define a diagonal on $C_*(P_{n+1})$.

Definition 11. *For each $n \geqslant 0$, define Δ_P on the top dimensional face $\underline{n+1} \in C_n(P_{n+1})$ by*

$$\Delta_P\left(\underline{n+1}\right) = \sum_{\substack{F \in \mathcal{C}^{q \times n-q+2} \\ 1 \leqslant q \leqslant n+1}} csgn(F)\, c(F) \otimes r(F); \tag{3.2}$$

extend Δ_P to proper faces $u = U_1|\cdots|U_p \in C_{n-p+1}(u) = C_{n_1}(U_1) \otimes \cdots \otimes C_{n_p}(U_p)$, $n_i = \#U_i - 1$, via the standard comultiplicative extension.

Example 3. *On P_3, all but two configuration matrices are step matrices:*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \hline\end{array} \longrightarrow R_3 \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline & 3 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline & 2 \\ \hline 1 & 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline & 1 \\ \hline 2 & 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \hline\end{array} \longrightarrow D_3 \longrightarrow \begin{array}{|c|c|}\hline 1 & \\ \hline 2 & 3 \\ \hline\end{array}$$

Consequently,

$$\begin{array}{rcccc} \Delta_P(\underline{3}) & = & 1|2|3 \otimes 123 & + & 123 \otimes 3|2|1 \\ & - & 1|23 \otimes 13|2 & + & 2|13 \otimes 23|1 \\ & - & 13|2 \otimes 3|12 & + & 12|3 \otimes 2|13 \\ & - & 1|23 \otimes 3|12 & + & 12|3 \otimes 23|1. \end{array}$$

There is a computational shortcut worth mentioning. Since $F \in \mathcal{C}$ if and only if $F^T \in \mathcal{C}$, we only need to derive half of the configuration matrices.

Definition 12. *For $F \in \mathcal{C}$, define the* <u>*transpose*</u> *of $c(F) \otimes r(F)$ to be*

$$[c(F) \otimes r(F)]^T = c\left(F^T\right) \otimes r\left(F^T\right).$$

Example 4. *Refer to Example 3 and note that each component in the right-hand column is the transpose of the component to its left. On P_4 we have:*

$$\begin{aligned} \Delta_P(\underline{4}) \ &= 1234 \otimes 4|3|2|1 + 123|4 \otimes (3|2|14 + 3|24|1 + 34|2|1) \\ & -12|34 \otimes (2|14|3 + 24|1|3) + 1|234 \otimes 14|3|2 \\ & -23|14 \otimes (3|24|1 + 34|2|1) + 13|24 \otimes (3|14|2 + 34|1|2) \\ & + (13|24 + 1|234 - 14|23 + 134|2) \otimes 4|3|12 \\ & - (12|34 + 124|3) \otimes (4|2|13 + 4|23|1) \\ & +3|124 \otimes 34|2|1 - 2|134 \otimes (24|3|1 + 4|23|1) \\ & +24|13 \otimes 4|23|1 + (1|234 - 14|23) \otimes 4|13|2 \\ & \pm \textit{(all transposes of the above).} \end{aligned}$$

We conclude this section with a proof of the fact that Δ_P is a chain map. First note that

$$\begin{aligned}\Delta_P\partial\left(\underline{n+1}\right) &= \sum \pm\Delta_P\left(M\right)|\Delta_P\left(N\right)\\ &= \sum \pm\left(u_i\otimes v_j\right)|\left(u^k\otimes v^\ell\right) = \sum \pm u_i|u^k\otimes v_j|v^\ell,\end{aligned}$$

where $u_i \otimes v_j = c\left(F_{j\times i}\right)\otimes r\left(F_{j\times i}\right),\ u^k\otimes v^\ell = c\left(F^{\ell\times k}\right)\otimes r\left(F^{\ell\times k}\right)$ and $F_{j\times i}$ and $F^{\ell\times k}$ range over all configurations matrices with entries from M and N, respectively. Although $u_i|u^k\otimes v_j|v^\ell$ is not a CP, there is the associated block matrix

$$\begin{array}{|c|c|}\hline 0 & F^{\ell\times k}\\ \hline F_{j\times i} & 0\\ \hline\end{array}\,. \tag{3.3}$$

Thus the components of $\Delta_P\partial\left(\underline{n+1}\right)$ lie in one-to-one correspondence with all such block matrices. Let $a_i\otimes b_j = A_1|\cdots|A_i\otimes B_j|\cdots|B_1$ and $a^k\otimes b^\ell = A^1|\cdots|A^k\otimes B^\ell|\cdots|B^1$ be the SCP's related to $u_i\otimes v_j$ and $u^k\otimes v^\ell$. Denoting a column (or row) by its set of non-zero entries, the step matrices

$$\begin{array}{|c|c|c|}\hline A_1 & \cdots & A_i\\ \hline\end{array} = \begin{array}{|c|}\hline B_1\\ \hline \vdots\\ \hline B_j\\ \hline\end{array} \quad\text{and}\quad \begin{array}{|c|c|c|}\hline A^1 & \cdots & A^k\\ \hline\end{array} = \begin{array}{|c|}\hline B^1\\ \hline \vdots\\ \hline B^\ell\\ \hline\end{array}$$

involve elements of M and N, respectively, and the block matrix associated with the pairing $a_i|a^k\otimes b_j|b^\ell$ is

$$\begin{array}{|c|c|c|c|c|c|}\hline \multicolumn{3}{|c|}{0} & A^1 & \cdots & A^k\\ \hline A_1 & \cdots & A_i & \multicolumn{3}{c|}{0}\\ \hline\end{array} = \begin{array}{|c|c|}\hline & B^1\\ \cline{2-2} 0 & \vdots\\ \cline{2-2} & B^\ell\\ \hline B_1 & \\ \cline{1-1} \vdots & 0\\ \cline{1-1} B_j & \\ \hline\end{array}\,.$$

Our main result combines the statements in Lemmas 1 and 2 below:

Theorem 1. *The cellular boundary map* $\partial : C_*\left(P_{n+1}\right)\to C_*\left(P_{n+1}\right)$ *is a* Δ_P*-coderivation for all* $n\geqslant 1$.

Corollary 1. *$(C_*(P_{n+1}), \Delta_P, \partial)$ is a DG coalgebra and the cellular projection $\rho_{n+1} : P_{n+1} \to I^n$ induces a DG coalgebra map*

$$(\rho_{n+1})_* : C_*(P_{n+1}) \to C_*(I^n).$$

Lemma 1. *Each non-zero component $(u_i \otimes v_j) \,|\, (u^k \otimes v^\ell)$ of $\Delta_P \partial (\underline{n+1})$ is a non-zero component of $(1 \otimes \partial + \partial \otimes 1)\, \Delta_P (\underline{n+1})$.*

Proof. Consider a component $(u_i \otimes v_j) \,|\, (u^k \otimes v^\ell)$ of $\Delta_P \partial (\underline{n+1})$, where $u_i \otimes v_j = U_1 | \cdots | U_i \otimes V_j | \cdots | V_1$ is a CP of partitions of $M = U_1 \cup \cdots \cup U_i$ and $u^k \otimes v^\ell = U^1 | \cdots | U^k \otimes V^\ell | \cdots | V^1$ is a CP of partitions of $N = \underline{n+1} \diagdown M$. The related SCP's $a_i \otimes b_j = A_1 | \cdots | A_i \otimes B_j | \cdots | B_1$ and $a^k \otimes b^\ell = A^1 | \cdots | A^k \otimes B^\ell | \cdots | B^1$ give the component $(a_i \otimes b_j) \,|\, (a^k \otimes b^\ell)$ of $\Delta_P \partial (\underline{n+1})$. Let $E = (e_{i,j})$ be the block matrix associated with $a_i | a^k \otimes b_j | b^\ell$. There are two cases:

<u>*Case 1*</u>: $e_{\ell+1,i} > e_{\ell,i+1}$.

Then $\min U_i = \min A_i > \max A^1 \geqslant \max U^1$ and the CP

$$u \otimes v = U_1 | \cdots | U_{i-1} | U^1 \cup U_i | U^2 | \cdots | U^k \otimes v_j | v^\ell$$

is a component of $\Delta_P (\underline{n+1})$ with associated configuration matrix

$$F = \begin{array}{|c|c|c|c|c|}\hline \multicolumn{2}{|c|}{0} & U^1 & \cdots & U^k \\ \hline U_1 & \cdots & U_i & \multicolumn{2}{c|}{0} \\ \hline \end{array} = \begin{array}{|c|}\hline v^\ell \\ \hline v_j \\ \hline \end{array}.$$

It follows that $u_i | u^k \otimes v = d^i_{U_i}(u) \otimes v$ is a component of $(1 \otimes \partial + \partial \otimes 1)\, \Delta_P (\underline{n+1})$. To check signs, we verify that the product of expressions (I) through (VI) below is 1. Let $V'_q | ... | V'_1 = v_j | v^\ell$ and note that $u \otimes v = c(F) \otimes r(F)$ is related to the SCP $a \otimes b = A_1 | \cdots | A_{i-1} | A^1 \cup A_i | A^2 | \cdots | A^k \otimes b_j | b^\ell = c(E) \otimes r(E)$.

I. $csgn(F) = \mathrm{I}_1 \cdot \mathrm{I}_2 \cdot \mathrm{I}_3 \cdot \mathrm{I}_4 \cdot \mathrm{I}_5 = (-1)^{\binom{q}{2}} \cdot [sgn_2 u \cdot sgn_2 a] \cdot rsgn(a) \cdot (-1)^{\epsilon'_1} \cdot psgn(v)$, where $\epsilon'_1 = \sum_{i=1}^{q-1} i \cdot \# V'_{i+1}$.

II. $sgn(d^i_{U_i}(u)) = \mathrm{II}_1 \cdot \mathrm{II}_2 = (-1)^{\# M + i + 1} \cdot (-1)^{\# U_i \# U^1}$, where the shuffle sign II_2 follows by assumption.

III. $sgn\,(d_M\,(\underline{n+1})) = \mathrm{III}_1 \cdot \mathrm{III}_2 = (-1)^{\# M} \cdot \text{shuff}(M; N)$.

IV. $csgn(F_{j \times i}) = \mathrm{IV}_1 \cdot \mathrm{IV}_2 \cdot \mathrm{IV}_3 \cdot \mathrm{IV}_4 \cdot \mathrm{IV}_5 = (-1)^{\binom{j}{2}} \cdot [sgn_2 u_i \cdot sgn_2 a_i] \cdot rsgn(a_i) \cdot (-1)^{\epsilon_1} \cdot psgn(v_j)$, where $\epsilon_1 = \sum_{i=1}^{j-1} i \cdot \# V_{i+1}$.

V. $csgn(F^{\ell \times k}) = \mathrm{V}_1 \cdot \mathrm{V}_2 \cdot \mathrm{V}_3 \cdot \mathrm{V}_4 \cdot \mathrm{V}_5 = (-1)^{\binom{\ell}{2}} \cdot [sgn_2 u^k \cdot sgn_2 a^k] \cdot rsgn(a^k) \cdot (-1)^{\epsilon^1} \cdot psgn(v^\ell)$, where $\epsilon^1 = \sum_{i=1}^{\ell-1} i \cdot \# V^{i+1}$.

VI. $(-1)^{\dim u^k \dim v_j} = (-1)^{(\ell-1)(i-1)}$ $(u_i \otimes v_j$ is a component of $\Delta_P(M)$; hence $\dim(u_i \otimes v_j) = \# M - 1$ and $\dim v_j = \# M - 1 - \dim u_i = i - 1)$.

Then by straightforward calculation,

(1) $\mathrm{I}_5 \cdot \mathrm{III}_2 \cdot \mathrm{IV}_5 \cdot \mathrm{V}_5 = 1$;

(2) $\mathrm{I}_2 \cdot \mathrm{IV}_2 \cdot \mathrm{V}_2 = \mathrm{I}_3 \cdot \mathrm{II}_2 \cdot \mathrm{IV}_3 \cdot \mathrm{V}_3 = (-1)^{\#A_i \#A^1 + \#U_i \#U^1}$;

(3) $\mathrm{I}_1 \cdot \mathrm{IV}_1 \cdot \mathrm{V}_1 = (\mathrm{I}_4 \cdot \mathrm{IV}_4 \cdot \mathrm{V}_4) \cdot (\mathrm{II}_1 \cdot \mathrm{III}_1) \cdot \mathrm{VI} = (-1)^{j\ell}$
($\#M = i + j - 1$ since $v_j = r(F_{j \times i})$).

<u>*Case 2*</u>: $e_{\ell+1,i} < e_{\ell,i+1}$.

Then $\max(V_1) \leqslant \max(B_1) < \min(B^\ell) = \min(V^\ell)$ and the CP

$$u \otimes v = u_i | u^k \otimes V_j | \cdots | V_1 \cup V^\ell | \cdots | V^1$$

is a component of $\Delta_P(\underline{n+1})$ with associated configuration matrix

$$F = \begin{array}{|c|c|} \hline & V^1 \\ \cline{2-2} 0 & \vdots \\ \hline V_1 & V^\ell \\ \hline \vdots & \\ \cline{1-1} V_j & 0 \\ \hline \end{array} = \begin{array}{|c|c|} \hline u_i & u^k \\ \hline \end{array}$$

It follows that $u_i | u^k \otimes v_j | v^\ell = u \otimes d_\ell^{V_1}(v)$ is a component of $(1 \otimes \partial + \partial \otimes 1)\, \Delta_P(\underline{n+1})$. The sign check is similar to the one in *Case 1* above and is left to the reader. □

Lemma 2. *Each non-zero component* $d_M^k(u) \otimes v$ *or* $u \otimes d_\ell^N(v)$ *of* $(1 \otimes \partial + \partial \otimes 1)\, \Delta_P(\underline{n+1})$ *is a non-zero component of* $\Delta_P \partial(\underline{n+1})$.

Proof. For simplicity we work with $\mathbb{Z}_2$ coefficients; sign checks with $\mathbb{Z}$ coefficients are straightforward calculations and left to the reader. Given an SCP $a \otimes b = c(E) \otimes r(E) = A_1 | \cdots A_p \otimes B_q | \cdots | B_1$ of partitions of $\underline{n+1}$, let $u \otimes v = c(F) \otimes r(F) = U_1 | \cdots | U_p \otimes V_q | \cdots | V_1$ be a related CP. Then there exist $M_j \subset A_j$ and $N_i \subset B_i$ with $\min M_j > \max A_{j+1}$ and $\min N_i > \max B_{i+1}$ such that

$$F = D_{N_{q-1}} \cdots D_{N_1} R_{M_{p-1}} \cdots R_{M_1} E. \tag{3.4}$$

Then $u \otimes v$ is a non-zero component of $\Delta_P(\underline{n+1})$. For each proper $M \subset U_k$, we prove that the component $d_M^k(u) \otimes v$ of $(1 \otimes \partial + \partial \otimes 1)\, \Delta_P(\underline{n+1})$ is a non-zero component of $\Delta_P \partial(\underline{n+1})$ if and only if the following conditions hold:

(1) $m = \min M \in A_k$;

(2) $(m, M] = (m, A_k \cup M_{k-1}]$;

(3) $m \in B_r$ implies $N_{r-1} = \varnothing$.

The dual statement for $u \otimes d_\ell^N(v)$ with $N \subset V_\ell$ and is also true; the proof follows by "mirror symmetry." Suppose conditions (1) - (3) hold. Set $M_0 = M_p = \varnothing$; then clearly, $U_i = (A_i \cup M_{i-1}) \setminus M_i$ for $1 \leqslant i \leqslant p$, and $M_{k-1} \subseteq M$ by conditions (1) and (2). Thus $U_1 \cup \cdots \cup U_{k-1} \cup M = A_1 \cup \cdots \cup A_{k-1} \cup M$ and it follows that $d_M^k(u) \otimes v$ is the non-zero component

$$\Delta_P (A_1 \cup \cdots \cup A_{k-1} \cup M \mid A_k \setminus M \cup A_{k+1} \cup \cdots \cup A_p)$$

of $\Delta_P \partial (\underline{n+1})$. Conversely, if conditions (1) - (3) fail to hold, we prove that there exists a unique CP $\bar{u} \otimes \bar{v} \neq u \otimes v$ such that $u \otimes v + \bar{u} \otimes \bar{v} \in \ker(\partial \otimes 1 + 1 \otimes \partial)$.

For existence, we consider all possible cases.

<u>*Case 1*</u>: Assume $(1)' : m \notin A_k$.

Let

$$\bar{u} = U_1 | \cdots | U_{k-1} \cup M | U_k \setminus M | \cdots | U_p;$$

then

$$d_{U_{k-1}}^{k-1}(\bar{u}) \otimes v = d_M^k(u) \otimes v.$$

Now $M \subset M_{k-1}$ since $m \in M_{k-1}$; hence $\bar{u} \otimes v$ may be obtained by replacing $R_{M_{k-1}}$ with $R_{M_{k-1} \setminus M}$ in (3.4) and $\bar{u} \otimes v$ is a CP related to $a \otimes b$.

<u>*Case 2*</u>: Assume $(1) \wedge (2)' : m \in A_k$ and $(m, M] \subset (m, A_k \cup M_{k-1}]$.

Let

$$\mu = \min (m, A_k \cup M_{k-1}] \setminus M \quad \text{and} \quad L = [A_k, m) \cup \mu.$$

Note that $\mu \in A_i$ for some $1 \leqslant i \leqslant k$.

<u>*Subcase 2A*</u>: *Assume* $\min L > \max A_{k+1}$, $k < p$.

Let

$$\bar{u} = U_1 | \cdots | M | (U_k \setminus M) \cup U_{k+1} | \cdots | U_p;$$

then

$$d_{U_k \setminus M}^{k+1}(\bar{u}) \otimes v = d_M^k(u) \otimes v.$$

Note that $\min A_k = m$ since $\min L > \max A_{k+1} > \min A_k$. Thus $L = \mu$. Now, $\min M_k > \max A_{k+1}$ by (3.4) and $\min U_k \setminus M = \min [(A_k \cup M_{k-1}) \setminus M_k] \setminus M \geqslant \min (A_k \cup M_{k-1}) \setminus M = \min (m, A_k \cup M_{k-1}] \setminus M = \mu = \min L > \max A_{k+1}$ so that $\min M_k \cup (U_k \setminus M) > \max A_{k+1}$. Hence $\bar{u} \otimes v$ can be obtained by replacing R_{M_k} with $R_{M_k \cup (U_k \setminus M)}$ in (3.4) and $\bar{u} \otimes v$ is a CP related to $a \otimes b$.

<u>*Subcase 2B*</u>: $\min L < \max A_{k+1}$ with $k \leqslant p$.

<u>*Subcase 2B1*</u>: *Assume* $\min A_{i-1} > \max A_i \setminus \mu$ *with* $\mu \in A_i$ and $1 < i \leqslant k$.

