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LIE ALGEBRA COHOMOLOGY AND GENERATING FUNCTIONS

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(*communicated by Hvedri Inassaridze*)

Abstract

Let $\mathfrak{g}$ be a simple Lie algebra, V an irreducible $\mathfrak{g}$-module, W the Weyl group and $\mathfrak{b}$ the Borel subalgebra of $\mathfrak{g}$, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$. The Borel-Weil-Bott theorem states that the dimension of $H^i(\mathfrak{n}; V)$ is equal to the cardinality of the set of elements of length i from W. Here a more detailed description of $H^i(\mathfrak{n}; V)$ as an $\mathfrak{h}$-module is given in terms of generating functions.

Results of Leger and Luks and Williams who described $H^i(\mathfrak{n}; \mathfrak{n})$ for $i \leqslant 2$ are generalized: $\dim H^*(\mathfrak{n}; \Lambda^*(\mathfrak{n}))$ and $\dim H^i(\mathfrak{n}; \mathfrak{n})$ for $i \leqslant 3$ are calculated and $\dim H^i(\mathfrak{n}; \mathfrak{n})$ as function of i and rank $\mathfrak{g}$ is described for the calssical series.

Introduction

The main field is $\mathbb{C}$ and all the algebras and modules considered are finite dimensional over $\mathbb{C}$. It is well known, that the standard cochain complex $C^*(\mathfrak{n}; V) = \oplus C^k(\mathfrak{n}; V)$ is isomorphic to the space of linear maps from $\Lambda^*\mathfrak{n}$ into V; so

$$C^*(\mathfrak{n}; V) \cong (\Lambda^*\mathfrak{n})' \otimes V, \tag{0.4}$$

where prime denotes the dualization and the sign * is reserved to denote the direct sum: for example, H^* denotes $\underset{k}{\oplus} H^k$, $E_1^{*,*} = \oplus_{i,j} E_1^{ij}$, $E_1^{i,*} = \oplus E_1^{i,j}$, etc.

Let $\operatorname{diag}(a_1, \dots, a_n)$ denote the matrix with the numbers a_i on its main diagonal, the other elements being 0.

Let $\mathfrak{g}$ be a (semi)simple Lie algebra, V an irreducible $\mathfrak{g}$-module, W the Weyl group of $\mathfrak{g}$ and $\mathfrak{b}$ its Borel subalgebra; $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. According to the Borel-Weil-Bott (BWB) theorem [**5**] (whose different proofs are presented, with increasing clarity, e.g., in [**11**], [**1**], [**4**]), we have

$$\dim H^i(\mathfrak{n}; V) = |W_i|, \tag{0.1}$$

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where $|S|$ denotes the cardinality of S and W_i is the set of elements from W of length i, see [**6**]. We hope there will be no confusion of the cardinality of a set with the modulus of a polynomial introduced below.

In addition to $\mathfrak{g}$-modules, $\mathfrak{n}$ has a lot of natural modules which are not $\mathfrak{g}$-modules but which admit a $\mathfrak{b}$-module structure; and in this paper we will study such modules. The most interesting is the adjoint module $\mathfrak{n}$, and our main aim is to calculate cohomology with values in it. In particular, we will show (Theorem 7.6), that if $\mathfrak{g} = A_r$, i.e., $\mathfrak{g} = \mathfrak{sl}(r+1)$, then

$$\dim H^*(\mathfrak{n};\mathfrak{n}) = (r+1)!\frac{r^2+9r+2}{12}.$$

Our methods can be generalized to embrace other $\mathfrak{b}$-modules.

Let V be an arbitrary $\mathfrak{b}$-module; we will analyze the action of the Cartan subalgebra $\mathfrak{h}$ on the cochain complex $C^*(\mathfrak{n};V)$. This complex is a direct sum of its weight subspaces C^*_ν where $\nu \in \mathfrak{h}'$. In §2 we will derive a formula for the generating function:

$$F(t,x) = \sum_{k,\nu} \dim C^k_\nu \cdot t^k \cdot x^\nu. \tag{2.4}$$

The guiding idea in what follows is that, first, the coboundary operator d preserves each subcomplex $C^*_\nu = \oplus_k C^k_\nu$ and, second,

$$\sum_k (-1)^k \dim H^k_\nu = \sum_k (-1)^k \dim C^k_\nu. \tag{0.2}$$

Therefore,

$$\dim H^* \geqslant \sum_\nu \left| \sum_k (-1)^k \dim C^k_\nu \right|. \tag{0.3}$$

Inequality (0.3) coincides with the inequality (2.5) and we will investigate for which modules this inequality becomes an equality. The modules for which the equality is attained are called *blue* ones.[1]

In §4 we will show that any irreducible $\mathfrak{g}$-module V is blue, and in §6 we prove the same for $V = \mathfrak{n}$, if $\mathfrak{g}$ is of type A_r.

If V is a blue module, one can calculate its cohomology simply by counting the coefficients of the function $G(x) = F(-1,x)$. In §7 we will calculate in this way $\dim H^*(\mathfrak{n};\mathfrak{n})$ and $\dim H^k(\mathfrak{n};\mathfrak{n})$ for $k \leqslant 3$. In §8 we will establish some properties of $\dim H^k(\mathfrak{n};\mathfrak{n})$ regarded as a function of k and $\operatorname{rk}\mathfrak{g}$.

In §9 we will discuss the results and problems for $\mathfrak{g} \neq A_r$.

I do not know any other investigations concerning $\dim H^*(\mathfrak{n};V)$, where V is not a $\mathfrak{g}$-module, except calculations of Leger and Luks [**12**] and Williams [**21**], where $H^2(\mathfrak{n};\mathfrak{n})$ and $H^1(\mathfrak{n};V)$ for V equal to either $\mathfrak{n}$, or its dual, $\mathfrak{n}'$, or $\mathfrak{g}/\mathfrak{n}$ are calculated.

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[1] As opposed to *yellow* modules, the colors chosen to match the two colors of the banner of the independent Ukraine.

Some results of this paper were announced in [**15**], see also [**16**]–[**19**]. They were delivered at the seminars of A. L. Onishchik-É. B. Vinberg and D. A. Leites in 1978/79.

§1. Generating Functions

1.1

Let $x = (x_1, \dots, x_r)$ be an r-tuple. For $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{R}^r$, we set $x^\nu = x_1^{\nu_1} \dots x_r^{\nu_r}$. In this paper, except for §§8–9, a *polynomial* is an expression $P(x) = \sum\limits_\nu a_\nu \cdot x^\nu$ with real or complex coefficients, where ν runs over some finite set of real vectors.

For a polynomial P, we define its support as $N_P = \{\nu \in \mathbb{R}^r \mid a_\nu \neq 0\}$ and its *modulus* as $|P(x)| = \sum\limits_{\nu \in N_p} |a_\nu|$. We call P a *convex polynomial* if N_P is a set of vertices of a convex polyhedral.

If $P(t, x)$ is a polynomial in two groups of indeterminates and $t^0 = (t_1^0, \dots, t_s^0)$ a vector with numerical coordinates, we denote by $|P(t^0, x)|$ the modulus of the polynomial obtained from $P(t, x)$ by replacing t with t^0.

1.2. Proposition. *If $P = \sum a_\nu x^\nu$ and $Q = \sum b_\nu x^\nu$ are polynomials, then*

(i) $|P \cdot Q| \leqslant |P| \cdot |Q|$;

(ii) *For any monomial $Q = x^\nu$, we have $|P \cdot Q| = |P|$.*

(iii) *If P is convex and b_ν are integers, then $|P \cdot Q| \geqslant |P|$.*

Proof. Statements (i) and (ii) are obvious. Now let P be convex and $\mu \in N_P$. Then there exists a linear form $\varphi \in (\mathbb{R}^r)'$ such that $\varphi(\mu) > \varphi(\nu)$ for all $\nu \in N_P$, $\nu \neq \mu$. We can find $\lambda \in N_Q$ such that $\varphi(\lambda) \geqslant \varphi(\nu')$ for any $\nu' \in N_Q$; then $\varphi(\lambda + \mu) > \varphi(\nu + \nu')$ and, therefore, $\lambda + \mu \neq \nu + \nu'$ for any $\nu \in N_P$, $\nu' \in M_Q$. So if $P \cdot Q = \sum c_\nu x^\nu$, then $|c_{\lambda+\mu}| = |a_\mu \cdot b_\lambda| \geqslant |a_\mu|$.

Thus, for any $\mu \in N_P$ we have found a vector $\lambda + \mu \in N_{P \cdot Q}$ such that $|c_{\lambda+\mu}| \geqslant |a_\mu|$. Clearly, if $\mu \neq \mu_1$ then $\mu + \lambda \neq \mu_1 + \lambda_1$. Hence,

$$|P \cdot Q| = \sum_{\nu \in N_{P \cdot Q}} |c_\nu| \geqslant \sum_{\mu \in N_P} |c_{\lambda+\mu}| \geqslant \sum_{\mu \in N_P} |a_\mu| = |P|. \quad \square$$

1.3

If a polynomial Q is obtained from P by an affine change of indeterminates, we will write $P \sim Q$.