When $i = k$ let

$$\bar{u} = U_1|\cdots|U_{k-1} \cup M|U_k \setminus M|\cdots|U_p;$$

and when $1 < i < k$, let

$$\bar{u} = U_1|\cdots|U_{i-1} \cup U_i|\cdots|M|U_k \setminus M|\cdots|U_p.$$

Then for all $i \leqslant k$,

$$d_M^k(u) \otimes v = d_{U_{i-1}}^{i-1}(\bar{u}) \otimes v.$$

When $i = k$, $\min A_{k-1} \cup (A_k \cap M) \leqslant \min A_k \cap M < \mu = \max A_k = \max A_k \setminus M$ so that

$$\bar{a} \otimes \bar{b} = A_1|\cdots|A_{k-1} \cup (A_k \cap M)|A_k \setminus M|\cdots|A_p \otimes b$$

is an SCP; let $\bar{E}$ be the associated step matrix and let

$$\bar{F} = D_{N_{q-1}} \cdots D_{N_1} R_{M_{p-1}} \cdots R_{M_k} R_{M_{k-1} \setminus M} \cdots R_{M_1} \bar{E}.$$

When $i < k$, we have $\mu = \max A_i > \max A_k \geqslant \max A_k \cap M$ so that $\min A_k \cap M < \max L$; furthermore, $\max A_k = \max A_k \cap M$ by the minimality of μ so that $\min A_{k-1} < \max A_k \cap M$. And finally, $\min L < \max A_{k+1}$ by assumption 2B. Thus

$$\bar{a} \otimes \bar{b} = A_1|\cdots|A_{i-1} \cup A_i \setminus \mu|\cdots|A_k \cap M|L|\cdots|A_p \otimes B_q|\cdots|B_{r+1}|B_{r-1}|\cdots|B_j \cup \mu|\cdots|B_1$$

is an SCP; let $\bar{E}$ be the associated step matrix. Note that $U_{i-1} \cup U_i = (A_{i-1} \cup M_{i-2} \cup A_i \setminus \mu) \setminus (M_i \setminus \mu)$ and $\mu \in M_j$ for $i \leqslant j \leqslant k-1$. Let

$$\bar{F} = D_{N_{q-1}} \cdots D_{N_{r-1} \cup \mu} \cdots D_{N_j \cup \mu} \cdots D_{N_1} R_{M_{p-1}} \cdots R_{M_k} R^{k-1}_{(M_{k-1} \setminus \mu) \setminus M} R^{k-2}_{M_{k-1} \setminus \mu} \cdots R^{i-1}_{M_i \setminus \mu} \cdots R_{M_1} \bar{E},$$

where $\mu \in B_r, \bar{B}_j$. Then for all $i \leqslant k$, $\bar{u} \otimes v = c\left(\bar{F}\right) \otimes r\left(\bar{F}\right)$ is a CP related to $\bar{a} \otimes \bar{b}$.

<u>*Subcase 2B2*</u>: *Assume* $\min A_{i-1} < \max A_i \setminus \mu$ *with* $\mu \in A_i$ and $1 < i \leqslant k$.

Let

$$\bar{u} \otimes \bar{v} = U_1|\cdots|M|U_k \setminus M|\cdots|U_p \otimes V_q|\cdots|V_r \cup V_{r-1}|\cdots|V_1,$$

where $\mu \in B_r, \bar{B}_j$. Then

$$d_M^k(u) \otimes v = \bar{u} \otimes d_{r-1}^{V_{r-1}}(\bar{v}).$$

When $i = k$, $\max L = \mu \in A_k$ so that $\min A_{k-1} < \max A_k \setminus \mu = \max A_k \setminus L$. Furthermore, $\min A_k \setminus L = m < \mu = \max L$; and finally, $\min L < \max A_{k+1}$ by assumption 2B. Thus

$$\bar{a} \otimes \bar{b} = A_1|\cdots|M|L|\cdots|A_p \otimes B_q|\cdots|B_{r+1}|B_{r-1}|\cdots|B_j \cup \mu|\cdots|B_1$$

is an SCP; let $\bar{E}$ be the associated step matrix. Since $\min\left(\mu, A_k \cup M_{k-1}\right] \setminus M > \mu = \max L$, the operator $R^k_{(\mu, A_k \cup M_{k-1}] \setminus M}$ is defined. Note that $M_k \subset L \cup (\mu, A_k \cup M_{k-1}] \setminus M$ and let

$$\bar{F} = D^{q-2}_{N_{q-1}} \cdots D^{r-1}_{N_r} D_{N_{r-2} \cup \mu} \cdots D_{N_j \cup \mu} \cdots D_{N_1} \\ R^p_{M_{p-1}} \cdots R^{k+1}_{M_k} R^k_{(\mu, A_k \cup M_{k-1}] \setminus M} \cdots R_{M_1} \bar{E}.$$

When $1 < i < k$ we have $\min A_{i-1} < \max A_i \setminus \mu$ by assumption 2B2, and $\min A_i \setminus \mu < \max A_{i+1}$ since $\mu \in A_i \cap M_{k-1}$ implies $\mu > \min A_i$. Next, $\min A_{k-1} < \max A_k = \max A_k \cap M$ since $\max A_k < \mu \in A_i$, and $\min A_k \cap M < \mu = \max L$. Finally, $\min L < \max A_{k+1}$ by assumption 2B. Thus

$$\bar{a} \otimes \bar{b} = A_1 | \cdots | A_i \setminus \mu | \cdots | A_k \cap M | L | \cdots | A_p \\ \otimes B_q | \cdots | B_{r+1} | B_{r-1} | \cdots | B_j \cup \mu | \cdots | B_1$$

is an SCP; let $\bar{E}$ be the associated step matrix. Since $\min M_{k-1} = \min M_{k-1} \setminus M = \mu > \max A_k$, both $R_{M_{k-1} \setminus \mu}$ and $R_{(M_{k-1} \setminus \mu) \setminus M}$ are defined, so let

$$\bar{F} = D^{q-2}_{N_{q-1}} \cdots D^{r-1}_{N_r} D_{N_{r-2} \cup \mu} \cdots D_{N_j \cup \mu} \cdots D_{N_1} \\ R^p_{M_{p-1}} \cdots R^{k+1}_{M_k} R^k_{(M_{k-1} \setminus \mu) \setminus M} R_{M_{k-1} \setminus \mu} \cdots R_{M_i \setminus \mu} \cdots R_{M_1} \bar{E}.$$

Then for all $i \leqslant k$, $\bar{u} \otimes v = c\left(\bar{F}\right) \otimes r\left(\bar{F}\right)$ is a CP related to $\bar{a} \otimes \bar{b}$.

<u>*Case 3*</u>: Assume $(1) \wedge (2) \wedge (3)' : m \in A_k \cap B_r$, $(m, M] = (m, A_k \cup M_{k-1}]$ and $N_{r-1} \neq \varnothing$.

Note that $M_k \subset (A_k \cup M_{k-1}) \setminus M = [A_k, m)$ by conditions (1) and (2) so that $U_k \setminus M = [A_k, m) \setminus M_k$. Let $\nu = \min N_{r-1}$; then $\nu \in B_i \cap A_j$ for some $1 \leqslant i \leqslant r-1$ and

$$j = k + \#[B_i, \nu) + \sum_{s=i+1}^{r-1} \left(\#B_s - 1\right).$$

<u>*Subcase 3A*</u>: *Assume* $A_j = \nu$.

In subcases 3A1 and 3A2, $\bar{u}$ is defined so that

$$d^{k+1}_{[A_k, m)}(\bar{u}) \otimes v = d^k_M(u) \otimes v.$$

<u>*Subcase 3A1*</u>: $j = k+1$.

Let

$$\bar{u} = U_1 | \cdots | M | \left(U_k \setminus M\right) \cup U_{k+1} | U_{k+2} | \cdots | U_p.$$

But $\nu > m$ since $\nu \in A_{k+1} \cap N_{r-1}$, consequently $M_k = \varnothing$ so that $U_k \setminus M = [A_k, m)$ and $U_{k+1} = A_{k+1} = \nu$; thus $M_{k+1} = \varnothing$. Clearly

$$\bar{a} \otimes \bar{b} = A_1 | \cdots | A_k \cap M | \left[A_k, m\right) \cup \nu | A_{k+2} | \cdots | A_p \\ \otimes\ B_q | \cdots | B_r \cup \nu | \cdots | B_i \setminus \nu | \cdots | B_1$$

is an SCP; let $\bar{E}$ be the associated step matrix and let

$$\bar{F} = D_{N_{q-1}} \cdots D_{N_{r-1}\setminus\nu} \cdots D_{N_i\setminus\nu} \cdots D_{N_1} R_{M_{p-1}} \cdots R^{k+1}_{\varnothing} R^{k}_{\varnothing} \cdots R_{M_1} \bar{E};$$

then $\bar{u} \otimes v = c\left(\bar{F}\right) \otimes r\left(\bar{F}\right)$ is a CP related to $\bar{a} \otimes \bar{b}$.

Subcase 3A2: $j > k+1$.

Let

$$\bar{u} = U_1|\cdots|M|U_k \setminus M|\cdots|U_{j-1} \cup U_j|U_{j+1}|\cdots|U_p.$$

Again, $\nu > m$ implies that $M_{j-1} = \varnothing$ and $U_j = A_j = \nu$. Clearly

$$\begin{aligned} \bar{a} \otimes \bar{b} = A_1|\cdots|A_k \cap M|\left[A_k, m\right) \cup \nu|\cdots|A_{j-1}|A_{j+1}|\cdots|A_p \\ \otimes\ B_q|\cdots|B_r \cup \nu|\cdots|B_i \setminus \nu|\cdots|B_1 \end{aligned}$$

is an SCP; let $\bar{E}$ be the associated step matrix and let

$$\begin{aligned} \bar{F} = D_{N_{q-1}} \cdots D_{N_{r-1}\setminus\nu} \cdots D_{N_i\setminus\nu} \cdots D_{N_1} \\ R_{M_{p-1}} \cdots R^{j}_{\varnothing} R^{j-1}_{M_{j-2}\cup\nu} \cdots R^{k+2}_{M_{k+1}\cup\nu} R^{k+1}_{M_k\cup\nu} R^{k}_{\varnothing} \cdots R_{M_1} \bar{E}; \end{aligned}$$

then $\bar{u} \otimes v = c\left(\bar{F}\right) \otimes r\left(\bar{F}\right)$ is a CP related to $\bar{a} \otimes \bar{b}$.

Subcase 3B: *Assume* $A_j \neq \nu$.

Note that $i > 1$ by assumption and let

$$\bar{u} \otimes \bar{v} = U_1|\cdots|M|U_k \setminus M|\cdots|U_p \otimes V_q|\cdots|V_i \cup V_{i-1}|\cdots|V_1;$$

then

$$d^k_M(u) \otimes v = \bar{u} \otimes d^{V_{i-1}}_{i-1}\left(\bar{v}\right).$$

Note that $\nu > m$ implies $M_{j-1} = \varnothing$ and $U_j = A_j \setminus M_j$. Clearly

$$\begin{aligned} \bar{a} \otimes \bar{b} = A_1|\cdots|A_k \cap M|\left[A_k, m\right) \cup \nu|\cdots|A_{j-1}|A_j \setminus \nu|A_{j+1}|\cdots|A_p \\ \otimes\ B_q|\cdots|B_r \cup \nu|\cdots|(B_i \cup B_{i-1}) \setminus \nu|\cdots|B_1 \end{aligned}$$

is a SCP; let $\bar{E}$ be the associated step matrix and let

$$\begin{aligned} \bar{F} = D_{N_{q-1}} \cdots D_{N_{r-1}\setminus\nu} \cdots D_{N_i\setminus\nu} \cdots D_{N_1} \\ R^{p}_{M_{p-1}} \cdots R^{j+1}_{M_j} R^{j}_{M_{j-1}\cup\nu} \cdots R^{k+1}_{M_k\cup\nu} R^{k}_{\varnothing} \cdots R_{M_1} \bar{E}; \end{aligned}$$

then $\bar{u} \otimes \bar{v} = c\left(\bar{F}\right) \otimes r\left(\bar{F}\right)$ is a CP related to $\bar{a} \otimes \bar{b}$.

For uniqueness of each pair $\bar{u} \otimes \bar{v}$ constructed above, note the transformations R and D fix minimal elements, i.e., if $\bar{u} \otimes \bar{v} = R(\bar{a}) \otimes D(\bar{b})$, then necessarily $\min \bar{U}_i = \min \bar{A}_i$ and $\min \bar{V}_i = \min \bar{B}_i$ for all i; in particular, if $R(\bar{a}) = \tilde{R}(a')$ or $D(\bar{b}) = \tilde{D}(b')$ then $\min \bar{A}_i = \min A'_i$ or $\min \bar{B}_i = \min B'_i$. Consequently, for $d^k_M(u) \otimes v$ or $u \otimes d^N_\ell(v)$ in the cases above, there is exactly one way to construct a step matrix $\bar{E}$ so that $\bar{a}$ is step increasing and $\bar{b}$ is step decreasing (it is straightforward to check that a construction with distinct $u \otimes v$, $\bar{u} \otimes \bar{v}$, and $u' \otimes v'$ would contradict the necessary condition above either for a and a' or for b and b'). This completes the proof. □

4. Permutahedral Sets

This section introduces the notion of a permutahedral set $\mathcal{Z}$, which is a combinatorial object generated by permutahedra and equipped with appropriate face and degeneracy operators. We construct the generating category $\mathbf{P}$ and show how to lift the diagonal on the permutahedra P constructed above to a diagonal on $\mathcal{Z}$. Naturally occurring examples of permutahedral sets include the double cobar construction, i.e., Adams' cobar construction [**1**] on the cobar with coassociative coproduct [**2**], [**3**], [**8**] (see Subsection 4.5 below). Permutahedral sets are distinguished from simplicial or cubical sets by their higher order structure relations. While our construction of $\mathbf{P}$ follows the analogous (but not equivalent) construction for polyhedral sets given by D.W. Jones in [**7**], there is no mention of structure relations in [**7**].

4.1. Singular Permutahedral Sets

By way of motivation we begin with constructions of two singular permutahedral sets–our universal examples. Whereas the first emphasizes coface and codegeneracy operators, the second emphasizes cellular chains and is appropriate for homology theory. We begin by constructing the various maps we need to define singular coface and codegeneracy operators.

Fix a positive integer n. For $0 \leqslant p \leqslant n$, let

$$\underline{p} = \begin{cases} \varnothing, & p=0 \\ \{1,\dots,p\}, & 1 \leqslant p \leqslant n \end{cases} \quad \text{and} \quad \overline{p} = \begin{cases} \varnothing, & p=0 \\ \{n-p+1,\dots,n\}, & 1 \leqslant p \leqslant n; \end{cases}$$

then $\underline{p}$ and $\overline{p}$ contain the first and last p elements of $\underline{n}$, respectively; note that $\underline{p} \cap \overline{q} = \{p\}$ whenever $p+q=n+1$. Given integers $r,s \in \underline{n}$ such that $r+s=n+1$, there is a canonical projection $\Delta_{r,s} : P_n \to P_r \times P_s$ whose restriction to a vertex $v = a_1|\cdots|a_n \in P_n$ is given by

$$\Delta_{r,s}(v) = b_1|\cdots|b_r \times c_1|\cdots|c_s,$$

where $(b_1,\dots,b_r; c_1,\dots,c_{k-1},c_{k+1},\dots,c_s)$ is the unshuffle of $(a_1,\dots,a_n)$ with $b_i \in \underline{r}$, $c_j \in \overline{s}$, $c_k = r$. For example, $\Delta_{2,3}(2|4|1|3) = 2|1 \times 2|4|3$ and $\Delta_{3,2}(2|4|1|3) = 2|1|3 \times 4|3$. Since the image of the vertices of a cell of P_n uniquely determines a cell in $P_r \times P_s$ the map $\Delta_{r,s}$ is well-defined and cellular. Furthermore, the restriction of $\Delta_{r,s}$ to an $(n-k)$-cell $A_1|\cdots|A_k \subset P_n$ is given by

$$\Delta_{r,s}(A_1|\cdots|A_k) = \begin{cases} \underline{r} \times (A_1|\cdots|A_i \setminus \underline{r-1}\,|\cdots|A_k), & \text{if } \underline{r} \subseteq A_i, \text{ some } i, \\ (A_1|\cdots|A_j \setminus \overline{s-1}\,|\cdots|A_k) \times \overline{s}, & \text{if } \overline{s} \subseteq A_j, \text{ some } j, \\ (A_1 \setminus \overline{s-1}\,|\cdots|A_k \setminus \overline{s-1}) \\ \qquad \times (A_1 \setminus \underline{r-1}\,|\cdots|A_k \setminus \underline{r-1}), & \text{otherwise.} \end{cases}$$

Note that $\Delta_{r,s}$ acts homeomorphically in the first two cases and degeneratively in the third when $1<k<n$. When $n=3$ for example, $\Delta_{2,2}$ maps the edge $1|23$ onto the edge $1|2 \times 23$ and the edge $13|2$ onto the vertex $1|2 \times 3|2$ (see Figure 3).

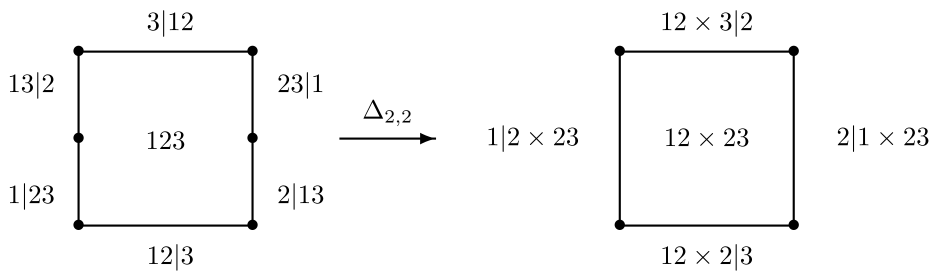


Figure 3: The projection $\Delta_{2,2} : P_3 \to I^2$.

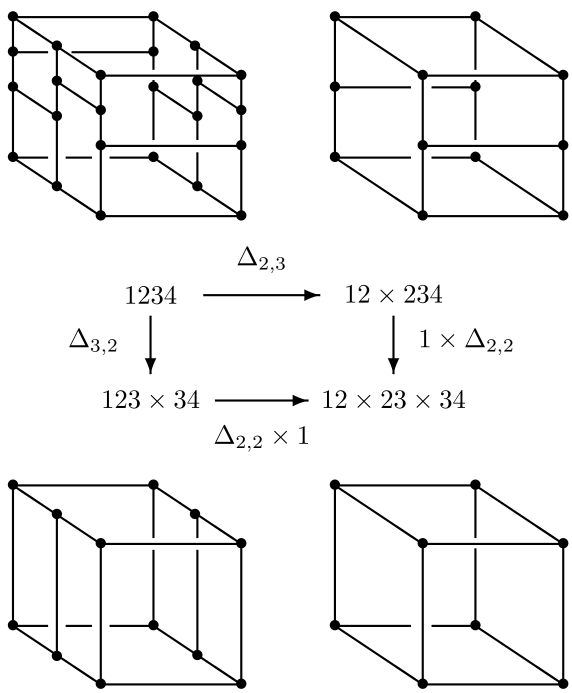


Figure 4: The projection $\rho_4 : P_4 \to I^3$.