Proposition. *If $P \sim Q$, then $|P| = |Q|$.*

Proof. The translation of the origin to the point ν corresponds to the multiplication of P by x^ν, so we may use Proposition 1.2(ii). Now, given a homogeneous change of indeteminates, let $\pi = (\pi_1, \dots, \pi_r)$, where $\pi_i = \sum\limits_j a_{ij} \nu_j$. Then we may denote:

$$y_i = \prod_j x_j^{b_{ij}}, \text{ where } (b_{ij}) = (a_{ij})^{-1}.$$

Clearly, $y^\pi = x^\nu$. This means that the change of indeteminates transforms a monomial $a_\nu x^\nu$ into some other monomial with the same coefficient a_ν. Now our statement is obvious. $\square$

It is easy to see that the correspondence between the monomial x^ν and the δ-function δ_ν establishes an isomorphism of the polynomial ring and the group ring of $\mathbb{R}^r$. It will be more convenient for us to operate with the polynomial ring: to substitute numerical values of variables is more natural for polynomials.

1.4

Now let $\mathfrak{h}$ be an r-dimensional commutative Lie algebra and V an $\mathfrak{h}$-module. Let V be semisimple, i.e., let V have a basis $v_1, \dots, v_m$ such that $h \cdot v_i = \mu_i(h) \cdot v_i$ for any $h \in \mathfrak{h}$ and some $\mu = (\mu_1, \dots, \mu_m) \in (\mathfrak{h}')^m$.

We will also assume that $\mathfrak{h}$ has a basis $h_1, \dots, h_r$ such that all the numbers $\mu_i(h_j)$ are real. Then we may consider the generating function of the $\mathfrak{h}$-module V:

$$A_V(x) = \sum_{i=1}^{m} x^{\mu_i} \tag{1.1}$$

This generating function coincides with the character of module V in the terminology of [7]: if $V = \oplus V_\nu$ is the decomposition of V into its weight subspaces, then, obviously,

$$A_V(x) = \sum_\nu \dim V_\nu \cdot x^\nu = \mathrm{ch}\, V. \tag{1.2}$$

If $V = \underset{k\in\mathbb{Z}}{\oplus} V^k$ is a $\mathbb{Z}$-graded module, we will consider the generating function

$$A_V(t, x) = \sum_{k,\nu} \dim V_\nu^k \cdot t^k \cdot x^\nu \tag{1.3}$$

where t is one more (one-dimensional) indeterminate.

1.5. Proposition. *Let U and V be two semisimple $\mathfrak{h}$-modules, $A_U(x)$ and $A_V(x)$ their generating functions* (1.1). *Then the $\mathfrak{h}$-modules $U \oplus V$, $U \otimes V$, V' and $\Lambda^* V = \oplus \Lambda^k V$ are also semisimple and their generating functions are given by the formulas*

$$A_{U\oplus V} = A_U + A_V, \tag{1.4}$$

$$A_{U\otimes V} = A_U \cdot A_V, \tag{1.5}$$

$$A_{V'}(x) = A_V(x^{-1}) = \sum_i x^{-\mu_i}, \tag{1.6}$$

$$A_{\Lambda^* V}(t, x) = \prod_i (1 + t \cdot x^{\mu_i}). \tag{1.7}$$

Proof is obvious.

§2. $\mathfrak{b}$-$\mathfrak{n}$-modules

2.1

Now let $\mathfrak{n}$ be an arbitrary Lie algebra and V any $\mathfrak{n}$-module. A pair (D, A) is a *derivation* of V compatible with D if D is a derivation of $\mathfrak{n}$ and $A : V \longrightarrow V$ is a linear transformation such that

$$A(n \cdot v) = n \cdot Av + Dn \cdot v \text{ for any } n \in \mathfrak{n},\ v \in V. \tag{2.1}$$

Obviously such derivations form a Lie algebra, which we denote by $\mathfrak{Der}(V)$. There is a projection: $(D, A) \mapsto D$ from $\mathfrak{Der}(V)$ on $\mathfrak{Der}(\mathfrak{n})$; its kernel consists of all pairs $(0, A)$, where $A : V \longrightarrow V$ commutes with the $\mathfrak{n}$-action on V. This projection gives us the action of $\mathfrak{Der}(V)$ on $\mathfrak{n}$, and, therefore, we can define the action of $\mathfrak{Der}(V)$ on the cochain complex $C^*(\mathfrak{n}; V)$ by the standard formula

$$((D, A)f)(n_1, \dots, n_k) = Af((n_1, \dots, n_k) - \sum f((n_1, \dots, Dn_i, \dots, n_k). \tag{2.2}$$

The standard calculation shows that the coboundary operator $d : C^k \longrightarrow C^{k+1}$ commutes with this action of $\mathfrak{Der}(V)$ on $C^*(\mathfrak{n}; V)$. So this action induces an action of $\mathfrak{Der}(V)$ on $H^*(\mathfrak{n}; V)$.

2.2

Now, let $\mathfrak{b}$ be Lie algebra, $\mathfrak{n}$ - ideal in $\mathfrak{b}$, and let V be a $\mathfrak{b}$-module (so V is an $\mathfrak{n}$-module as well). For any $b \in \mathfrak{b}$ let D_b be the restriction of ad b onto $\mathfrak{n}$ and A_b the action of b on V. Then

$$b \mapsto (D_b, A_b) \tag{2.3}$$

is a homomorphism $\mathfrak{b} \longrightarrow \mathfrak{Der}(V)$. So $\mathfrak{b}$ naturally acts on $C^*(\mathfrak{n}; V)$ and on $H^*(\mathfrak{n}; V)$. It is well known that every Lie algebra trivially acts on its (co)homology, so $\mathfrak{n}$ trivially acts on $H^*(\mathfrak{n}; V)$, and we may regard the above $\mathfrak{b}$-action as the action of $\mathfrak{b}/\mathfrak{n} = \mathfrak{h}$ on $H^*(\mathfrak{n}; V)$.

2.3

Henceforth we will assume that $\mathfrak{b}$ is a semidirect sum of its ideal $\mathfrak{n}$ and a commutative subalgebra $\mathfrak{h}$. Then any $\mathfrak{b}$-module is at the same time an $\mathfrak{h}$-module and an $\mathfrak{n}$-module. We will call V a $\mathfrak{b}$-$\mathfrak{n}$-*module* if

(i) V is a $\mathfrak{b}$-module and

(ii) V has a basis of $\mathfrak{h}$-eigenvectors $v_1, .., v_m$ with real weights in some fixed basis of $\mathfrak{h}$.

The weight of v_i will be denoted by $\mu_i \in \mathfrak{h}'$. We will assume that all the modules to be studied are $\mathfrak{b}$-$\mathfrak{n}$-modules; in particular, we will assume that the adjoint representation of $\mathfrak{h}$ on $\mathfrak{n}$ is semisimple; let $\lambda_1, \dots, \lambda_n \in \mathfrak{h}'$ be the weights of this representation.

2.4

Let $C^* = \underset{\nu}{\oplus} C^*_\nu = \underset{k,\nu}{\oplus} C^k_\nu$ be the decomposition of the graded $\mathfrak{h}$-module $C^* = C^*(\mathfrak{n}; V)$. Set $a_{k\nu} = \dim C^k_\nu$.

Theorem. *Let* $F_V(t,x) = \sum\limits_{k,\nu} a_{k\nu} \cdot t^k \cdot x^\nu$ *be the generating function of* $C^*(\mathfrak{n}; V)$. *Then*

$$F_V(t,x) = \prod_i (1 + t \cdot x^{-\lambda_i}) \cdot A_V(x). \tag{2.4}$$

Proof. Since $C^*(\mathfrak{n}; V)$ is isomorphic to $(\Lambda^* \mathfrak{n})' \otimes V$ as a graded $\mathfrak{h}$-module (formula (0.3)), our theorem follows from Proposition1.5. □

2.5

Set $b_\nu = \sum\limits_{0 \leqslant k \leqslant n} (-1)^k a_{k\nu}$ and $c_{k\nu} = a_{k\nu} - a_{k+1\nu} - a_{k-1\nu}$; let $G_V(x) = \sum b_\nu \cdot x^\nu$. Obviously, $G_V(x) = F_V(-1, x)$.