Now identify the set $U = \{u_1 < \cdots < u_n\}$ with P_n and the ordered partitions of U with the faces of P_n in the obvious way. Then $(\Delta_{r,s} \times 1) \circ \Delta_{r+s-1,t} = (1 \times \Delta_{s,t}) \circ \Delta_{r,s+t-1}$ whenever $r + s + t = n + 2$ so that $\Delta_{*,*}$ acts coassociatively with respect to Cartesian product. It follows that each k-tuple $(n_1, \ldots, n_k) \in \mathbb{N}^k$ with $k \geqslant 2$ and $n_1 + \cdots + n_k = n + k - 1$ uniquely determines a cellular projection $\Delta_{n_1 \cdots n_k} : P_n \to P_{n_1} \times \cdots \times P_{n_k}$ given by the composition

$$\Delta_{n_1 \cdots n_k} = \left(\Delta_{n_1,n_2} \times 1^{\times k-2}\right) \circ \cdots \circ \left(\Delta_{n_{(k-2)}-k+3,n_{k-1}} \times 1\right) \circ \Delta_{n_{(k-1)}-k+2,n_k},$$

where $n_{(q)} = n_1 + \cdots + n_q$; and in particular,

$$\Delta_{n_1 \cdots n_k}(\underline{n}) = \underline{n_1} \times \underline{n_{(2)} - 1} \setminus \underline{n_1 - 1} \times \cdots \times \underline{n_{(k)} - (k-1)} \setminus \underline{n_{(k-1)} - (k-1)}. \quad (4.1)$$

Note that formula 4.1 with $k = n - 1$ and $n_i = 2$ for all i defines a projection $\rho_n : P_n \to I^{n-1}$

$$\rho_n(\underline{n}) = \Delta_{2 \cdots 2}(\underline{n}) = 12 \times 23 \times \cdots \times \{n-1, n\}$$

(see Figure 4) acting on a vertex $u = u_1 | \cdots | u_n$ as follows: For each $i \in \underline{n-1}$, let $\{u_j, u_k \mid j < k\} = \{u_1, \ldots, u_n\} \cap \{i, i+1\}$ and set $v_i = u_j$, $v_{i+1} = u_k$; then $\rho_n(u) = v_1 | v_2 \times \cdots \times v_{n-1} | v_n$.

Now choose a (non-cellular) homeomorphism $\gamma_n : I^{n-1} \to P_n$ whose restriction to a vertex $v = v_1 | v_2 \times \cdots \times v_{n-1} | v_n$ can be expressed inductively as follows: Set $A_2 = v_1 | v_2$; if A_{k-1} has been obtained from $v_1 | v_2 \times \cdots \times v_{k-2} | v_{k-1}$, set

$$A_k = \begin{cases} A_{k-1} | k, & \text{if } v_k = k, \\ k | A_{k-1}, & \text{otherwise.} \end{cases}$$

For example, $\gamma_4(2|1 \times 3|2 \times 3|4) = 3|2|1|4$. Then γ_n sends the vertices of I^{n-1} to cubical vertices of P_n and the vertices of P_n fixed by $\gamma_n \rho_n$ are exactly its cubical vertices. Given a codimension 1 face $A|B \subset P_n$, index the elements of A and B as follows: If $n \in A$, write $A = \{a_1 < \cdots < a_m\}$ and $B = \{b_1 < \cdots < b_\ell\}$; if $n \in B$, write $A = \{a_1 < \cdots < a_\ell\}$ and $B = \{b_1 < \cdots < b_m\}$. Then $A|B$ uniquely embeds in P_n as the subcomplex

$$P_\ell \times P_m = \begin{cases} a_1 | \cdots | a_m | B \times A | b_1 | \cdots | b_\ell, & \text{if } n \in A \\ A | b_1 | \cdots | b_m \times a_1 | \cdots | a_\ell | B, & \text{if } n \in B. \end{cases}$$

For example, $14|23$ embeds in P_4 as $1|4|23 \times 14|2|3$. Let $\iota_{A|B} : A|B \hookrightarrow P_\ell \times P_m$ denote this embedding and let $h_{A|B} = \iota_{A|B}^{-1}$; then $h_{A|B} : P_\ell \times P_m \to A|B$ is an orientation preserving homeomorphism. Also define the cellular projection

$$\phi_{A|B} : P_n \to P_\ell \times P_m = \begin{cases} b_1 \cdots b_\ell \times a_1 \cdots a_m, & \text{if } n \in A \\ a_1 \cdots a_\ell \times b_1 \cdots b_m, & \text{if } n \in B \end{cases}$$

on a vertex $c = c_1 | \cdots | c_n$ by $\phi_{A|B}(c) = u_1 | \cdots | u_\ell \times v_1 | \cdots | v_m$, where $(u_1, \ldots, u_\ell; v_1, \ldots, v_m)$ is the unshuffle of $(c_1, \ldots, c_n)$ with $u_i \in B$, $v_j \in A$ when $n \in A$ or with $u_i \in A$, $v_j \in B$ when $n \in B$. Note that unlike $\Delta_{r,s}$, the projection $\phi_{A|B}$ always degenerates on the top cell; furthermore, $\phi_{A|B} \circ h_{A|B} = \phi_{B|A} \circ h_{A|B} = 1$. We note that when A or B is a singleton set, the projection $\phi_{A|B}$ was defined by R.J. Milgram in [**14**].

The *singular codegeneracy operator associated with* $A|B$ is the map $\beta_{A|B} : P_n \to P_{n-1}$ given by the composition

$$P_n \xrightarrow{\phi_{A|B}} P_\ell \times P_m \xrightarrow{\rho_\ell \times \rho_m} I^{\ell-1} \times I^{m-1} = I^{n-2} \xrightarrow{\gamma_{n-1}} P_{n-1};$$

the *singular coface operator associated with* $A|B$ is the map $\delta_{A|B} : P_{n-1} \to P_n$ given by the composition

$$P_{n-1} \xrightarrow{\rho_{n-1}} I^{n-2} = I^{\ell-1} \times I^{m-1} \xrightarrow{\gamma_\ell \times \gamma_m} P_\ell \times P_m \xrightarrow{h_{A|B}} A|B \xrightarrow{i} P_n.$$

Unlike the simplicial or cubical case, $\delta_{A|B}$ need not be injective. We shall often abuse notation and write $h_{A|B} : P_\ell \times P_m \to P_n$ when we mean $i \circ h_{A|B}$.

We are ready to define our first universal example. For future reference and to emphasize the fact that our definition depends only on positive integers, let $(n_1, \ldots, n_k) \in \mathbb{N}^k$ such that $n_{(k)} = n$ and denote

$$\mathcal{P}_{n_1 \cdots n_k}(n) = \{Partitions\ A_1|\cdots|A_k\ of\ \underline{n} \mid \#A_i = n_i\}.$$

Definition 13. *Let Y be a topological space. The* <u>*singular permutahedral set of Y*</u> *consists of the singular set*

$$Sing_*^P Y = \bigcup_{n \geqslant 1} \left[Sing_n^P Y = \{Continuous\ maps\ P_n \to Y\}\right]$$

together with singular face and degeneracy operators

$$d_{A|B} : Sing_n^P Y \to Sing_{n-1}^P Y \quad and \quad \varrho_{A|B} : Sing_{n-1}^P Y \to Sing_n^P Y$$

defined respectively for each $n \geqslant 2$ and $A|B \in \mathcal{P}_{**}(n)$ as the pullback along $\delta_{A|B}$ and $\beta_{A|B}$, i.e., for $f \in Sing_n^P Y$ and $g \in Sing_{n-1}^P Y$,

$$d_{A|B}(f) = f \circ \delta_{A|B} \quad and \quad \varrho_{A|B}(g) = g \circ \beta_{A|B}.$$

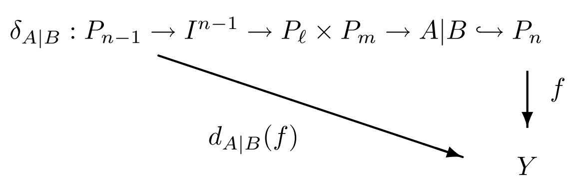


Figure 5: The singular face operator associated with $A|B$.

Although coface operators $\delta_{A|B} : P_{n-1} \to P_n$ need not be inclusions, the top cell of P_{n-1} is always non-degenerate (c.f. Definition 20); however, the top cell of P_{n-2} may degenerate under quadratic compositions $\delta_{A|B}\delta_{C|D} : P_{n-2} \to P_n$. For example, $\delta_{12|34}\delta_{13|2} : P_2 \to P_4$ is a constant map, since $\delta_{12|34} : P_3 \to P_2 \times P_2 \hookrightarrow P_4$ sends the edge $13|2$ to the vertex $1|2 \times 3|2$.

Definition 14. *A quadratic composition of face operators $d_{C|D}d_{A|B}$* <u>*acts on P_n*</u> *if the top cell of P_{n-2} is non-degenerate under the composition*

$$\delta_{A|B}\delta_{C|D} : P_{n-2} \to P_n.$$

Theorem 3 below gives the conditions under which a quadratic composition acts on P_n. For comparison, quadratic compositions of simplicial or cubical face operators always act on the simplex or cube. When $d_{C|D}d_{A|B}$ acts on P_n, we assign the label $d_{C|D}d_{A|B}$ to the codimension 2 face $\delta_{A|B}\delta_{C|D}(\underline{n})$. The various paths of descent from the top cell to a cell in codimension 2 gives rise to relations among compositions of face and degeneracy operators (see Figure 6).

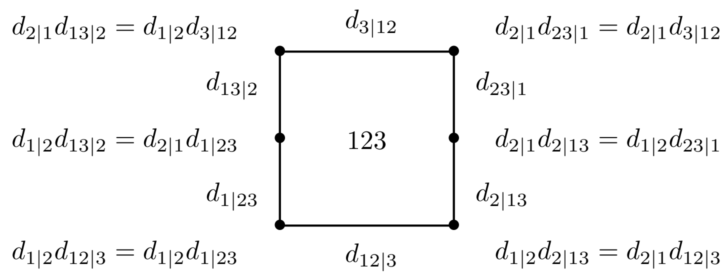


Figure 6: Quadratic relations on the vertices of P_3.

It is interesting to note that singular permutahedral sets have higher order structure relations, an example of which appears below in Figure 7 (see also (4.4)). This distinguishes permutahedral sets from simplicial or cubical sets in which relations are strictly quadratic. Our second universal example, called a "singular multipermutahedral set," specifies a singular permutahedral set by restricting to maps $f = \bar{f} \circ \Delta_{n_1 \cdots n_k}$ for some continuous $\bar{f} : P_{n_1} \times \cdots \times P_{n_k} \to Y$. Face and degeneracy operators satisfy those relations above in which $\Delta_{n_1 \cdots n_k}$ plays no essential role.

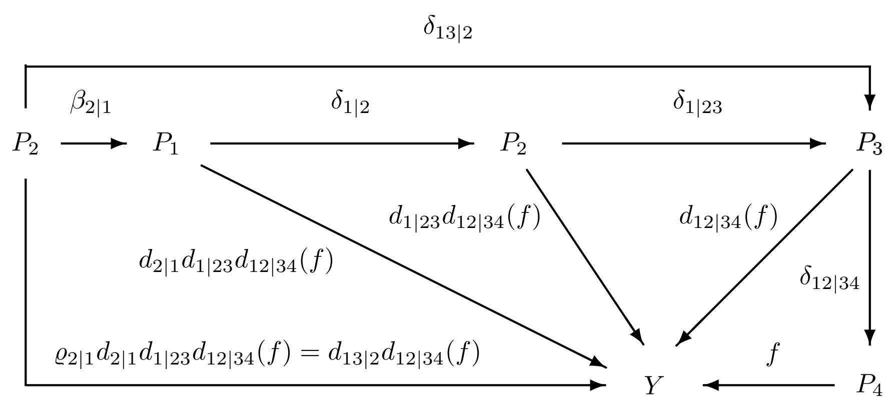


Figure 7: A quartic relation in $Sing_*^P Y$.

Once again, fix a positive integer n, but this time consider $(n_1, \ldots, n_k) \in (\mathbb{N} \cup 0)^k$ with $n_{(k)} = n - 1$ and the projection $\Delta_{n_1+1 \cdots n_k+1} : P_n \to P_{n_1+1} \times \cdots \times P_{n_k+1}$ with $\Delta_n : P_n \to P_n$ defined to be the identity. Given a topological space Y, let

$$Sing^{n_1 \cdots n_k} Y = \left\{ \bar{f} \circ \Delta_{n_1+1 \cdots n_k+1} : P_n \to Y \mid \bar{f} \text{ is continuous} \right\};$$

define $f, f' \in Sing^{n_1 \cdots n_k} Y$ to be equivalent if there exists $g : P_{n_1+1} \times \cdots \times P_{n_{i-1}+1} \times P_1 \times P_{n_{i+1}+1} \times \cdots \times P_{n_k+1} \to Y$ for some $i < k$ such that

$$f = g \circ (1^{\times i-1} \times \phi_{\underline{n_i+1}|n_i+1} \times 1^{\times k-i-1}) \circ \Delta_{n_1+1 \cdots n_{i-1}+1, n_i+2, n_{i+2}+1 \cdots n_k+1}$$

and

$$f' = g \circ (1^{\times i} \times \phi_{1|\underline{n_{i+2}+1}\backslash 1} \times 1^{\times k-i-2}) \circ \Delta_{n_1+1 \cdots n_i+1, n_{i+2}+2, n_{i+3}+1 \cdots n_k+1},$$

in which case we write $f \sim f'$. The geometry of the cube motivates this equivalence; the degeneracies in the product of cubical sets implies the identification (c.f. [**10**] or the definition of the cubical set functor $\mathbf{\Omega}X$ in the Appendix).

Define the singular set

$$Sing_n^M Y = \bigcup_{\substack{(n_1,\dots,n_k) \in (\mathbb{N}\cup 0)^k \\ n_{(k)} = n-1}} Sing^{n_1 \cdots n_k} Y / \sim .$$

Singular face and degeneracy operators

$$d_{A|B} : Sing_n^M Y \to Sing_{n-1}^M Y \quad \text{and} \quad \varrho_{A|B} : Sing_{n-1}^M Y \to Sing_n^M Y$$

are defined piece-wise for each $n \geqslant 2$ and $A|B \in \mathcal{P}_{*,*}(n)$, depending on the form of $A|B$. More precisely, for each pair of integers (p_i, q_i), $1 \leqslant i \leqslant k$, with

$$p_i = 1 + \sum_{j=1}^{i-1} n_j \quad \text{and} \quad q_i = 1 + \sum_{j=i+1}^{k} n_j, \quad \text{let}$$

$$\mathcal{Q}_{p_i,q_i}(n) = \left\{ U|V \in \mathcal{P}_{*,*}(n) \mid \left(\underline{p_i} \subseteq U \text{ or } \underline{p_i} \subseteq V\right) \text{ and } \left(\overline{q_i} \subseteq U \text{ or } \overline{q_i} \subseteq V\right) \right\};$$

in particular, when $r + s = n + 1$, set $k = 2$, $p_1 = q_2 = 1$, $p_2 = r$ and $q_1 = s$, then

$$\mathcal{Q}_{r,1}(n) = \{U|V \in \mathcal{P}_{*,*}(n) \mid \underline{r} \subseteq U \text{ or } \underline{r} \subseteq V\} \text{ and}$$

$$\mathcal{Q}_{1,s}(n) = \{U|V \in \mathcal{P}_{*,*}(n) \mid \overline{s} \subseteq U \text{ or } \overline{s} \subseteq V\}.$$

Since we identify $\underline{r}|\overline{s} \subset P_{n+1}$ with $P_r \times P_s = \Delta_{r,s}(P_n)$, it follows that $A|B \in \mathcal{Q}_{p_i,q_i}(n)$ for some i if and only if $\delta_{A|B}\delta_{\underline{r}|\overline{s}} : P_{n-1} \to P_{n+1}$ is non-degenerate; consequently we consider cases $A|B \in \mathcal{Q}_{p_i,q_i}(n)$ for some i and $A|B \notin \mathcal{Q}_{p_i,q_i}(n)$ for all i.

Since our definitions of $d_{A|B}$ and $\varrho_{A|B}$ are independent in the first case and interdependent in the second, we define both operators simultaneously. But first we need some notation: Given an increasingly ordered set $M = \{m_1 < \cdots < m_k\} \subset \mathbb{N}$, let $I_M : M \to \underline{\#M}$ denote the *indexing map* $m_i \mapsto i$ and let $M + z = \{m_i + z\}$ denote *translation by* $z \in \mathbb{Z}$. Of course, $M - z$ and $M + z$ are left and right translations when $z > 0$; we adopt the convention that translation takes preference over set operations.

Assume $A|B \in \mathcal{Q}_{p_i,q_i}(n)$ for some i, and let

$$C_i = \{p_i, p_i + 1, \dots, p_i + n_i\};$$

$$A_i = (C_i \cap A) - n_{(i-1)}, \quad B_i = (C_i \cap B) - n_{(i-1)};$$

$$n_i' = \#(A \cap C_i) - 1, \quad n_i'' = \#(B \cap C_i) - 1. \tag{4.2}$$

For example, $n = 6$, $n_1 = 3$ and $n_2 = 2$ determines the projection $\Delta_{4,3} : P_6 \to 1234 \times 456$ and pairs $(p_1, q_1) = (1, 3)$ and $(p_2, q_2) = (4, 1)$. Thus $A|B = 1234|56 \in$

$\mathcal{Q}_{3,2}(6)$ and the composition $\delta_{\underline{4}|\overline{3}}\delta_{A|B} : P_5 \to P_7$ is non-degenerate. Furthermore, $C_2 = 456$, $A_2 = (456 \cap 1234) - 3 = 1$, $B_2 = 23$, $n_i' = 0$, $n_i'' = 1$ and we may think of $d_{A|B}$ acting on 1234×456 as $1 \times d_{1|23}$.

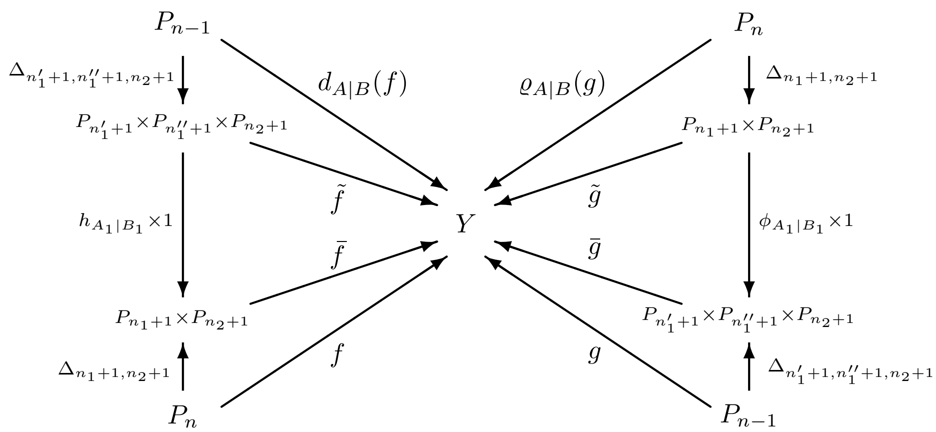


Figure 8: Face and degeneracy operators when $i = 1$ and $k = 2$.

For $f = \bar{f} \circ \Delta_{n_1+1\cdots n_k+1} \in Sing_n^M Y$, let $\tilde{f} = \bar{f} \circ (1^{\times i-1} \times h_{A_i|B_i} \times 1^{\times k-i})$ and define

$$d_{A|B}(f) = \tilde{f} \circ \Delta_{n_1+1\cdots n_i'+1, n_i''+1\cdots n_k+1}.$$

Dually, note that $n_i'+n_i'' = n_i - 1$ implies the sum of coordinates $(n_1, \ldots, n_{i-1}, n_i', n_i'', n_{i+1}, \ldots, n_k) \in (\mathbb{N} \cup 0)^{k+1}$ is $n - 2$. So for $g = \bar{g} \circ \Delta_{n_1+1\cdots n_i'+1, n_i''+1\cdots n_k+1} \in Sing_{n-1}^M Y$, let $\tilde{g} = \bar{g} \circ (1^{\times i-1} \times \phi_{A_i|B_i} \times 1^{\times k-i})$ and define

$$\varrho_{A|B}(g) = \tilde{g} \circ \Delta_{n_1+1\cdots n_k+1}$$

(see Figure 8).

On the other hand, assume that $A|B \notin \mathcal{Q}_{p_i,q_i}(n)$ for all i and define $d_{A|B}$ inductively as follows: When $k = 2$, set $r = n_1 + 1$, $s = n_2 + 1$ and let

$$K|L = \begin{cases} (\underline{r} \cap A) \cup \overline{s} \,|\, \underline{r} \cap B, & r \in A \\ \underline{r} \cap A \,|\, (\underline{r} \cap B) \cup \overline{s}, & r \in B \end{cases}$$

$$M|N = \begin{cases} (\overline{s} \cap A) - 1 \,|\, \underline{n-1} \setminus (\overline{s} \cap A) - 1, & r \in B \\ \underline{n-1} \setminus (\overline{s} \cap B) - \#L \,|\, (\overline{s} \cap B) - \#L, & r \in A, \quad n \in A \\ I_{\underline{n}\setminus L}(A) \,|\, \underline{n-1} \setminus I_{\underline{n}\setminus L}(A), & r \in A, \quad n \in B \end{cases}$$

$$C|D = \begin{cases} I_{\underline{n}\setminus B}(\underline{r} \cap A) \,|\, \underline{n-1} \setminus I_{\underline{n}\setminus B}(\underline{r} \cap A), & r \in B, \quad n \in B \\ I_{\underline{n}\setminus A}(\overline{s} \cap B) \,|\, \underline{n-1} \setminus I_{\underline{n}\setminus A}(\overline{s} \cap B), & r \in A, \quad n \in B \\ & \\ \underline{n-1} \setminus I_{\underline{n}\setminus B}(\overline{s} \cap A) \,|\, I_{\underline{n}\setminus B}(\overline{s} \cap A), & r \in B, \quad n \in A \\ \underline{n-1} \setminus I_{\underline{n}\setminus A}(\underline{r} \cap B) \,|\, I_{\underline{n}\setminus A}(\underline{r} \cap B), & r \in A, \quad n \in A. \end{cases}$$

Then define

$$d_{A|B} = \varrho_{C|D} d_{M|N} d_{K|L}. \tag{4.3}$$

Remark 1. *This definition makes sense since* $K|L \in \mathcal{Q}_{p_1,q_1}(n)$, $M|N \in \mathcal{Q}_{p_3,q_3}(n-1)$, $C|D \in \mathcal{Q}_{p_1,q_1}(n-1)$ *with either* $r, n \in B$ *or* $r, n \in A$ *and* $C|D \in \mathcal{Q}_{p_3,q_3}(n-1)$ *with either* $r \in B$, $n \in A$ *or* $r \in A$, $n \in B$. *Of course,* $\mathcal{Q}_{**}(n-1)$ *is considered with respect to the decomposition* $n-2 = m_1 + m_2 + m_3$ *fixed after the action of* $d_{K|L}(\underline{r} \times \underline{s})$.

If $k = 3$, consider the pair $(r, s) = (n_1 + 1, n - n_1)$, then $(r_1, s_1) = (n_2 + 1, n - n_1 - n_2 - 1)$ for $A_1|B_1 = I_{\underline{n}\setminus\underline{r}}(\bar{s} \cap A)|I_{\underline{n}\setminus\underline{r}}(\bar{s} \cap B) \in \mathcal{P}_{p_1,q_1}(n-r)$, and so on. Now dualize and use the same formulas above to define the degeneracy operator $\varrho_{A|B}$.

Definition 15. *Let* Y *be a topological space. The* <u>*singular multipermutahedral set of* Y</u> *consists of the singular set* $Sing_*^M Y$ *together with the singular face and degeneracy operators*

$$d_{A|B} : Sing_n^M Y \to Sing_{n-1}^M Y \quad \text{and} \quad \varrho_{A|B} : Sing_{n-1}^M Y \to Sing_n^M Y$$

defined respectively for each $n \geqslant 2$ *and* $A|B \in \mathcal{P}_{**}(n)$.

Remark 2. *The operator* $d_{A|B}$ *defined in (4.3) applied to* $d_{U|V}$ *for some* $U|V \in \mathcal{P}_{r,s}(n+1)$ *yields the higher order structural relation*

$$d_{A|B} d_{U|V} = \varrho_{C|D} d_{M|N} d_{K|L} d_{U|V} \tag{4.4}$$

discussed in our first universal example.

Now $Sing_*^M Y$ determines the singular (co)homology of a space Y in the following way: Let R be a commutative ring with identity. For $n \geqslant 1$, let $C_{n-1}(Sing^M Y)$ denote the R-module generated by $Sing_n^M Y$ and form the "chain complex"

$$(C_*(Sing^M Y), d) = \bigoplus_{\substack{n_{(k)}=n-1 \\ n\geqslant 1}} (C_{n-1}(Sing^{n_1\cdots n_k} Y), d_{n_1\cdots n_k}),$$

where

$$d_{n_1\cdots n_k} = \sum_{A|B \in \bigcup_{i=1}^k \mathcal{Q}_{p_i,q_i}(n)} -(-1)^{n_{(i-1)}+n_i'} \ \mathit{shuff}(C_i \cap A; C_i \cap B)\ d_{A|B}.$$

Refer to the example in Figure 7 and note that for $f \in C_4(Sing^M Y)$ with $d_{13|2} d_{12|34}(f) \neq 0$, the component $d_{13|2} d_{12|34}(f)$ of $d^2(f) \in C_2(Sing^M Y)$ is not cancelled and $d^2 \neq 0$. Hence d is not a differential. To remedy this, form the quotient

$$C_*^{\diamondsuit}(Y) = C_*\left(Sing^M Y\right)/DGN,$$

where DGN is the submodule generated by the degeneracies, and obtain the *singular permutahedral chain complex* $\left(C_*^{\diamondsuit}(Y), d\right)$. Because the signs in d are determined by the index i, which is missing in our first universal example, we are unable to use our first example to define a chain complex with signs. However, we could use it to define a unoriented theory with $\mathbb{Z}_2$-coefficients.

The singular homology of Y is recovered from the composition

$$C_*(SingY) \to C_*(Sing^I Y) \to C_*(Sing^M Y) \to C_*^{\diamondsuit}(Y)$$

arising from the canonical cellular projections

$$P_{n+1} \to I^n \to \Delta^n.$$

Since this composition is a chain map, there is a natural isomorphism

$$H_*(Y) \approx H_*^{\diamondsuit}(Y) = H_*(C_*^{\diamondsuit}(Y), d).$$

The fact that our diagonal on P and the A-W diagonal on simplices commute with projections allows us to recover the singular cohomology ring of Y as well. Finally, we remark that a cellular projection f between polytopes induces a chain map between corresponding singular chain complexes whenever chains on the target are normalized. Here $C_*(SingY)$ and $C_*(Sing^I Y)$ are non-normalized and the induced map f^* is not a chain map; but fortunately $d^2 = 0$ does not depend $df^* = f^*d$.

4.2. Abstract Permutahedral Sets

We begin by constructing a generating category $\mathbf{P}$ for permutahedral sets similar to that of finite ordered sets and monotonic maps for simplicial sets. The objects of $\mathbf{P}$ are the sets $n! = S_n$ of permutations of $\underline{n}$, $n \geqslant 1$. But before we can define the morphisms we need some preliminaries. First note that when P_n is identified with its vertices $n!$, the maps ρ_n and γ_n defined above become

$$\rho_n : n! \to 2!^{n-1} \quad \text{and} \quad \gamma_n : 2!^{n-1} \to n!.$$

Given a non-empty increasingly ordered set $M = \{m_1 < \cdots < m_k\} \subset \mathbb{N}$, let $M!$ denote the set of all permutations of M and let $J_M : M! \to k!$ be the map defined for $a = \left(m_{\sigma(1)}, ..., m_{\sigma(k)}\right) \in M!$ by $J_M(a) = \sigma$. For $n, m \in \mathbb{N}$ and partitions $A_1|\cdots|A_k \in \mathcal{P}_{n_1\cdots n_k}(n)$ and $B_1|\cdots|B_\ell \in \mathcal{P}_{m_1\cdots m_\ell}(m)$ with $n - k = m - \ell = \varkappa$, define the morphism

$$f_{A_1|\cdots|A_k}^{B_1|\cdots|B_\ell} : m! \to n!$$

by the composition

$$m! \overset{sh_B}{\to} \prod_{j=1}^{\ell} B_j \overset{\sigma_{\max}}{\to} \prod_{r=1}^{\ell} B_{j_r} \overset{J_B}{\to} \prod_{j=r}^{\ell} m_{j_r}! \overset{\rho_*}{\to} 2!^{\varkappa} \overset{\gamma_*}{\to} \prod_{s=1}^{k} n_{i_s}! \overset{J_A^{-1}}{\to} \prod_{s=1}^{k} A_{i_s} \overset{\sigma_{\max}^{-1}}{\to} \prod_{i=1}^{k} A_i \overset{\iota_A}{\to} n!$$

where sh_B is a surjection defined for $b = \{b_1, ..., b_m\} \in m!$ by

$$sh_B(b) = (b_{1,1}, .., b_{m_1,1}; ...; b_{1,\ell}, .., b_{m_\ell,\ell}),$$

in which the right-hand side is the unshuffle of b with $b_{r,t} \in B_t$, $1 \leqslant r \leqslant m_t$, $1 \leqslant t \leqslant \ell$; $\sigma_{\max} \in S_\ell$ is a permutation defined by $j_r = \sigma_{\max}(r)$, $\max B_{j_r} = \max(B_1 \cup B_2 \cup \cdots \cup B_{j_r})$; $J_B = \prod_{r=1}^{\ell} J_{B_{j_r}}$; $\rho_* = \prod_{r=1}^{\ell} \rho_{j_r}$ and $\gamma_* = \prod_{s=1}^{k} \gamma_{i_s}$; finally, ι_A is the inclusion. It is easy to see that

$$f_{A_1|\cdots|A_k}^{B_1|\cdots|B_\ell} = f_{A_1|\cdots|A_k}^{\underline{\varkappa+1}} \circ f_{\underline{\varkappa+1}}^{B_1|\cdots|B_\ell} \quad \text{and} \quad f_{\underline{n}}^{\underline{n}} = \gamma_n \circ \rho_n.$$

In particular, the maps $f^{n-1}_{A|B} : (n-1)! \to n!$ and $f^{A|B}_{\underline{n-1}} : n! \to (n-1)!$ are generator morphisms denoted by $\delta_{A|B}$ and $\beta_{A|B}$, respectively (see Theorem 2 below, the statement of which requires some new set operations).

Definition 16. *Given non-empty disjoint subsets $A, B, U \subset \underline{n+1}$ with $A \cup B \subseteq U$, define the* <u>*lower and upper disjoint unions*</u> *(with respect to U) by*

$$A \underline{\sqcup} B = \begin{cases} I_{U \smallsetminus A}(B) + \#A - 1, & \text{if } \min B > \min(U \smallsetminus A) \\ I_{U \smallsetminus A}(B) + \#A - 1 \cup \underline{\#A}, & \text{if } \min B = \min(U \smallsetminus A) \end{cases}$$

and

$$A \bar{\sqcup} B = \begin{cases} I_{U \smallsetminus B}(A), & \text{if } \max A < \max(U \smallsetminus B) \\ I_{U \smallsetminus B}(A) \cup \overline{\#B} - 1, & \text{if } \max A = \max(U \smallsetminus B). \end{cases}$$

If either A or B is empty, define $A \underline{\sqcup} B = A \bar{\sqcup} B = A \cup B$. Furthermore, given non-empty disjoint subsets $A, B_1, \ldots, B_k \subset \underline{n+1}$ with $k \geqslant 1$, set $U = A \cup B_1 \cup \cdots \cup B_k$ and define

$$A \square (B_1 | \cdots | B_k) = (B_1 | \cdots | B_k) \square A = \begin{cases} A \underline{\sqcup} B_1 | \cdots | A \underline{\sqcup} B_k, & \text{if } \max A < \max U \\ B_1 \bar{\sqcup} A | \cdots | B_k \bar{\sqcup} A, & \text{if } \max A = \max U. \end{cases}$$

Note that if $A|B$ is a partition of $\underline{n+1}$, then

$$A \underline{\sqcup} B = A \bar{\sqcup} B = \underline{n}.$$

Given a partition $A_1 | \cdots | A_{k+1}$ of $\underline{n}$, define $A^1_1 | \cdots | A^1_{k+1} = A^1_1 | \cdots | A^{k+1}_1 = A_1 | \cdots | A_{k+1}$; inductively, given $A^i_1 | \cdots | A^i_{k-i+2}$ the partition of $\underline{n-i+1}$, $1 \leqslant i < k$, let

$$A^{i+1}_1 | \cdots | A^{i+1}_{k-i+1} = A^i_1 \square (A^i_2 | \cdots | A^i_{k-i+2})$$

be the partition of $\underline{n-i}$; and given $A^1_i | \cdots | A^{k-i+2}_i$ the partition of $\underline{n-i+1}$, $1 \leqslant i < k$, let

$$A^1_{i+1} | \cdots | A^{k-i+1}_{i+1} = (A^1_i | \cdots | A^{k-i+1}_i) \square A^{k-i+2}_i$$

be the partition of $\underline{n-i}$.

Theorem 2. *For $A_1 | \cdots | A_{k+1} \in \mathcal{P}_{n_1 \cdots n_{k+1}}(n)$, $2 \leqslant k \leqslant n$, the map $f^{n-k}_{A_1|\cdots|A_{k+1}} : (n-k)! \to n!$ can be expressed as a composition of δ's two ways:*

$$f^{n-k}_{A_1|\cdots|A_{k+1}} = \delta_{A^1_1 \,|\, A^1_2 \cup \cdots \cup A^1_{k+1}} \cdots \delta_{A^k_1 \,|\, A^k_2} = \delta_{A^1_1 \cup \cdots \cup A^k_1 \,|\, A^{k+1}_1} \cdots \delta_{A^1_k \,|\, A^2_k}.$$

Proof. The proof is straightforward and omitted. □

There is also the dual set of relations among the β's.

Example 5. *Theorem 2 defines structure relations among the δ's, the first of which is*

$$\delta_{A|B \cup C}\, \delta_{A \square (B|C)} = \delta_{A \cup B|C}\, \delta_{(A|B) \square C} \tag{4.5}$$

when $k = 2$. In particular, let $A|B|C = 12|345|678$. Since $A \underline{\sqcup} B = \{1234\}$, $A \underline{\sqcup} C = \{567\}$, $A \bar{\sqcup} C = \{12\}$ and $B \bar{\sqcup} C = \{34567\}$, we obtain the following quadratic relation on $12|345|678$:

$$\delta_{12|345678} \delta_{1234|567} = \delta_{12345|678} \delta_{12|34567};$$

similarly, on 345|12|678 *we have*

$$\delta_{345|12678}\delta_{1234|567} = \delta_{12345|678}\delta_{34567|12}.$$

Theorem 3. *Let* $A|B \in \mathcal{P}_{p,q}(n+1)$ *and* $C|D \in \mathcal{P}_{**}(n)$. *Then* $\delta_{A|B}\delta_{C|D}$ *coincides with a map* $f^{n-1}_{X|Y|Z} : (n-1)! \to (n+1)!$ *if and only if*

$$C|D \in \begin{cases} \mathcal{Q}_{q,1}(n) \cup \mathcal{Q}_{1,p}(n), & \text{if } n+1 \in A \\ \mathcal{Q}_{p,1}(n) \cup \mathcal{Q}_{1,q}(n), & \text{if } n+1 \in B. \end{cases} \tag{4.6}$$

Proof. If $\delta_{A|B}\delta_{C|D}$ coincides with $f^{n-1}_{X|Y|Z}$, then according to relation (4.7) we have either

$$A|B = X|Y \cup Z \text{ and } C|D = X \Box (Y|Z)$$

or

$$A|B = X \cup Y|Z \text{ and } C|D = (X|Y) \Box Z.$$

Hence there are two cases.

<u>*Case 1:*</u> $A|B = X|Y \cup Z$.

<u>*Subcase 1a:*</u> Assume $n+1 \in A$. If $\max Y = \max(Y \cup Z)$, then $\overline{p} \subseteq Y \overline{\sqcup} X$; otherwise $\max(Y \cup Z) = \max Z$ and $\overline{p} \subseteq Z \overline{\sqcup} X$. In either case, $C|D = Y \overline{\sqcup} X | Z \overline{\sqcup} X \in \mathcal{Q}_{1,p}(n)$.

<u>*Subcase 1b:*</u> Assume $n+1 \in B$. If $\min Y = \min(Y \cup Z)$, then $\underline{p} \subseteq X \underline{\sqcup} Y$; otherwise $\min(Y \cup Z) = \min Z$ and $\underline{p} \subseteq X \underline{\sqcup} Z$. In either case, $C|D = X \underline{\sqcup} Y | X \underline{\sqcup} Z \in \mathcal{Q}_{p,1}(n)$.

<u>*Case 2:*</u> $A|B = X \cup Y|Z$.

<u>*Subcase 2a:*</u> Assume $n+1 \in A$. If $\min X = \min(X \cup Y)$, then $\underline{q} \subseteq Z \underline{\sqcup} X$; otherwise $\min(X \cup Y) = \min Y$ and $\underline{q} \subseteq Z \underline{\sqcup} Y$. In either case, $C|D = Z \underline{\sqcup} X | Z \underline{\sqcup} Y \in \mathcal{Q}_{q,1}(n)$.

<u>*Subcase 2b:*</u> Assume $n+1 \in B$. If $\max X = \max(X \cup Y)$, then $\overline{q} \subseteq X \overline{\sqcup} Z$; otherwise $\max(X \cup Y) = \max Y$ and $\overline{q} \subseteq Y \overline{\sqcup} Z$. In either case, $C|D = X \overline{\sqcup} Z | Y \overline{\sqcup} Z \in \mathcal{Q}_{1,q}(n)$.