Theorem. *In the above notations*

$$\dim H^*(\mathfrak{n}; V) \geqslant |G_V|; \tag{2.5}$$

$$\dim H^k_\nu \geqslant c_{k\nu}; \tag{2.6}$$

$$\dim H^k \geqslant \sum_\nu \max(0, c_{k\nu}). \tag{2.7}$$

Proof. Denote by H^k_ν is the cohomology of the complex

$$\ldots \longrightarrow C^{k-1}_\nu \longrightarrow C^k_\nu \longrightarrow C^{k+1}_\nu \longrightarrow \ldots$$

Clearly,

$$\dim H^k_\nu \geqslant \dim C^k_\nu - \dim C^{k-1}_\nu - \dim C^{k+1}_\nu = c_{k\nu},$$

and we obtain (2.6). It immediately implies (2.7).

To prove formula (2.5), let us apply (0.2) to the complex C^*_ν. We get

$$|b_\nu| = |\sum_k (-1)^k \dim C^k_\nu| = |\sum_k (-1)^k \dim H^k_\nu| \leqslant \sum_k \dim H^k_\nu = \dim H^*_\nu. \tag{2.8}$$

Hence, $\dim H^* = \sum\limits_\nu \dim H^*_\nu \geqslant \sum |b_\nu| = |G|$. □

2.6. Corollary. *If* $\lambda_1 \neq 0, \ldots, \lambda_n \neq 0$ *and* V *is a* $\mathfrak{b}$-$\mathfrak{n}$*-module, then* $H^*(\mathfrak{n}; V) \neq 0$.

Proof. If $H^*(\mathfrak{n}; V) = 0$, then $|G| = 0$ and so $G(x) \equiv 0$. Since $G(x) = F(-1, x)$, we see from (2.4) that some of the vectors $\lambda_1, \ldots, \lambda_n$ vanish. □

The assumption on V to be a $\mathfrak{b}$-$\mathfrak{n}$-module is essential. If $\mathfrak{b}$ is a two-dimensional non-commutative Lie algebra, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the one-dimensional Lie algebra and V is one-dimensional non-trivial $\mathfrak{n}$-module, then $H^*(\mathfrak{n}; V) = 0$ even if $\lambda \neq 0$.

2.7. Corollary. *Let* $z^0 = (z^0_1, \ldots, z^0_s)$ *be a set of complex numbers,* $|z^0_i| \leqslant 1$ *for all* i, *and* $y = (y_1, \ldots, y_{r-s})$ *some variables. Then* $\dim H^*(\mathfrak{n}; V) \geqslant |G(z^0, y)|$.

§3. The spectral sequence. The term E_1^{ij}

3.1

If $V = \mathbb{C}$ is the trivial $\mathfrak{b}$-module, then $A_V(x) = 1$. So $F_{\mathbb{C}}(t,x) = \prod\limits_i (1 + t \cdot x^{\lambda_i})$, and we can write

$$F_V(t,x) = F_{\mathbb{C}}(t,x) A_V(x). \tag{3.1}$$

Our purpose in this section is to set an analogous formula (3.5) for the generating functions of cohomology.

3.2

Let V be a $\mathfrak{b}$-$\mathfrak{n}$-module. Assume that V admits a filtration

$$\begin{aligned} &V = V_m \supset V_{m-1} \supset \ldots \supset V_1 \supset V_0 = \{0\} \text{ such that } \dim V_i = i \text{ and} \\ &\mathfrak{b} V_i \subset V_i; \quad \mathfrak{n} V_i \subset V_{i-1}. \end{aligned} \tag{3.2}$$

If (3.2) is fulfilled, V is said to be a *nilmodule*. One can easily see that if there exists an $h \in \mathfrak{h}$ such that $\lambda_i(h) > 0$ for all i, then any such $\mathfrak{b}$-$\mathfrak{n}$-module V is a nilmodule.

Denote: $C^{(i)} = \{\varphi \in C^* \mid \mathrm{Im}\varphi \subset V_i\}$. Then the sets $C^{(i)}$ forms a filtration of C^*, and this filtration generates a spectral sequence [**9**]. Let us calculate its first term E_1^{ij}.

3.3. Theorem. $E_1^{ij} \cong H^i(\mathfrak{n})$ *as linear spaces for all* j *such that* $0 \leqslant j \leqslant \dim V - 1$.

Proof. The main rule for calculating E_1 is to replace the filtered complex with the graded one. In our case this means to substitute the graded module V_{gr} in place of the filtered module V. So $E_1^{ij} \cong H^i(\mathfrak{n}; V_{j+1}/V_j) \cong H^i(\mathfrak{n})$. □

Usually, the differentials d_k in the spectral sequence map E_k^{ij} to $E_k^{i+1-k,j+k}$, and $\bigoplus\limits_j E_\infty^{i-j,j}$ is the graded module associated with the filtered module H^i. But our method of filtration does not coincide with the standard one: here d_k maps E_k^{ij} to $E_k^{i+1,j+k}$, and the graded module associated with $H^i(\mathfrak{n}; V)$ is $\bigoplus\limits_{j=0}^{m-1} E_\infty^{ij} = E_\infty^{i*}$.