Conversely, given $A|B \in \mathcal{P}_{p,q}(n+1)$ and $C|D$ satisfying conditions (4.6) above, let

$$[A|B;\, C|D] = \begin{cases} A \,|\, I_B^{-1}(\overline{q} \cap C - p + 1) \,|\, I_B^{-1}(\overline{q} \cap D - p + 1), & C|D \in \mathcal{Q}_{p,1}(n) \\ I_A^{-1}\left(\underline{p} \cap C\right) \,|\, I_A^{-1}\left(\underline{p} \cap D\right) \,|\, B, & C|D \in \mathcal{Q}_{1,q}(n) \end{cases}$$

and

$$[A|B;\, C|D] = \begin{cases} A \,|\, I_B^{-1}\left(\underline{q} \cap C\right) \,|\, I_B^{-1}\left(\underline{q} \cap D\right), & C|D \in \mathcal{Q}_{1,p}(n) \\ I_A^{-1}(\overline{p} \cap C - q + 1) \,|\, I_A^{-1}(\overline{p} \cap D - q + 1) \,|\, B, & C|D \in \mathcal{Q}_{q,1}(n). \end{cases}$$

A straightforward calculation shows that

$$[X|Y \cup Z;\; X \Box (Y|Z)] = X|Y|Z = [X \cup Y|Z;\; (X|Y) \Box Z].$$

Consequently, if $X|Y|Z = [A|B;\, C|D]$, either

$$A|B = X|\; Y \cup Z \text{ and } C|D = X \Box (Y|Z)$$

when $C|D \in \mathcal{Q}_{p,1}(n) \cup \mathcal{Q}_{1,p}(n)$ or

$$A|B = X \cup Y|Z \text{ and } C|D = (X|Y)\Box Z$$

when $C|D \in \mathcal{Q}_{q,1}(n) \cup \mathcal{Q}_{1,q}(n)$. $\Box$

On the other hand, if $C|D \notin \mathcal{Q}_{p,1}(n) \cup \mathcal{Q}_{1,p}(n) \cup \mathcal{Q}_{q,1}(n) \cup \mathcal{Q}_{1,q}(n)$, higher order structure relations involving both coface and codegeneracy operators appear.

Definition 17. *Let $\mathcal{C}$ be the category of sets. A permutahedral set is a contravariant functor*

$$\mathcal{Z} : \mathbf{P} \to \mathcal{C}.$$

Thus a permutahedral set $\mathcal{Z}$ is a graded set $\mathcal{Z} = \{\mathcal{Z}_n\}_{n\geqslant 1}$ endowed with face and degeneracy operators

$$d_{A|B} = \mathcal{Z}(\delta_{A|B}) : \mathcal{Z}_n \to \mathcal{Z}_{n-1} \quad \text{and} \quad \varrho_{M|N} = \mathcal{Z}(\beta_{M|N}) : \mathcal{Z}_n \to \mathcal{Z}_{n+1}$$

satisfying an appropriate set of relations, which includes quadratic relations such as

$$d_{A\Box(B|C)} d_{A|B\cup C} = d_{(A|B)\Box C} d_{A\cup B|C} \tag{4.7}$$

induced by (4.5) and higher order relations such as

$$d_{A|B} d_{U|V} = \varrho_{C|D} d_{M|N} d_{K|L} d_{U|V}$$

discussed in (4.4).

Let us define the abstract analog of a singular multipermutahedral set, which leads to a singular chain complex with arbitrary coefficients.

Definition 18. *For $n \geqslant 1$, let $X_n = \bigcup_{n_{(k)}=n-1,\ n_k\geqslant 0} X^{n_1\cdots n_k}$ and $X_{n-1} = \bigcup_{m_{(\ell)}=n-2,\ m_\ell\geqslant 0} X^{m_1\cdots m_\ell}$ be filtered sets; let $A|B \in \mathcal{Q}_{p_i,q_i}(n)$ for some i. A map $g : X_n \to X_{n-1}$ acts as an $A|B$-formal derivation if*

$$g|_{X^{n_1\cdots n_k}} : X^{n_1\cdots n_k} \to X^{n_1\cdots n_i',n_i''\cdots n_k},$$

where (n_i', n_i'') is given by (4.2).

Let $\mathcal{C}_M$ denote the category whose objects are positively graded sets X_* filtered by subsets $X_n = \bigcup_{n_{(k)}=n-1,\ n_k\geqslant 0} X^{n_1\cdots n_k}$ and whose morphisms are filtration preserving set maps.

Definition 19. *A multipermutahedral set is a contravariant functor $\mathcal{Z} : \mathbf{P} \to \mathcal{C}_M$ such that*

$$\mathcal{Z}(\delta_{A|B}) : \mathcal{Z}(n!) \to \mathcal{Z}((n-1)!)$$

acts as an $A|B$-formal derivation for each $A|B \in \mathcal{Q}_{p_i,q_i}$, all $i \geqslant 1$.

Thus a multipermutahedral set $\mathcal{Z}$ is a graded set $\{\mathcal{Z}_n\}_{n\geqslant 1}$ with

$$\mathcal{Z}_n = \bigcup_{\substack{n_{(k)}=n-1 \\ n_k\geqslant 0}} \mathcal{Z}^{n_1\cdots n_k},$$

together with face and degeneracy operators

$$d_{A|B} = \mathcal{Z}(\delta_{A|B}) : \mathcal{Z}_n \to \mathcal{Z}_{n-1} \quad \text{and} \quad \varrho_{M|N} = \mathcal{Z}(\beta_{M|N}) : \mathcal{Z}_n \to \mathcal{Z}_{n+1}$$

satisfying the relations of a permutahedral set and the additional requirement that $d_{A|B}$ respect underlying multigrading. This later condition allows us to form the chain complex of $\mathcal{Z}$ with signs mimicking the cellular chain complex of permutahedra (see below). Note that the chain complex of a permutahedral set is only defined with $\mathbb{Z}_2$-coefficients in general.

4.3. The Cartesian product of permutahedral sets

The objects and morphisms in the category $\mathbf{P} \times \mathbf{P}$ are the sets and maps

$$n!! = \bigcup_{r+s=n} r! \times s! \quad \text{and} \quad \bigcup_{f,g \in \mathbf{P}} f \times g : m!! \to n!!$$

all $m, n \geqslant 1$. There is a functor $\Delta : \mathbf{P} \to \mathbf{P} \times \mathbf{P}$ defined as follows. If $A|B \in \mathcal{Q}_{r,1}(n) \cup \mathcal{Q}_{1,s}(n)$, define $\Delta_{r,s}(A|B) = A_1|B_1 \times A_2|B_2 \in r! \times s!$ and define $\delta_{A|B} : (n-1)! \to n!$ by

$$\Delta(\delta_{A|B}) = \delta_{A_1|B_1} \times \delta_{A_2|B_2},$$

where $\delta_{A_i|B_i} = 1$ for either $i = 1$ or $i = 2$. Define $\Delta(\beta_{A|B})$ similarly. On the other hand, if $A|B \notin \mathcal{Q}_{r,1}(n) \cup \mathcal{Q}_{1,s}(n)$, define

$$\Delta(\delta_{A|B}) = \Delta(\delta_{K|L})\Delta(\delta_{M|N})\Delta(\beta_{C|D}),$$

where $K|L$, $M|N$, $C|D$ are given by the formulas in (4.3). Dually, define $\Delta(\beta_{M|N})$. It is easy to check that Δ is well defined.

Given multipermutahedral sets $\mathcal{Z}', \mathcal{Z}'' : \mathbf{P} \to \mathcal{C}_M$, first define a functor

$$\mathcal{Z}' \tilde{\times} \mathcal{Z}'' : \mathbf{P} \times \mathbf{P} \to \mathcal{C}_M$$

on an object $n!!$ by

$$(\mathcal{Z}' \tilde{\times} \mathcal{Z}'')(n!!) = \bigcup_{r+s=n} \mathcal{Z}'(r!) \times \mathcal{Z}''(s!) \big/ \sim,$$

where $(a, b) \sim (c, e)$ if and only if $a = \varrho'_{\underline{r}|r+1}(c)$ and $e = \varrho''_{1|\underline{s+1}\setminus\underline{1}}(b)$. On a map $h = \bigcup(f \times g) : m!! \to n!!$,

$$(\mathcal{Z}' \tilde{\times} \mathcal{Z}'')(h) : (\mathcal{Z}' \tilde{\times} \mathcal{Z}'')(n!!) \to (\mathcal{Z}' \tilde{\times} \mathcal{Z}'')(m!!)$$

is the map induced by $\bigcup(\mathcal{Z}'(f) \times \mathcal{Z}''(g))$. Now define the product $\mathcal{Z}' \times \mathcal{Z}''$ to be the composition of functors

$$\mathcal{Z}' \times \mathcal{Z}'' = \mathcal{Z}' \tilde{\times} \mathcal{Z}'' \circ \Delta : \mathbf{P} \to \mathcal{C}_M.$$

The face operator $d_{A|B}$ on $\mathcal{Z}' \times \mathcal{Z}''$ is given by

$$d_{A|B}(a \times b) = \begin{cases} d'_{\underline{r} \cap A | \underline{r} \cap B}(a) \times b, & \textit{if } A|B \in \mathcal{Q}_{1,s}(n), \\ a \times d''_{(\overline{s} \cap A) - r+1 \,|\, (\overline{s} \cap B) - r+1}(b), & \textit{if } A|B \in \mathcal{Q}_{r,1}(n), \\ \varrho_{C|D} d_{M|N} d_{K|L}(a \times b), & \textit{otherwise}, \end{cases} \tag{4.8}$$

with $M|N$, $K|L$, $C|D$ given by the formulas in (4.3).

Example 6. *The canonical map* $\iota : Sing^P X \times Sing^P Y \to Sing^P(X \times Y)$ *defined for* $(f, g) \in Sing^P_r X \times Sing^P_s Y$ *by*

$$\iota(f, g) = (f \times g) \circ \Delta_{r,s}$$

is a map of permutahedral sets. Consequently, if X *is a topological monoid, the singular permutahedral complex* $Sing^P X$ *inherits a canonical monoidal structure.*

4.4. The diagonal on a permutahedral set

Let $\mathcal{Z} = (\mathcal{Z}_n, d_{A|B}, \varrho_{M|N})$ be a multipermutahedral set. The chain complex of $\mathcal{Z}$ is

$$C^{\diamondsuit}_*(\mathcal{Z}) = C_*(\mathcal{Z})/DGN,$$

where DGN is the submodule generated by the degeneracies;

$$(C_*(\mathcal{Z}), d) = \bigoplus_{\substack{n_{(k)}+1=n \\ n \geqslant 1}} (C_{n-1}(\mathcal{Z}^{n_1 \cdots n_k}), d_{n_1 \cdots n_k})$$

and

$$d_{n_1 \cdots n_k} = \sum_{A|B \in \bigcup_{i=1}^k \mathcal{Q}_{p_i, q_i}(n)} -(-1)^{n_{(i-1)} + n'_i} \; shuff(C_i \cap A; C_i \cap B) \; d_{A|B}.$$

The explicit diagonal

$$\Delta : C^{\diamondsuit}_*(\mathcal{Z}) \to C^{\diamondsuit}_*(\mathcal{Z}) \otimes C^{\diamondsuit}_*(\mathcal{Z})$$

on $a \subset \mathcal{Z}_n$ is given by

$$\Delta(a) = \sum_{\substack{F \in \mathcal{C}^{q \times n-q+2} \\ 0 \leqslant q \leqslant n}} csgn(F) \;\; d_{c(F)}(a) \otimes d_{r(F)}(a), \tag{4.9}$$

where $d_{A_1|\ldots|A_{k+1}} = \mathcal{Z}(f^{n-k}_{A_1|\ldots|A_{k+1}})$.

4.5. The double cobar-construction $\Omega^2 C_*(X)$

Given a simplicial, cubical or a permutahedral set W with base point $*$, let $C_*(W)$ denote the quotient $C_*(W)/C_{>0}(*)$. Say that W is *k-reduced* if W_i contains exactly one element for each $i \leqslant k$ and let ΩC denote the cobar construction on a 1-reduced DG coalgebra C. In [**8**] and [**9**], Kadeishvili and Saneblidze construct functors from the category of 1-reduced simplicial sets to the category of cubical sets and from the category of 1-reduced cubical sets to the category of multipermutahedral sets (denoted by $\mathbf{\Omega}$ in either case) for which the following statements hold (c.f. [**3**], [**2**]):

Theorem 4. *[8] Given a 1-reduced simplicial set* X*, there is a canonical identification isomorphism*

$$\Omega C_*(X) \approx C^{\square}_*(\mathbf{\Omega} X).$$

Theorem 5. *[9] Given a 1-reduced cubical set* Q*, there is a canonical identification isomorphism*

$$\Omega C^{\square}_*(Q) \approx C^{\diamondsuit}_*(\mathbf{\Omega} Q).$$

For completeness, definitions of these two functors appear in the appendix. Since the chain complex of any cubical set Q is a DG coalgebra with strictly coassociative coproduct, setting $Q = \mathbf{\Omega} X$ in Theorem 5 immediately gives:

Theorem 6. *For a 2-reduced simplicial set X there is a canonical identification isomorphism*

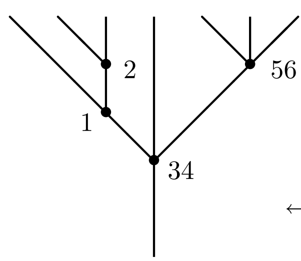
$$\Omega^2 C_*(X) \approx C_*^{\diamondsuit}(\mathbf{\Omega^2} X).$$

Now if $X = Sing^1 Y$, then $\Omega C_*(X)$ is Adams' cobar construction for the space Y [**1**]; consequently, there is a canonical (geometric) coproduct on $\Omega^2 C_*(Sing^1 Y)$. We shall extend this canonical coproduct to an "A_∞-Hopf algebra" structure in the sequel [**16**].

5. The Multiplihedra and Associahedra

The multiplihedron J_n and the associahedron K_{n+1} are cellular projections of P_n defined in terms of planar trees. Consequently, we shall need to index the faces of P_n four ways: (1) by partitions of $\underline{n}$ (see Section 2), (2) by (planar rooted) p-leveled trees with $n+1$ leaves (PLT's), (3) by parenthesized strings of $n+1$ indeterminants with $p-1$ levels of subscripted parentheses and (4) by $(p-1)$-fold compositions of face operators acting on $n+1$ indeterminants. The second and third serve as transitional intermediaries between the first and fourth.

Define a correspondence between PLT's and partitions of $\underline{n}$ as follows: Let T^p_{n+1} be a PLT with $n+1$ leaves, p-levels and root in level p. Number the leaves from left to right and assign the label i to the node at which the branch of leaf i meets the branch of leaf $i+1$, $1 \leqslant i \leqslant n$ (a node may have multiple labels). Let $U_j = \{\text{labels assigned to } j\text{-level nodes}\}$ and identify T^p_{n+1} with the partition $U_1|\cdots|U_p$ of $\underline{n}$ (see Figure 9). Thus binary n-leveled trees parametrize the vertices of P_n. Loday and Ronco constructed a map from S_n to binary n-leveled trees [**12**]; its extension to faces of P_n was given by Tonks [**19**]. Note that the map from PLT's to partitions defined above gives an inverse.



$\longleftrightarrow \quad 256|1|34$

$\longleftrightarrow \quad d_{(0,1)} d_{(1,1)(4,2)}$

$\longleftrightarrow \quad \big((\bullet\,(\bullet\,\bullet)_1)_2\,\bullet\,(\bullet\,\bullet\,\bullet)_1\big)$

Figure 9: Various representations of the face 256|1|34.

To define the correspondence between PLT's and subscripted parenthesizations of $n+1$ indeterminants, begin by identifying the top cell of P_n with the $(n+1)$-leaf corolla and the (unsubscripted) parenthesized string $(x_1 x_2 \cdots x_{n+1})$. Let T^p_{n+1} be a

PLT with $p > 1$. If the branches meeting at a level 1 node contain leaves $i, \ldots, i+k$, enclose the corresponding indeterminants in a pair of parentheses with subscript 1; if the branches meeting at a level 2 node contain leaves $i_1, \ldots, i_k$, enclose the corresponding indeterminants in a pair of parentheses with subscript 2; and so on for $p-1$ steps (see Example 5).

Compositions of face operators encode this parenthesization procedure. For $s \geqslant 1$, choose s pairs of indices $(i_1, \ell_1) \cdots (i_s, \ell_s)$ such that $0 \leqslant i_r < i_{r+1} \leqslant n-1$ and $i_r + \ell_r + 1 \leqslant i_{r+1}$. The face operator

$$d_{(i_1,\ell_1)\cdots(i_s,\ell_s)} : P_n \to \partial P_n$$

acts on $(x_1 x_2 \cdots x_{n+1})$ by simultaneously inserting s disjoint (non-nested) pairs of inner parentheses with subscript 1, the first enclosing $x_{i_1+1} \cdots x_{i_1+\ell_1+1}$, the second enclosing $x_{i_2+1} \cdots x_{i_2+\ell_2+1}$, and so on. Thus,

$$d_{(i_1,\ell_1)\cdots(i_s,\ell_s)}\,(x_1 x_2 \cdots x_{n+1}) =$$
$$(x_1 \cdots (x_{i_1+1} \cdots x_{i_1+\ell_1+1})_1 \cdots (x_{i_s+1} \cdots x_{i_s+\ell_s+1})_1 \cdots x_{n+1}).$$

A composition of face operators

$$d_{\left(i_1^k,\ell_1^k\right)\cdots\left(i_{s_k}^k,\ell_{s_k}^k\right)} \cdots d_{\left(i_1^1,\ell_1^1\right)\cdots\left(i_{s_1}^1,\ell_{s_1}^1\right)} : P_n \to \partial^k P_n \tag{5.1}$$

continues this process inductively: If the j^{th} operator inserted parentheses with subscript j, treat each such pair and its contents as a single indeterminant and apply the $(j+1)^{st}$ as above, inserting parentheses subscripted by $j+1$.

Refer to Figure 9 above. The composition $d_{(0,1)}d_{(1,1)(4,2)}$ acts on $(\bullet\bullet\bullet\bullet\bullet\bullet\bullet)$ in the following way: First, $d_{(1,1)(4,2)}$ simultaneously inserts two inner pairs of parentheses with subscript 1:

$$(\bullet\,(\bullet\bullet)_1\,\bullet\,(\bullet\bullet\bullet)_1).$$

Next, $d_{(0,1)}$ inserts the single pair with subscript 2:

$$\left((\bullet\,(\bullet\bullet)_1)_2\,\bullet\,(\bullet\bullet\bullet)_1\right).$$

We summarize the discussion above as a proposition.

Proposition 5. *The following correspondences (defined above) preserve combinatorial structure:*

$$\begin{aligned}
\{\textit{Faces of } P_n\} \quad &\leftrightarrow \quad \{\textit{Partitions of } \underline{n}\} \\
&\leftrightarrow \quad \{\textit{Leveled trees with } n+1 \text{ leaves}\} \\
&\leftrightarrow \quad \left\{ \begin{array}{c} \textit{Strings of } n+1 \textit{ indeterminants} \\ \textit{with subscripted parentheses} \end{array} \right\} \\
&\leftrightarrow \quad \left\{ \begin{array}{c} \textit{Compositions of face operators} \\ \textit{acting on } n+1 \textit{ indeterminants.} \end{array} \right\}
\end{aligned}$$

Assign the identity face operator Id to the top dimensional face of P_{n+1} and use the correspondences above to assign compositions of faces operators to lower dimensional faces (see Figure 10). For faces in codimension 1 we have:

Face of P_{n+1}	**Face operator**
$P_n \times 0$	$d_{(0,n)}$
$P_n \times 1$	$d_{(n,1)}$
$d_{(i_1,\ell_1)\cdots(i_k,\ell_k)} \times \left[0, 1-2^{\ell_1+\cdots+\ell_k-n}\right]$	$d_{(i_1,\ell_1)\cdots(i_k,\ell_k)}$
$d_{(i_1,\ell_1)\cdots(i_k,\ell_k)} \times \left[1-2^{\ell_1+\cdots+\ell_k-n}, 1\right]$	$\begin{cases} d_{(i_1,\ell_1)\cdots(i_k,\ell_k)(n,1)}, & i_k+\ell_k<n \\ d_{(i_1,\ell_1)\cdots(i_k,\ell_k+1)}, & i_k+\ell_k=n. \end{cases}$

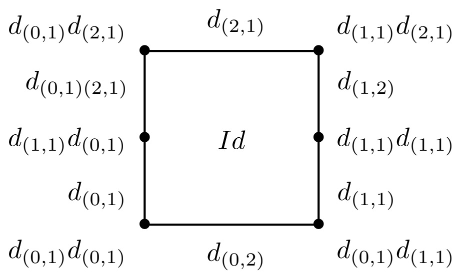


Figure 10: P_3 with face-operator labeling.

Since compositions of face operators are determined by the correspondence between faces and partitions, we only label the codimension 1 faces of the related polytopes below.

The associahedra $\{K_n\}$ serve as parameter spaces for higher homotopy associativity. In his seminal papers of 1963 [**18**], J. Stasheff constructed K_n in the following way: Let $K_2 = *$; if K_{n-1} has been constructed, define K_n to be the cone on the set

$$\bigcup_{\substack{r+s=n+1 \\ 1\leqslant k\leqslant n-s+1}} \left(K_r \times K_s\right)_k .$$

Thus, K_n is an $(n-2)$-dimensional convex polytope.

Stasheff's motivating example of higher homotopy associativity in [**18**] is the singular chain complex on the (Poincarè) loop space of a connected CW-complex. Here associativity holds up to homotopy, the homotopies between the various associations are homotopic, the homotopies between these homotopies are homotopic, and so on. An abstract A_∞-algebra is a DGA in which associativity behaves as in Stasheff's motivating example. If $\varphi^2 : A\otimes A \to A$ is the multiplication on an A_∞-algebra A, the homotopies $\varphi^n : A^{\otimes n} \to A$ are multilinear operations such that φ^3 is a chain homotopy between the associations $(ab)\,c$ and $a\,(bc)$ thought of as quadratic compositions $\varphi^2\left(\varphi^2\otimes 1\right)$ and $\varphi^2\left(1\otimes\varphi^2\right)$ in three variables, φ^4 is a chain homotopy bounding the cycle of five quadratic compositions in four variables involving φ^2 and

φ^3, and so on. Let $C_*(K_r)$ denote the cellular chains on K_r. The natural correspondence between faces of K_r and the various compositions of the φ^n's in r variables (modulo an appropriate equivalence) induces a chain map $C_*(K_r) \to Hom(A^{\otimes r}, A)$ that determines the relations among the compositions of φ^n's. This chain map together with our diagonal on K_n leads to the tensor product of A_∞-algebras (see Section 5).

Now if we disregard levels, a PLT is simply a planar rooted tree (PRT). Quite remarkably, A. Tonks [**19**] showed that K_n is the identification space $P_{n-1}/\sim$ in which all faces indexed by isomorphic PRT's are identified. Since the quotient map $\theta : P_{n-1} \to K_n$ is cellular, the faces of K_n are indexed by PRT's with n leaves. The correspondence between PRT's with n leaves and parenthesizations of n indeterminants is simply this: Given a node N, parenthesize the indeterminants that correspond to leaves on all branches that meet at node N.

Example 7. *With one exception, all classes of faces of P_3 consist of a single element. Elements of the exceptional class*

$$[1|3|2, 13|2, 3|1|2]$$

represent the parenthesization $((\bullet\bullet)(\bullet\bullet))$. Whereas $1|3|2$ and $3|1|2$ insert inner parentheses in the opposite order, the element $13|2$ inserts inner parentheses simultaneously and represents a homotopy between $1|3|2$ and $3|1|2$. Tonks' projection θ sends the exceptional class to the vertex of K_4 represented by the parenthesization $((\bullet\bullet)(\bullet\bullet))$. The classes of faces of P_4 with more than one element and their projections to K_5 are:

$$\begin{array}{l}
[12|4|3, 124|3, 4|12|3] \xrightarrow{\theta} ((\bullet\bullet\bullet)(\bullet\bullet)) \\
[1|3|24, 13|24, 3|1|24] \longrightarrow ((\bullet\bullet)(\bullet\bullet)\bullet) \\
[1|4|23, 14|23, 4|1|23] \longrightarrow ((\bullet\bullet)\bullet(\bullet\bullet)) \\
[2|4|13, 24|13, 4|2|13] \longrightarrow (\bullet(\bullet\bullet)(\bullet\bullet)) \\
[1|34|2, 134|2, 34|1|2] \longrightarrow ((\bullet\bullet)(\bullet\bullet\bullet))
\end{array}$$

$$\begin{array}{l}
[1|3|2|4, 13|2|4, 3|1|2|4] \longrightarrow (((\bullet\bullet)(\bullet\bullet))\bullet) \\
[2|4|3|1, 24|3|1, 4|2|3|1] \longrightarrow (\bullet((\bullet\bullet)(\bullet\bullet))) \\
[1|2|4|3, 1|24|3, 1|4|2|3, 14|2|3, 4|1|2|3] \longrightarrow (((\bullet\bullet)\bullet)(\bullet\bullet)) \\
[1|3|4|2, 13|4|2, 3|1|4|2, 3|14|2, 3|4|1|2] \longrightarrow ((\bullet\bullet)((\bullet\bullet)\bullet)) \\
[1|4|3|2, 14|3|2, 4|1|3|2, 4|13|2, 4|3|1|2] \longrightarrow ((\bullet\bullet)(\bullet(\bullet\bullet))) \\
[2|1|4|3, 2|14|3, 2|4|1|3, 24|1|3, 4|2|1|3] \longrightarrow ((\bullet(\bullet\bullet))(\bullet\bullet)).
\end{array}$$

Faces and edges represented by elements of the first five classes project to edges; edges and vertices represented by elements of the next six classes project to vertices.

The multiplihedra $\{J_{n+1}\}$, which serve as parameter spaces for homotopy multiplicative morphisms of A_∞-algebras, lie between the associahedra and permutahedra (see [**18**], [**6**]). If $f^1 : A \to B$ is such a morphism, there is a chain homotopy f^2 between the quadratic compositions $f^1\varphi_A^2$ and $\varphi_B^2(f^1 \otimes f^1)$ in two variables, there is a chain homotopy f^3 bounding the cycle of the six quadratic compositions in three variables involving f^1, f^2, φ_A^2, φ_A^3, φ_B^2 and φ_B^3, and so on. The natural correspondence between faces of J_r and the various compositions of f^i, φ_A^j and φ_B^k in r variables (modulo an appropriate equivalence) induces a chain map $C_*(J_r) \to Hom(A^{\otimes r}, B)$.

The multiplihedron J_{n+1} can also be realized as a subdivision of the cube I^n. For $n = 0, 1, 2$, set $J_{n+1} = P_{n+1}$. If J_n has been constructed, J_{n+1} is the subdivision of $J_n \times I$ given below and its various $(n-1)$-faces are labeled as indicated:

Face of J_{n+1}	**Face operator**
$J_n \times 0$	$d_{(0,n)}$
$J_n \times 1$	$d_{(n,1)}$
$d_{(i,\ell)} \times I$	$d_{(i,\ell)}, \quad 1 \leqslant i < n - \ell$
$d_{(i,\ell)} \times \left[0, 1 - 2^{-i}\right]$	$d_{(i,\ell)}, \quad 1 \leqslant i = n - \ell$
$d_{(i,\ell)} \times \left[1 - 2^{-i}, 1\right]$	$d_{(i,\ell+1)}, \ 1 \leqslant i = n - \ell$
$d_{(0,\ell_1)\cdots(i_k,\ell_k)} \times \left[0, 1 - 2^{k-n}\right]$	$d_{(0,\ell_1)\cdots(i_k,\ell_k)}$
$d_{(0,\ell_1)\cdots(i_k,\ell_k)} \times \left[1 - 2^{k-n}, 1\right]$	$\begin{cases} d_{(0,\ell_1)\cdots(i_k,\ell_k)(n,1)}, & i_k < n - \ell_k \\ d_{(0,\ell_1)\cdots(i_k,\ell_k+1)}, & i_k = n - \ell_k. \end{cases}$

Thus faces of J_{n+1} are indexed by compositions of face operators of the form

$$d_{(i_m,\ell_m)} \cdots d_{(i_{k_1},\ell_{k_1})\cdots(i_{k_s},\ell_{k_s})} \cdots d_{(i_1,\ell_1)}. \tag{5.2}$$

In terms of trees and parenthesizations this says the following: Let T be a $(k+1)$-leveled tree with left-most branch attached at level p. For $1 \leqslant j < p$, insert level j parentheses one pair at a time without regard to order as in K_{n+2}; next, insert all level p parentheses simultaneously as in P_{n+1}; finally, for $j > p$, insert level j parentheses one pair at a time without regard to order. Thus multiple lower indices in a composition of face operators may only occur when the left-most branch is attached above the root. This suggests the following equivalence relation on the set of $(k+1)$-leveled trees with $n+2$ leaves: Let T and T' be p-leveled trees with $n+2$ nodes whose p-level meets U_p and U_p' contain 1. Then $T \sim T'$ if T and T' are isomorphic as PLT's and $U_p = U_p'$. This equivalence relation induces a cellular projection $\pi : P_{n+1} \to J_{n+1}$ under which J_{n+1} can be realized as an identification space of P_{n+1}. Furthermore, the projection $J_{n+1} \to K_{n+2}$ given by identifying faces of J_{n+1} indexed by isomorphic PLT's gives the factorization $P_{n+1} \xrightarrow{\pi} J_{n+1} \to K_{n+2}$ of Tonks' projection.

It is interesting to note the role of the indices ℓ_j in compositions of face operators representing the faces of J_{n+1} as in (5.2). With one exception, each U_j in the corresponding partition $U_1|\cdots|U_{m+1}$ is a set of consecutive integers; this holds without exception for all U_j on K_{n+2}. The exceptional set U_p is a union of s sets of consecutive integers with maximal cardinality, as is typical of sets U_j on P_{n+1}. Thus J_{n+1} exhibits characteristics of both combinatorial structures.

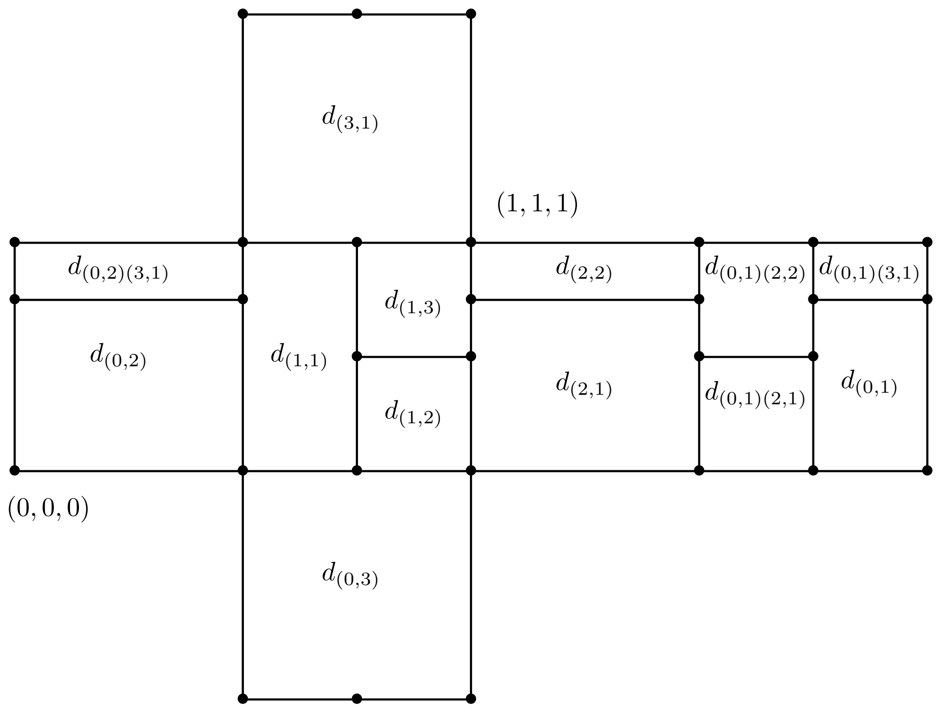


Figure 11: J_4 as a subdivision of $J_3 \times I$.

We realize the associahedron K_{n+2} in a similar way. For $n = 0, 1$, set $K_{n+2} = P_{n+1}$. If K_{n+1} has been constructed, let $e_{i,\epsilon}$ denote the face $(x_1, \ldots, x_{i-1}, \epsilon, x_{i+1}, \ldots, x_n) \subset I^n$, where $\epsilon = 0, 1$ and $1 \leqslant i \leqslant n$. Then K_{n+2} is the subdivision of $K_{n+1} \times I$ given below and its various $(n-1)$-faces are labeled as indicated:

Face of K_{n+2}	**Face operator**	
$e_{\ell,0}$	$d_{(0,\ell)}$,	$1 \leqslant \ell \leqslant n$
$e_{n,1}$	$d_{(n,1)}$,	
$d_{(i,\ell)} \times I$	$d_{(i,\ell)}$,	$1 \leqslant i < n - \ell$
$d_{(i,\ell)} \times \left[0, 1 - 2^{-i}\right]$	$d_{(i,\ell)}$,	$1 \leqslant i = n - \ell$
$d_{(i,\ell)} \times \left[1 - 2^{-i}, 1\right]$	$d_{(i,\ell+1)}$,	$1 \leqslant i = n - \ell$

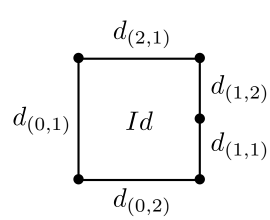


Figure 12: K_4 as a subdivision of $K_3 \times I$.

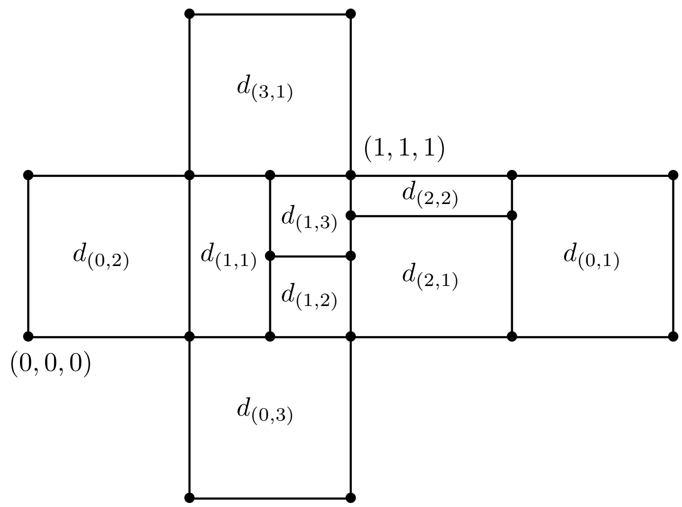


Figure 13: K_5 as a subdivision of $K_4 \times I$.

6. Diagonals on the Associahedra and Multiplihedra

The diagonal Δ_P on $C_*(P_{n+1})$ descends to diagonals Δ_J on $C_*(J_{n+1})$ and Δ_K on $C_*(K_{n+2})$ via the cellular projections $\pi : P_{n+1} \to J_{n+1}$ and $\theta : P_{n+1} \to K_{n+2}$ discussed in Section 2 above. This fact is an immediate consequence of Proposition 7.

Definition 20. *Let $f : W \to X$ be a cellular map of CW-complexes, let Δ_W be a diagonal on $C_*(W)$ and let $X^{(r)}$ denote the r-skeleton of X. A k-cell $e \subseteq W$ is degenerate under f if $f(e) \subseteq X^{(r)}$ with $r < k$. A component $a \otimes b$ of Δ_W is degenerate under f if either a or b is degenerate under f.*

Let us identify the non-degenerate cells of P_{n+1} under π and θ.

Definition 21. *Let $A_1|\cdots|A_p$ be a partition of $\underline{n+1}$ with $p > 1$ and let $1 \leqslant k < p$. The subset A_k is exceptional if for $k < j \leqslant p$, there is an element $a_{i,j} \in A_j$ such that $\min A_k < a_{i,j} < \max A_k$.*

Proposition 6. *Let $a = A_1|\cdots|A_p$ be a face of P_{n+1} and let*

$$d_{(i_1^{p-1},\ell_1^{p-1})\cdots(i_{s_{p-1}}^{p-1},\ell_{s_{p-1}}^{p-1})}\cdots d_{(i_1^1,\ell_1^1)\cdots(i_{s_1}^1,\ell_{s_1}^1)}$$

be its unique representation as a composition of face operators.

(1) The following are all equivalent:

(1a) The face a is degenerate under π.

(1b) $\min A_j > \min(A_{j+1} \cup \cdots \cup A_p)$ with A_j exceptional for some $j < p$.

(1c) $i_1^k > 0$ and $s_k > 1$ for some $k < p$.

(2) The following are all equivalent:

(2a) The face a is degenerate under θ.

(2b) A_j is exceptional for some $j < p$.

(2c) $s_k > 1$ for some $k < p$.

Proof. Obvious. □

Example 8. *The subset $A_1 = \{13\}$ in the partition $a = 13|24$ is exceptional and the face $a \subset P_4$ is degenerate under θ. In terms of compositions of face operators, the face a corresponds to $d_{(0,1)(2,1)}(x_1 \cdots x_5)$ with $s_1 = 2$. Furthermore, a is also non-degenerate under π since $i_1^1 = 0$ (equivalently, $\min A_1 < \min A_2$).*

Next, we apply Tonks' projection and obtain an explicit formula for the diagonal Δ_K on the associahedra.

Proposition 7. *Let $f : W \to X$ be a surjective cellular map and let Δ_W be a diagonal on $C_*(W)$. Then Δ_W uniquely determines a diagonal Δ_X on $C_*(X)$ given by the non-degenerate components of Δ_W under f. Moreover, Δ_X is the unique map that commutes the following diagram:*

$$\begin{array}{ccc} C_*(W) & \xrightarrow{\Delta_W} & C_*(W) \otimes C_*(W) \\ f \downarrow & & \downarrow f \otimes f \\ C_*(X) & \underset{\Delta_X}{\longrightarrow} & C_*(X) \otimes C_*(X). \end{array}$$

Proof. Obvious. □

In Section 2 we established correspondences between faces of P_{n+1} and PLT's with $n+2$ leaves and between faces of K_{n+2} and PRT's with $n+2$ leaves. Since a PRT can be viewed as a PLT, faces of K_{n+2} can be viewed as faces of P_{n+1}.