3.4. The action of $\mathfrak{b}$ on the spectral sequence

Theorem. *The action of* $\mathfrak{b}$ *on* C^* *induces an action of* $\mathfrak{b}$ *on* E_k^{ij} *for all* i, j, k. *This action commutes with* d_k *and its restriction on* $\mathfrak{n}$ *is trivial.*

Proof. Substitution of V_{gr} for V means that we regard the elements of V_{i-1} as "infinitesimals" as compared with the elements of V_i. (One can use the language of contracted representations [13], [2] to give the formal proof.) We know that d commutes with the action of $\mathfrak{b}$. So it commutes "even more so" if we neglect infinitesimals, and E_1^* admits a $\mathfrak{b}$-module structure. The second of the formulas (3.2) shows that $\mathfrak{n}$ acts trivially on V_{gr} and, therefore, $\mathfrak{n}$ acts trivially on E_1^* (hence, $\mathfrak{n}$ acts trivially on E_k^*). Thus we have proved the theorem for $k = 1$. It remains to regard the infinitesimals up to the k-th order as negligible ones which gives the proof for any k. □

3.5. Main result of this section

Let $h_{k\nu} = \dim H^k_\nu(\mathfrak{n})$ and let

$$P(t,x) = \sum_{k,\nu} h_{k\nu} \cdot t^k \cdot x^\nu \tag{3.3}$$

be the generating function for the graded $\mathfrak{h}$-module $H^*(\mathfrak{n})$.

Theorem. *Let V be a $\mathfrak{b}$-$\mathfrak{n}$-nilmodule and V_i the terms of the corresponding filtration. Then the generating function $D(t,s,x)$ of the graded $\mathfrak{b}$-module $E_1^* = \oplus E_{1\nu}^{ij}$ is given by the formula*

$$D_V(t,s,x) = P(t,x) \cdot \sum_j s^j \cdot x^{\mu_j}, \tag{3.4}$$

where μ_j is the weight of V_{j+1}/V_j.

Proof. Obviously, $D(t,s,x) = \sum_j s^j \cdot D_j(t,x)$, where D_j is the generating function for E_1^{*j}. Now let u_1 be the weight vector from E_1^{*j} and u_0 its representative in $E_0^{*j} = (\Lambda^*\mathfrak{n})' \otimes (V_{j+1}/V_j)$.

We see that the weights of u_1 and u_0 coincide, and are equal to $\lambda = \mu_j$, where λ is the weight of some cohomology class from $H^*(\mathfrak{n})$. So, by multiplying $P(t,x)$ by x^{μ_j}, we obtain $D_j(t,x)$. This implies (3.4). □

3.6. Corollary. *E_1^* is isomorphic to $H^*(\mathfrak{n}) \otimes V$ as $\mathfrak{h}$-modules.*

Proof follows from the fact that the generating functions of these two $\mathfrak{h}$-modules coincide. □

3.7

The function $D(t,s,x)$ is not invariant since one can consider filtration of V with respect to different s's. So hereafter we will suppress the parameter s and use the function $Q(t,x) = D(t,1,x)$. Obviously,

$$Q_V(t,x) = P(t,x) \cdot A_V(x). \tag{3.5}$$

So Q only depends on the algebra $\mathfrak{b}$ and module V over it.

3.8. Proposition. $Q_V(-1,x) = G_V(x)$.

Proof. Applying formula (2.8) to the complex $C^*(\mathfrak{n})$ we see that $P(-1,x) = F_{\mathbb{C}}(-1,x)$. Now it remains to use the definition of $G(x)$ and formulas (3.1), (3.5). □

Therefore, hereafter we will use the function Q_V instead of F_V.

3.9

We will also consider the function $R(x) = Q(1,x)$. Clearly, $|R(x)| = |Q(t,x)| = \dim E_1^*$. So we see that

$$|G_V(x)| \leqslant \dim H^*(\mathfrak{n}; V) \leqslant |R_V(x)|. \tag{3.6}$$

Let $r_\nu = \dim E_{1\nu}^*$. Then $R(x) = \sum r_\nu \cdot x^\nu$. We will call $r_\nu = r_\nu(V)$ the *ν-multiplicity of the module V*, and $r(V) = \max_\nu r_\nu(V)$ the *multiplicity* of V.