Definition 22. *For $n \geqslant 0$, let Δ_P be the diagonal on $C_*(P_{n+1})$ and let $\theta : P_{n+1} \to K_{n+2}$ be Tonks' projection. View each face e of the associahedron K_{n+2} as a face of P_{n+1} and define $\Delta_K : C_*(K_{n+2}) \to C_*(K_{n+2}) \otimes C_*(K_{n+2})$ by*

$$\Delta_K(e) = (\theta \otimes \theta)\Delta_P(e).$$

Corollary 2. *The map Δ_K given by Definition 22 is the diagonal on $C_*(K_{n+2})$ induced by Δ_P.*

Proof. This is an immediate application of Proposition 7. □

Consider a CP $u \otimes v = c(F) \otimes r(F)$ related to SCP $a \otimes b = c(E) \otimes r(E)$ via $F = D_{N_{q-1}} \cdots D_1 R_{M_{p-1}} \cdots R_1 E$. Note that both factors of $u \otimes v$ are non-degenerate under θ if and only if b is non-degenerate and each M_j has maximal cardinality. Alternatively, if $d^p \cdots d^1 \otimes d^q \cdots d^1 (e^n \otimes e^n)$ is a component of $\Delta_K(e^n)$, factors in the corresponding pairing $T_p \otimes T_q$ of PRT's have $n+2$ leaves, $p+1$ and $q+1$ nodes and respective dimensions $n-p$ and $n-q$. Hence $p+q = n$ and $T_p \otimes T_q$

has exactly $n+2$ nodes. But if $T_u \otimes T_v$ is the pairing of PLT's corresponding to $u \otimes v$, forgetting levels in $T_u \otimes T_v$ gives the pairing of PRT's corresponding to $\theta(u) \otimes \theta(v)$. Since the number of nodes in $T_u \otimes T_v$ is at least $n+2$, $\theta(a) \otimes \theta(b)$ is non-degenerate in $\Delta_K(e^n)$ if and only if the total number of nodes in $T_u \otimes T_v$ is exactly $n+2$.

Choose a system of generators $e^n \in C_n(K_{n+2})$, $n \geqslant 0$. The signs in (6.1) below follow from (3.2).

Definition 23 ([15]). *For each $n \geqslant 0$, define Δ_K on $e^n \in C_n(K_{n+2})$ by*

$$\Delta_K(e^n) = \sum_{0 \leqslant p \leqslant p+q=n+2} (-1)^{\epsilon} \; d_{(i_{p-1},\ell_{p-1})} \cdots d_{(i_1,\ell_1)} \otimes d_{(i'_{q-1},\ell'_{q-1})} \cdots d_{(i'_1,\ell'_1)} (e^n \otimes e^n), \tag{6.1}$$

where

$$\epsilon = \sum_{j=1}^{p-1} i_j(\ell_j+1) + \sum_{k=1}^{q-1} (i'_k + k + q)\ell'_k,$$

and lower indices $\big((i_1,\ell_1),\ldots,(i_{p-1},\ell_{p-1});(i'_1,\ell'_1),\ldots,(i'_{q-1},\ell'_{q-1})\big)$ range over all solutions of the following system of inequalities:

$$\left\{\begin{array}{ll} 1 \leqslant i'_j < i'_{j-1} \leqslant n+1 & (1) \\ 1 \leqslant \ell'_j \leqslant n+1-i'_j-\ell'_{(j-1)} & (2) \\ 0 \leqslant i_k \leqslant \min\limits_{o'(t_k)<r<k} \left\{ i_r,\ i'_{t_k} - \ell_{(o'(t_k))} \right\} & (3) \\ 1 \leqslant \ell_k = \epsilon_k - i_k - \ell_{(k-1)}, & (4) \end{array}\right\}_{\substack{1 \leqslant k \leqslant p-1 \\ 1 \leqslant j \leqslant q-1}} \tag{6.2}$$

where

$$\{\epsilon_1 < \cdots < \epsilon_{q-1}\} = \{1,\ldots,n\} \setminus \left\{i'_1,\ldots,i'_{q-1}\right\};$$

$$\epsilon_0 = \ell_0 = \ell'_0 = i_p = i'_q = 0;$$

$$i_0 = i'_0 = \epsilon_q = \ell_{(p)} = \ell'_{(q)} = n+1;$$

$$\ell_{(u)} = \textstyle\sum_{j=0}^{u} \ell_j \text{ for } 0 \leqslant u \leqslant p;$$

$$\ell'_{(u)} = \textstyle\sum_{k=0}^{u} \ell'_k \text{ for } 0 \leqslant u \leqslant q;$$

$$t_u = \min\left\{ r \mid i'_r + \ell'_{(r)} - \ell'_{(o(u))} > \epsilon_u > i'_r \right\};$$

$$o(u) = \max\{r \mid i'_r \geqslant \epsilon_u\}; \text{ and}$$

$$o'(u) = \max\{r \mid \epsilon_r \leqslant i'_u\}.$$

Extend Δ_K to proper faces of K_{n+2} via the standard comultiplicative extension.

Theorem 7. *The map Δ_K given by Definition 23 is the diagonal induced by θ.*

Proof. If $v = L_\beta(v')$ is non-degenerate in some component $u \otimes v$ of Δ_P, then so is v', and we immediately obtain inequality (1) of (6.2). Next, each non-degenerate decreasing b uniquely determines an SCP $a \otimes b$. Although a may be degenerate, there is a unique non-degenerate $u = R_{M_{p-1}} \cdots R_{M_1}(a)$ obtained by choosing each M_j with maximal cardinality (the case $M_j = \varnothing$ for all j may nevertheless occur); then $u \otimes b$ is a non-degenerate CP associated with $a \otimes b$ in Δ_P. As a composition of face operators, straightforward examination shows that u has form $u = d_{(i_{p-1},\ell_{p-1})} \cdots d_{(i_1,\ell_1)}(e^n)$ and is related to $b = d_{(i'_{q-1},\ell'_{q-1})} \cdots d_{(i'_1,\ell'_1)}(e^n)$ by

$$i_k = \min_{o'(t_k)<r<k} \left\{ i_r,\ i'_{t_k} - \ell_{(o'(t_k))} \right\},\ 1 \leqslant k < p;$$

and equality holds in (4) of (6.2). Finally, let $b = L_\beta(\bar{b})$. As we vary $\bar{b}$ in all possible ways, each $\bar{b}$ is non-degenerate and we obtain all possible non-degenerate CP's $\bar{u} \otimes \bar{b}$ associated with $\bar{a} \otimes \bar{b}$ ($\bar{u} = u$ when $\bar{b} = b$ and $\beta = \varnothing$). For each such $\bar{u} = d_{(i_{p-1},\ell_{p-1})} \cdots d_{(i_1,\ell_1)}(e^n)$ we have both inequality (3) and equality in (4) of (6.2). Hence, the theorem is proved. □

Example 9. *On K_4 we obtain:*

$$\Delta_K\left(e^2\right) = \Big\{ d_{(0,1)}d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)}d_{(2,1)} + d_{(0,2)} \otimes d_{(1,1)} + d_{(0,2)} \otimes d_{(1,2)} + d_{(1,1)} \otimes d_{(1,2)} - d_{(0,1)} \otimes d_{(2,1)} \Big\} (e^2 \otimes e^2).$$

7. Application: Tensor Products of A_∞-(co)algebras

In this section, we use Δ_K to define the tensor product of A_∞-(co)algebras in maximal generality. We note that a special case was given by J. Smith [**17**] for certain objects with a richer structure than we have here. We also mention that Lada and Markl [**11**] defined an A_∞ tensor product structure on a construct different from the tensor product of graded modules.

We adopt the following notation and conventions: Let R be a commutative ring with unity; R-modules are assumed to be $\mathbb{Z}$-graded, tensor products and Hom's are defined over R and all maps are R-module maps unless otherwise indicated. If an R-module V is connected, $\overline{V} = V/V_0$. The symbol $1 : V \to V$ denotes the identity map; the suspension and desuspension maps $\uparrow$ and $\downarrow$ shift dimension by $+1$ and -1, respectively. Define $V^{\otimes 0} = R$ and $V^{\otimes n} = V \otimes \cdots \otimes V$ with $n > 0$ factors; then $TV = \oplus_{n \geqslant 0} V^{\otimes n}$ and T^aV (respectively, T^cV) denotes the free tensor algebra (respectively, cofree tensor coalgebra) of V. Given R-modules $V_1, \ldots, V_n$, a permutation $\sigma \in S_n$ induces an isomorphism $\sigma : V_1 \otimes \cdots \otimes V_n \to V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$ by $\sigma(x_1 \cdots x_n) = \pm\, x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}$, where $\pm$ is the Koszul sign. In particular, $\sigma_{2,n} = (1\ 3\ \cdots\ (2n-1)\ 2\ 4\ \cdots\ 2n) : (A \otimes B)^{\otimes n} \to A^{\otimes n} \otimes B^{\otimes n}$ and $\sigma_{n,2} = \sigma_{2,n}^{-1}$ induce isomorphisms $(\sigma_{2,n})^* : Hom\left(A^{\otimes n} \otimes B^{\otimes n}, A \otimes B\right) \to Hom\left((A \otimes B)^{\otimes n}, A \otimes B\right)$ and $(\sigma_{n,2})_* : Hom\left(A \otimes B, A^{\otimes n} \otimes B^{\otimes n}\right) \to Hom\left(A \otimes B,\ (A \otimes B)^{\otimes n}\right)$. The map $\iota : Hom(U,V) \otimes Hom(U',V') \to Hom(U \otimes U', V \otimes V')$ is the canonical isomorphism. If $f : V^{\otimes p} \to V^{\otimes q}$ is a map, we let $f_{i,n-p-i} = 1^{\otimes i} \otimes f \otimes 1^{\otimes n-p-i} : V^{\otimes n} \to$

$V^{\otimes n-p+q}$, where $0 \leqslant i \leqslant n-p$. The abbreviations *DGM, DGA,*and *DGC* stand for *differential graded R-module, DG R-algebra* and *DG R-coalgebra*, respectively.

We begin with a review of A_∞-(co)algebras paying particular attention to the signs. Let A be a connected R-module equipped with operations $\{\varphi^k \in Hom^{k-2}\left(A^{\otimes k}, A\right)\}_{k\geqslant 1}$. For each k and $n \geqslant 1$, linearly extend φ^k to $A^{\otimes n}$ via

$$\sum_{i=0}^{n-k} \varphi^k_{i,n-k-i} : A^{\otimes n} \to A^{\otimes n-k+1},$$

and consider the induced map of degree -1 given by

$$\sum_{i=0}^{n-k} \left(\uparrow \varphi^k \downarrow^{\otimes k}\right)_{i,n-k-i} : \left(\uparrow \overline{A}\right)^{\otimes n} \to \left(\uparrow \overline{A}\right)^{\otimes n-k+1}.$$

Let $\widetilde{B}A = T^c\left(\uparrow \overline{A}\right)$ and define a map $d_{\widetilde{B}A} : \widetilde{B}A \to \widetilde{B}A$ of degree -1 by

$$d_{\widetilde{B}A} = \sum_{\substack{1\leqslant k\leqslant n\\ 0\leqslant i\leqslant n-k}} \left(\uparrow \varphi^k \downarrow^{\otimes k}\right)_{i,n-k-i}. \tag{7.1}$$

The identities $(-1)^{[n/2]} \uparrow^{\otimes n}\downarrow^{\otimes n} = 1^{\otimes n}$ and $[n/2] + [(n+k)/2] \equiv nk + [k/2] \pmod 2$ imply that

$$d_{\widetilde{B}A} = \sum_{\substack{1\leqslant k\leqslant n\\ 0\leqslant i\leqslant n-k}} (-1)^{[(n-k)/2]+i(k+1)} \uparrow^{\otimes n-k+1} \varphi^k_{i,n-k-i} \downarrow^{\otimes n}. \tag{7.2}$$

Definition 24. *$(A, \varphi^n)_{n\geqslant 1}$ is an* <u>*A_∞-algebra*</u> *if $d^2_{\widetilde{B}A} = 0$.*

Proposition 8. *For each $n \geqslant 1$, the operations $\{\varphi^n\}$ on an A_∞-algebra satisfy the following quadratic relations:*

$$\sum_{\substack{0\leqslant \ell\leqslant n-1\\ 0\leqslant i\leqslant n-\ell-1}} (-1)^{\ell(i+1)} \varphi^{n-\ell}\varphi^{\ell+1}_{i,n-\ell-1-i} = 0. \tag{7.3}$$

Proof. For $n \geqslant 1$,

$$\begin{aligned} 0 &= \sum_{\substack{1\leqslant k\leqslant n\\ 0\leqslant i\leqslant n-k}} (-1)^{[(n-k)/2]+i(k+1)} \uparrow \varphi^{n-k+1} \downarrow^{\otimes n-k+1}\uparrow^{\otimes n-k+1} \varphi^k_{i,n-k-i} \downarrow^{\otimes n} \\ &= \sum_{\substack{1\leqslant k\leqslant n\\ 0\leqslant i\leqslant n-k}} (-1)^{n-k+i(k+1)} \varphi^{n-k-1}\varphi^k_{i,n-k-i} \\ &= -(-1)^n \sum_{\substack{0\leqslant \ell\leqslant n-1\\ 0\leqslant i\leqslant n-\ell-1}} (-1)^{\ell(i+1)} \varphi^{n-\ell}\varphi^{\ell+1}_{i,n-\ell-1-i}. \end{aligned}$$

□

It is easy to prove that

Proposition 9. *If $(A, \varphi^n)_{n\geqslant 1}$ is an A_∞-algebra, then $\left(\widetilde{B}A, d_{\widetilde{B}A}\right)$ is a DGC.*

Definition 25. *Let $(A, \varphi^n)_{n\geqslant 1}$ be an A_∞-algebra. The tilde bar construction on A is the DGC $\left(\widetilde{B}A, d_{\widetilde{B}A}\right)$.*

Definition 26. *Let A and C be A_∞-algebras. A chain map $f = f^1 : A \to C$ is a map of A_∞-algebras if there is a sequence of maps $\{f^k \in Hom^{k-1}\left(A^{\otimes k}, C\right)\}_{k\geqslant 2}$ such that*

$$\widetilde{f} = \sum_{n\geqslant 1}\left(\sum_{k\geqslant 1}\uparrow f^k \downarrow^{\otimes k}\right)^{\otimes n} : \widetilde{B}A \to \widetilde{B}C$$

is a DGC map.

Dually, consider a sequence of operations $\{\psi^k \in Hom^{k-2}\left(A, A^{\otimes k}\right)\}_{k\geqslant 1}$. For each k and $n \geqslant 1$, linearly extend each ψ^k to $A^{\otimes n}$ via

$$\sum_{i=0}^{n-1}\psi^k_{i,n-1-i} : A^{\otimes n} \to A^{\otimes n+k-1},$$

and consider the induced map of degree -1 given by

$$\sum_{i=0}^{n-1}\left(\downarrow^{\otimes k}\psi^k\uparrow\right)_{i,n-1-i} : \left(\downarrow \overline{A}\right)^{\otimes n} \to \left(\downarrow \overline{A}\right)^{\otimes n+k-1}.$$

Let $\widetilde{\Omega}A = T^a\left(\downarrow \overline{A}\right)$ and define a map $d_{\widetilde{\Omega}A} : \widetilde{\Omega}A \to \widetilde{\Omega}A$ of degree -1 by

$$d_{\widetilde{\Omega}A} = \sum_{\substack{n,k\geqslant 1\\ 0\leqslant i\leqslant n-1}}\left(\downarrow^{\otimes k}\psi^k\uparrow\right)_{i,n-1-i},$$

which can be rewritten as

$$d_{\widetilde{\Omega}A} = \sum_{\substack{n,k\geqslant 1\\ 0\leqslant i\leqslant n-1}}(-1)^{[n/2]+i(k+1)+k(n+1)}\downarrow^{\otimes n+k-1}\psi^k_{i,n-1-i}\uparrow^{\otimes n}. \tag{7.4}$$

Definition 27. $(A, \psi^n)_{n\geqslant 1}$ *is an A_∞-coalgebra if $d^2_{\widetilde{\Omega}A} = 0$.*

Proposition 10. *For each $n \geqslant 1$, the operations $\left\{\psi^k\right\}$ on an A_∞-coalgebra satisfy the following quadratic relations:*

$$\sum_{\substack{0\leqslant \ell\leqslant n-1\\ 0\leqslant i\leqslant n-\ell-1}}(-1)^{\ell(n+i+1)}\psi^{\ell+1}_{i,n-\ell-1-i}\psi^{n-\ell} = 0. \tag{7.5}$$

Proof. The proof is similar to the proof of Proposition 8 and is omitted. □

Again, it is easy to prove that

Proposition 11. *If $(A, \psi^n)_{n\geqslant 1}$ is an A_∞-coalgebra , then $\left(\widetilde{\Omega}A, d_{\widetilde{\Omega}A}\right)$ is a DGA.*

Definition 28. *Let $(A, \psi^n)_{n\geqslant 1}$ be an A_∞-coalgebra. The tilde cobar construction on Ais the DGA $\left(\widetilde{\Omega}A, d_{\widetilde{\Omega}A}\right)$.*

Definition 29. *Let A and B be A_∞-coalgebras. A chain map $g = g^1 : A \to B$ is a* <u>*map of A_∞-coalgebras*</u> *if there is a sequence of maps $\{g^k \in Hom^{k-1}\left(A, B^{\otimes k}\right)\}_{k\geqslant 2}$ such that*

$$\widetilde{g} = \sum_{n\geqslant 1}\left(\sum_{k\geqslant 1} \downarrow^{\otimes k} g^k \uparrow\right)^{\otimes n} : \widetilde{\Omega}A \to \widetilde{\Omega}B, \tag{7.6}$$

is a DGA map.

The structure of an A_∞-(co)algebra is encoded by the quadratic relations among its operations (also called "higher homotopies"). Although the "direction," i.e., sign, of these higher homotopies is arbitrary, each choice of directions determines a set of signs in the quadratic relations, the "simplest" of which appears on the algebra side when no changes of direction are made; see (7.1) and (7.3) above. Interestingly, the "simplest" set of signs appear on the coalgebra side when ψ^n is replaced by $(-1)^{[(n-1)/2]}\,\psi^n$, $n \geqslant 1$, i.e., the direction of every third and fourth homotopy is reversed. The choices one makes will depend on the application; for us the appropriate choices are as in (7.3) and (7.5).