3.10

Recall [**4**] that for the nilradical $\mathfrak{n}$ of $\mathfrak{b}$, there exists a free resolution of the trivial $\mathfrak{b}$-module such that $Q_V(t,x)$ is the generating function for the corresponding cochain complex for any V. We will only use a much simpler, trivial fact that $\dim H^k_\nu \leqslant \dim E^{k*}_{1\nu}$; in particular, if $E^{k*}_{1\nu} = 0$, then $H^k_\nu = 0$.

§4. The Blue Modules and the Borel-Weil-Bott Theorem

4.1

Now we will study the following question: when the inequality (2.5) becomes equality? Let us give an appropriate definition.

A $\mathfrak{b}$-$\mathfrak{n}$-module V is a *ν-blue module* if

$$\dim H^*_\nu = |b_\nu| \tag{4.1}$$

and V is a *blue module* if it is ν-blue for all the ν.

So V is a blue module if and only if $\dim H^*(\mathfrak{n}; V) = |G_V|$.

4.2. Proposition. *Let V be a $\mathfrak{b}$-$\mathfrak{n}$-module and $\nu \in \mathfrak{h}'$ a weight of V. The module V is ν-blue if and only if $H^k_\nu = 0$ for all even k or for all odd k.*

Proof. Since $b_\nu = \sum\limits_k (-1)^k \dim H^k_\nu$, then V is ν-blue if and only if

$$\sum_k \dim H^k_\nu = |\sum_k (-1)^k \dim H^k_\nu|.$$

So all the nonzero terms in the right-hand side must have the same sign. □

4.3. Proposition. *In the above notations any of the following conditions is sufficient for V to be ν-blue:*

(i) $C^k_\nu \neq 0$ *for no more than one k.*
(ii) $H^k_\nu \neq 0$ *for no more than one k.*
(iii) $\dim C^*_\nu \leqslant 1$.
(iv) $r_\nu \leqslant 1$.
(v) $\dim H^*_\nu \leqslant 1$.

Proof. This is an easy corollary of Proposition 4.2 since any of the conditions (i)-(v) is stronger than the condition of Proposition 4.2.

4.4

Hereafter we will consider the situation discussed in Introduction: $\mathfrak{b}$ is the Borel subalgebra of some semisimple Lie algebra $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{n}$ are the Cartan subalgebra and the nilradical of $\mathfrak{b}$, respectively. Let $R_+ = \{\alpha_i\}_{i\in I}$ be the set of positive roots of $\mathfrak{g}$, $R_- = -R_+$, $\rho = \frac{1}{2}\sum\limits_{i\in I}\alpha_i$ and W the Weyl group of $\mathfrak{g}$.

Clearly, if U is a $\mathfrak{g}$-module, then U is a $\mathfrak{b}$-$\mathfrak{n}$-nilmodule. So we can apply all the previous results to U. The weights of U will be denoted by β_j for $j = 1, \dots, m$.

Theorem. *If U is an irreducible $\mathfrak{g}$-module, then U is a blue $\mathfrak{b}$-$\mathfrak{n}$-module.*

Proof will be given in sec. 4.5–4.8. (One can prove that any finite dimensional $\mathfrak{g}$-module is blue.)

4.5

We see that

$$G_V(x) = \prod_{i\in I}(1 - x^{-\alpha_i}) \cdot \sum_{j=1}^{m} x^{\beta_j}. \tag{4.2}$$

Let us change the variables by setting $\pi = 2\nu - \rho$. Then $G_V(x)$ transforms into an equivalent function $\tilde{G}_V$, and, clearly,

$$\tilde{G}_V(x) = \prod_{i\in I}(x^{\alpha_i} - x^{-\alpha_i}) \cdot \sum_{j=1}^{m} x^{2\beta_j}. \tag{4.3}$$

In what follows we will operate in these new variables.