Let $\mathcal{A}_\infty = \oplus_{n\geqslant 2} C_*\left(K_n\right)$ and let $\left(A, \varphi^n\right)_{n\geqslant 1}$ be an A_∞-algebra with quadratic relations as in (7.3). For each $n \geqslant 2$, associate $e^{n-2} \in C_{n-2}\left(K_n\right)$ with the operation φ^n via

$$e^{n-2} \mapsto (-1)^n\,\varphi^n \tag{7.7}$$

and each codimension 1 face $d_{(i,\ell)}\left(e^{n-2}\right) \in C_{n-3}\left(K_n\right)$ with the quadratic composition

$$d_{(i,\ell)}\left(e^{n-2}\right) \mapsto \varphi^{n-\ell}\varphi^{\ell+1}_{i,n-\ell-1-i}. \tag{7.8}$$

Then (7.7) and (7.8) induce a chain map

$$\zeta_A : \mathcal{A}_\infty \longrightarrow \oplus_{n\geqslant 2} Hom^*\left(A^{\otimes n}, A\right) \tag{7.9}$$

representing the A_∞-algebra structure on A. Dually, if $\left(A, \psi^n\right)_{n\geqslant 1}$ is an A_∞-coalgebra with quadratic relations as in (7.5), the associations

$$e^{n-2} \mapsto \psi^n \text{ and } d_{(i,\ell)}\left(e^{n-2}\right) \mapsto \psi^{\ell+1}_{i,n-\ell-1-i}\psi^{n-\ell}$$

induce a chain map

$$\xi_A : \mathcal{A}_\infty \longrightarrow \oplus_{n\geqslant 2} Hom^*\left(A, A^{\otimes n}\right) \tag{7.10}$$

representing the A_∞-coalgebra structure on A. The definition of the tensor product is now immediate:

Definition 30. *The* <u>*tensor product of A_∞-algebras*</u> *(A, ζ_A) and (B, ζ_B) is given by*

$$(A, \zeta_A) \otimes (B, \zeta_B) = \left(A \otimes B, \zeta_{A\otimes B}\right),$$

where $\zeta_{A\otimes B}$ is the sum of the compositions

$$\begin{array}{ccc} C_*(K_n) & \overset{\zeta_{A\otimes B}}{\longrightarrow} & Hom((A\otimes B)^{\otimes n}, A\otimes B) \\ {}_{\Delta_K}\downarrow & & \uparrow (\sigma_{2,n})^*\iota \\ C_*(K_n)\otimes C_*(K_n) & \underset{\zeta_A\otimes\zeta_B}{\longrightarrow} & Hom(A^{\otimes n}, A)\otimes Hom(B^{\otimes n}, B) \end{array}$$

over all $n \geqslant 2$; the A_∞-algebra operations Φ^n on $A\otimes B$ are given by

$$\Phi^n = (\sigma_{2,n})^*\,\iota\,(\zeta_A\otimes\zeta_B)\,\Delta_K\left(e^{n-2}\right).$$

Dually, the tensor product of A_∞-coalgebras (A,ξ_A) and (B,ξ_B) is given by

$$(A,\xi_A)\otimes(B,\xi_B) = (A\otimes B, \xi_{A\otimes B}),$$

where $\xi_{A\otimes B}$ is the sum of the compositions

$$\begin{array}{ccc} C_*(K_n) & \overset{\xi_{A\otimes B}}{\longrightarrow} & Hom(A\otimes B, (A\otimes B)^{\otimes n}) \\ {}_{\Delta_K}\downarrow & & \uparrow (\sigma_{n,2})_*\iota \\ C_*(K_n)\otimes C_*(K_n) & \underset{\xi_A\otimes\xi_B}{\longrightarrow} & Hom(A, A^{\otimes n})\otimes Hom(B, B^{\otimes n}) \end{array}$$

over all $n \geqslant 2$; the A_∞-coalgebra operations Ψ^n on $A\otimes B$ are given by

$$\Psi^n = (\sigma_{n,2})_*\,\iota\,(\xi_A\otimes\xi_B)\,\Delta_K\left(e^{n-2}\right).$$

Example 10. *If $(A,\psi^n)_{n\geqslant 1}$ is an A_∞-coalgebra, the following A_∞ operations arise on $A\otimes A$:*

$$\begin{aligned}
\Psi^1 &= \psi^1\otimes 1 + 1\otimes\psi^1 \\
\Psi^2 &= \sigma_{2,2}\left(\psi^2\otimes\psi^2\right) \\
\Psi^3 &= \sigma_{3,2}\left(\psi_0^2\psi_0^2\otimes\psi^3 + \psi^3\otimes\psi_1^2\psi_0^2\right) \\
\Psi^4 &= \sigma_{4,2}\big(\psi_0^2\psi_0^2\psi_0^2\otimes\psi^4 + \psi^4\otimes\psi_2^2\psi_1^2\psi_0^2 + \psi_0^3\psi_0^2\otimes\psi_1^2\psi_0^3 \\
&\qquad +\psi_0^3\psi_0^2\otimes\psi_1^3\psi_0^2 + \psi_1^2\psi_0^3\otimes\psi_1^3\psi_0^2 - \psi_0^2\psi_0^3\otimes\psi_2^2\psi_0^3\big) \\
&\ \ \vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

Note that the compositions in Definition 30 only use the operations ψ^n and not the quadratic relations (7.5). Indeed, one can iterate an arbitrary family of operations $\{\psi^n\}$ as in Example (10) to produce iterated operations $\Psi^n : A^{\otimes k} \to \left(A^{\otimes k}\right)^{\otimes n}$ whether or not (A,ψ^n) is an A_∞-coalgebra. Of course, the Ψ^n's define an A_∞-coalgebra structure on $A^{\otimes k}$ whenever $d^2_{\widetilde{\Omega}(A^{\otimes k})} = 0$, and we make extensive use of this fact in the sequel [**16**]. Finally, since Δ_K is homotopy coassociative (not strict), the tensor product only iterates up to homotopy. In the sequel we always coassociate on the extreme left.

8. Appendix

For completeness, we review the definitions of the functors given by Kadeishvili and Saneblidze in [**8**], [**9**] from the category of 1-reduced simplicial sets to the category of cubical sets and from the category of 1-reduced cubical sets to the category of permutahedral sets.

8.1. The cubical set functor $\mathbf{\Omega}X$

Given a 1-reduced simplicial set $X = \{X_n, \partial_i,\, s_i\}_{n\geqslant 0}$, define the graded set $\mathbf{\Omega}X$ as follows: Let X^c be the graded set of formal expressions

$$X^c_{n+k} = \{\eta_{i_k}\cdots\eta_{i_1}\eta_{i_0}(x) \mid x \in X_n\}_{n\geqslant 0;k\geqslant 0},$$

where $\eta_{i_0} = 1$, $i_1 \leqslant \cdots \leqslant i_k$, $1 \leqslant i_j \leqslant n+j-1$, $1 \leqslant j \leqslant k$, and let $\bar{X}^c = s^{-1}(X^c_{\geqslant 0})$ be the desuspension of X^c. Let $\Omega' X$ be the free graded monoid generated by $\bar{X}^c$; denote elements of $\Omega' X$ by $\bar{x}_1\cdots\bar{x}_k$, where $x_j \in X_{m_j+1},\ m_j \geqslant 0$. The total degree $m = |\bar{x}_1\cdots\bar{x}_k| = \sum|\bar{x}_j|$ and we write $\bar{x}_1\cdots\bar{x}_k \in (\Omega' X)_m$. The product of two elements $\bar{x}_1\cdots\bar{x}_k$ and $\bar{y}_1\cdots\bar{y}_\ell$ is given by concatenation $\bar{x}_1\cdots\bar{x}_k\bar{y}_1\cdots\bar{y}_\ell$; the only relation on $\Omega' X$ is strict associativity. Let $\mathbf{\Omega}X$ be the graded monoid obtained from $\Omega' X$ via

$$\mathbf{\Omega}X = \Omega' X \big/ \sim\ ,$$

where $\overline{\eta_n(x)} \sim \overline{s_n(x)}$ for $x \in X_{>0}$, and $\bar{x}_1\cdots\overline{\eta_{m_i+1}(x_i)}\cdot\bar{x}_{i+1}\cdots\bar{x}_k \sim \bar{x}_1\cdots\bar{x}_i\cdot\overline{\eta_1(x_{i+1})}\cdots\bar{x}_k$ for $x_i \in X^c_{m_i+1}$, $i < k$. Let MX denote the free monoid generated by $\bar{X} = s^{-1}(X_{>0})$; there is an inclusion of graded modules $MX \subset \Omega' X$.

Apparently $\Omega' X$ canonically admits the structure of a cubical set. Denote the components of Alexander-Whitney diagonal by

$$\nu_i : X_n \to X_i \times X_{n-i},$$

where $\nu_i(x) = \partial_{i+1}\cdots\partial_n(x)\times\partial_0\cdots\partial_{i-1}(x),\ 0 \leqslant i \leqslant n$, and let $x^n \in X_n$ denote an n-simplex simplex. Then

$$\nu_i(x^n) = (x')^i \times (x'')^{n-i} \in X_i \times X_{n-i}$$

for all $n > 0$. For $1 \leqslant i \leqslant n-1$, define face operators $d_i^0, d_i^1 : (\mathbf{\Omega}X)_{n-1} \to (\mathbf{\Omega}X)_{n-2}$ on a (monoidal) generator $\overline{x^n} \in \bar{X}_n \subset \bar{X}^c_n$ by

$$d_i^0\left(\overline{x^n}\right) = \overline{(x')^i}\cdot\overline{(x'')^{n-i}} \quad\text{and}\quad d_i^1\left(\overline{x^n}\right) = \overline{\partial_i\left(x^n\right)},$$

and extend to elements $\bar{x}_1\cdots\bar{x}_k \in MX$ via

$$d_i^0\left(\bar{x}_1\cdots\bar{x}_k\right) = \bar{x}_1\cdots\overline{(x_q')^{j_q}}\cdot\overline{(x_q'')^{m_q-j_q+1}}\cdots\bar{x}_k,$$

$$d_i^1\left(\bar{x}_1\cdots\bar{x}_k\right) = \bar{x}_1\cdots\overline{\partial_{j_q}\left(x_q\right)}\cdots\bar{x}_k,$$

where $i = m_{(q-1)}+j_q \leqslant m_{(q)},\ 1 \leqslant i \leqslant n-1,\ 1 \leqslant q \leqslant k$. Then the defining identities for a cubical set involving d_i^0 and d_i^1 can easily be checked on MX. In particular, the simplicial relations between the ∂_i's imply the cubical relations between d_i^1's; the associativity relations between ν_i's imply the cubical relations between d_i^0's, and the commutativity relations between ∂_i's and ν_j's imply the cubical relations between d_i^1's and d_j^0's. Next, define degeneracy operators $\eta_i : (\mathbf{\Omega} X)_{n-1} \to (\mathbf{\Omega} X)_n$ on a (monoidal) generator $\overline{x} \in (\bar{X}^c)_{n-1}$ by

$$\eta_i(\overline{x}) = \overline{\eta_i(x)};$$

and extend to elements $\bar{x}_1 \cdots \bar{x}_k \in \mathbf{\Omega} X$ via

$$\eta_i\left(\overline{x}_1 \cdots \overline{x}_k\right) = \overline{x}_1 \cdots \eta_{j_q}(\overline{x_q}) \cdots \overline{x}_k,$$

$$\eta_n\left(\overline{x}_1 \cdots \overline{x}_k\right) = \overline{x}_1 \cdots \overline{x}_{m_{k-1}} \cdot \eta_{m_k+1}\left(\overline{x}_k\right),$$

where $i = m_{(q-1)} + \ j_q \leqslant m,\ 1 \leqslant i \leqslant n-1,\ 1 \leqslant q \leqslant k$, and extend face operators on degenerate elements inductively so that the defining identities of a cubical set are satisfied. Then in particular, the following identities hold for all $x^n \in X_n$:

$$d_1^0\left(\overline{x^n}\right) = \overline{(x')^1} \cdot \overline{(x'')^{n-1}} = e \cdot \overline{(x'')^{n-1}} = \overline{(x'')^{n-1}} = \overline{\partial_0\left(x^n\right)},$$

$$d_{n-1}^0\left(\overline{x^n}\right) = \overline{(x')^{n-1}} \cdot \overline{(x'')^1} = \overline{(x'')^{n-1}} \cdot e = \overline{(x')^{n-1}} = \overline{\partial_n\left(x^n\right)},$$

where $e \in (\mathbf{\Omega} X)_0$ denotes the unit. It is easy to see that the cubical set $\{\mathbf{\Omega} X, d_i^0, d_i^1, \eta_i\}$ depends functorially on X.

8.2. The permutahedral set functor $\mathbf{\Omega} Q$

Let $Q = (Q_n, d_i^0, d_i^1, \eta_i)_{n \geqslant 0}$ be a 1-reduced cubical set. Recall that the diagonal

$$\Delta : C_*(Q) \to C_*(Q) \otimes C_*(Q)$$

on $C_*(Q)$ is defined on $a \in Q_n$ by

$$\Delta(a) = \Sigma\ (-1)^{\epsilon} d_B^0(a) \otimes d_A^1(a),$$

where $d_B^0 = d_{j_1}^0 \cdots d_{j_q}^0$, $d_A^1 = d_{i_1}^1 \cdots d_{i_p}^1$, summation is over all shuffles $(A; B) = (i_1 < \cdots < i_q; j_1 < \cdots < j_p)$ of $\underline{n}$ and ϵ is the sign of the shuffle. The primitive components of the diagonal are given by the extreme cases $A = \varnothing$ and $B = \varnothing$.

Let $\bar{Q} = s^{-1}(Q_{>0})$ denote the desuspension of Q, let $\mathbf{\Omega}'' Q$ be the free graded monoid generated by $\bar{Q}$ with the unit $e \in \bar{Q}_1 \subset \mathbf{\Omega}'' Q$ and let Υ be the set of formal expressions

$$\Upsilon = \{\varrho_{M_k|N_k}((\cdots \varrho_{M_2|N_2}(\varrho_{M_1|N_1}(\bar{a}_1 \cdot \bar{a}_2) \cdot \bar{a}_3) \cdots) \cdot \bar{a}_{k+1}) |\, a_i \in Q_{r_i}\}_{r_i \geqslant 1; k \geqslant 2},$$

$M_i|N_i \in \mathcal{P}_{r_{(i)}, r_{i+1}}(r_{(i+1)})$ or $M_i|N_i \in \mathcal{P}_{r_{i+1}, r_{(i)}}(r_{(i+1)}),\ 1 \leqslant i \leqslant k$. Note that one or more of the a_i's can be the unit e. Adjoin the elements of Υ to $\mathbf{\Omega}'' Q$ and obtain the graded monoid $\mathbf{\Omega}' Q$ and let $\mathbf{\Omega} Q$ be the monoid

$$\mathbf{\Omega} Q = \mathbf{\Omega}' Q / \sim,$$

where $\varrho_{M|N}(\bar{a} \cdot \bar{b}) \sim \varrho_{N|M}(\bar{a} \cdot \bar{b}),\ \varrho_{j|\underline{n} \setminus j}(e \cdot \bar{a}) \sim \varrho_{\underline{n} \setminus j|j}(\bar{a} \cdot e) \sim \overline{\eta_j(a)}$, $a, b \in Q_{>0}$, and $\bar{a}_1 \cdots \varrho_{\underline{r_i}|r_i+1}(\bar{a}_i \cdot e) \cdot \bar{a}_{i+2} \cdots \bar{a}_{k+1} \sim \bar{a}_1 \cdots \bar{a}_i \cdot \varrho_{1|\underline{r_{i+2}+1} \setminus 1}(e \cdot \bar{a}_{i+2}) \cdots \bar{a}_{k+1}$ for $a_i \in Q_{r_i}, a_{i+1} = e$, $1 \leqslant i \leqslant k$. Then $\mathbf{\Omega} Q$ is canonically a multipermutahedral set in the following way: First, define the face operator $d_{A|B}$ on a monoidal generator $\bar{a} \in \bar{Q}_n$ by

$$d_{A|B}(\bar{a}) = \overline{d_B^0(a)} \cdot \overline{d_A^1(a)}, \quad A|B \in \mathcal{P}_{*,*}(n).$$

Next, use the formulas in the definition of a singular multipermutahedral set (4.3) to define $d_{A|B}$ and $\varrho_{M|N}$ on decomposables. In particular, the following identities hold for $1 \leqslant i \leqslant n$:

$$d_{i|\underline{n+1}\backslash i}\,(\overline{a}) = \overline{d_i^1(a)} \quad \text{and} \quad d_{\underline{n+1}\backslash i|i}\,(\overline{a}) = \overline{d_i^0(a)}.$$

It is easy to see that $(\mathbf{\Omega}Q,\, d_{A|B},\, \varrho_{M|N})$ is a multipermutahedral set that depends functorially on Q.

Remark 3. *The fact that the definition of* $\mathbf{\Omega}Q$ *uses all cubical degeneracies is justified geometrically by the fact that a degenerate singular* n*-cube in the base of a path space fibration lifts to a singular* $(n-1)$*-permutahedron in the fibre, which is degenerate with respect to Milgram's projections [14] (c.f., the definition of the cubical set* $\mathbf{\Omega}X$ *on a simplicial set* X*).*

References

[1] J. F. Adams, On the cobar construction, *Proc. Nat. Acad. Sci. (USA)*, **42** (1956), 409-412.

[2] H. J. Baues, The cobar construction as a Hopf algebra, *Invent. Math.*, **132** (1998), 467-489.

[3] G. Carlsson and R. J. Milgram, Stable homotopy and iterated loop spaces, *Handbook of Algebraic Topology (Edited by I. M. James)*, North-Holland (1995), 505-583.

[4] H.S.M. Coxeter, W.O.J. Moser, Generators and relations for discrete groups, Springer-Verlag, 1972.

[5] Matthias R. Gaberdiel and Barton Zwiebach, Tensor constructions of open string theories I: Foundations, *Nucl.Phys.* B505 (1997), 569-624.

[6] N. Iwase and M. Mimura, Higher homotopy associativity, *Lecture Notes in Math.*, 1370 (1986), 193-220.

[7] D. W. Jones, A general theory of polyhedral sets and corresponding T-complexes, Dissertationes Mathematicae, CCLXYI, Warszava (1988).

[8] T. Kadeishvili and S. Saneblidze, A cubical model of a fibration, *preprint*, AT/0210006.

[9] ——————, The twisted Cartesian model for the double path space fibration, preprint, AT/0210224.

[10] D. M. Kan, Abstract homotopy I, *Proc. Nat. Acad. Sci. U.S.A.*, **41** (1955), 1092-1096.

[11] T. Lada and M. Markl, Strongly homotopy Lie algebras, *Communications in Algebra* **23** (1995), 2147-2161.

[12] J.-L. Loday and M. Ronco, Hopf algebra of the planar binary trees, *Adv. in Math.* **139,** No. 2 (1998), 293-309.

[13] S. Mac Lane, "Homology," Springer-Verlag, Berlin/New York, 1967.

[14] R. J. Milgram, Iterated loop spaces, *Ann. of Math.*, **84** (1966), 386-403.

[15] S. Saneblidze and R. Umble, A diagonal on the associahedra, *preprint*, math. AT/0011065.

[16] ——————, The biderivative and A_∞-bialgebras, *J. Homology, Homotopy and Appl.*, to appear, *preprint*, math. AT/0011065.

[17] J. R. Smith, "Iterating the cobar construction", *Memiors of the Amer. Math. Soc.* **109**, Number 524, Providence, RI, 1994.

[18] J. D. Stasheff, Homotopy associativity of H-spaces I, II, *Trans. Amer. Math. Soc.,* **108** (1963), 275-312.

[19] A. Tonks, Relating the associahedron and the permutohedron, In "Operads: Proceedings of the Renaissance Conferences (Hartford CT / Luminy Fr 1995)" Contemporary Mathematics, **202** (1997), pp. 33-36.

[20] G. Ziegler, "Lectures on Polytopes," GTM 152, Springer-Verlag, New York, 1995.

This article may be accessed via WWW at http://www.rmi.acnet.ge/hha/
or by anonymous ftp at
ftp://ftp.rmi.acnet.ge/pub/hha/volumes/2004/n1a20/v6n1a20.(dvi,ps,pdf)

Samson Saneblidze `sane@rmi.acnet.ge`

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
M. Aleksidze st., 1
0193 Tbilisi, Georgia

Ronald Umble `ron.umble@millersville.edu`

Department of Mathematics
Millersville University of Pennsylvania
Millersville, PA. 17551