4.6. Lemma. *If $U = \mathbb{C}$ is the trivial $\mathfrak{g}$-module, then*

$$G_{\mathbb{C}}(x) = \sum_{w\in W} (-1)^{l(w)} \cdot x^{w\rho}, \tag{4.4}$$

*where $l(w)$ is the length of $w \in W$, see [**6**]. Moreover, $|G_{\mathbb{C}}| = |W|$.*

Proof. One sees that the Weyl formula [**7**]

$$\prod_{i\in I}(x^{\alpha_i} - x^{-\alpha_i}) = \sum_{w\in W} (-1)^{l(w)} \cdot x^{w\rho} \tag{4.5}$$

coincides with (4.4). To prove the second statement, we must show that $w_1\rho \neq w_2\rho$ if $w_1 \neq w_2$, because in this case there are $|W|$ different monomials in the right hand side of (4.4). But this follows from the well known fact that ρ lies strictly inside a Weyl chamber. □

4.7. Lemma. $G_V \geqslant |W|$ *for any* $\mathfrak{b}$-$\mathfrak{n}$-*module.*

Proof. Clearly, $|w\rho| = |\rho|$ for any w. So, by Lemma 4.6 the set $N_{G_{\mathbb{C}}}$ lies on the sphere of radius $|\rho|$ and, therefore, $G_{\mathbb{C}}$ is a convex polynomial. It remains to apply Proposition 1.2 (iii). □

4.8

Now, by the BWB theorem, if U is an irreducible $\mathfrak{g}$-module, then $\dim H^*(\mathfrak{n}; U) = |W|$. With Lemma 4.7, we have:

$$|W| \leqslant G_U \leqslant \dim H^*(\mathfrak{n}; U) = |W| \tag{4.6}$$

which implies that all the inequalities are equalities. So U is blue, and the proof of Theorem 4.4 is completed. □

4.9. Corollary. $\dim H^*(\mathfrak{n}; V) \geqslant |W|$ *for any* $\mathfrak{b}$-$\mathfrak{n}$-*module* V.

Proof follows from Lemma 4.7. □

4.10

Now we can give a convenient formula for the introduced in sec. 3.5 function $P(t, x)$. It follows from formula (4.4) and the fact that $\mathbb{C}$ is a blue module that for a function $m(w)$ we have

$$P(t, x) = \sum_{w \in W} x^{w\rho} \cdot t^{m(w)}. \tag{4.7}$$

Proposition. *$m(w)$ in formula (4.7) coincides with $l(w)$.*

Proof. It is well known [6] that the length of w coincides with card $(wR_+ \cap R_-)$. So let $l = l(w)$ and let $w\{\alpha_1, \dots, \alpha_n\} = \{-\alpha_1, -\alpha_2, \dots, -\alpha_l, \alpha_{l+1}, \dots, \alpha_n\}$. Then

$$w\rho = \frac{1}{2}(-\alpha_1 - \dots - \alpha_l + \alpha_{l+1} + \dots + \alpha_n).$$

Let us open the brackets in the left hand side of (4.5). We can get the monomial $x^{w\rho}$ only if we take each second summand from the first l factors, and each first summand from all the other factors since the degree of any other monomial is less than that corresponding to $w\rho$ with respect to the Weyl chamber containing $w(\rho)$. Now, consider the corresponding to G function $\tilde{F}_{\mathbb{C}}(t, x) = \prod (x^{\alpha_i} + tx^{-\alpha_i})$; similarly, if we want to get the term $x^{w\rho}t^m$, we must take each second summand from the first l factors. Thus, we get the term $x^{w\rho}t^l$. □

Finally, we obtain:

$$P(t, x) = \sum_{w} t^{l(w)} x^{w\rho}. \tag{4.8}$$

4.11. Theorem. *Let U be an irreducible $\mathfrak{g}$-module, ν an arbitrary weight. Then*

(i) $\dim H^*_\nu(\mathfrak{n}; U) \leqslant 1$.

(ii) $H^*_\nu(\mathfrak{n}; U) \neq 0 \iff \dim C^*_\nu(\mathfrak{n}; U) = 1$. *So* $\dim C^*_\nu(\mathfrak{n}; U)$ *is either even or is equal to* 1 *for any* ν.

(iii) *Inequality* (2.7) *is an equality.*

(iv) *Let λ_+ be the highest weight of U, and $\sigma = \rho + \lambda_+$. Then the generating function $P_U(t, x) := \sum\limits_{k,\nu} \dim H^k_\nu(\mathfrak{n}; U) \cdot t^k \cdot x^\nu$ is given by the formula*

$$P_U(t, x) = \sum_{w \in W} x^{w\sigma} \cdot t^{l(w)}. \tag{4.9}$$

Proof is an easy consequence of the proof of Theorem 4.4 and Proposition 4.10. □

4.12

It might be useful sometimes to have an explicit formula for $P(t, x)$ when $\mathfrak{g}$ is any classical simple Lie algebra. Taking W and ρ from [**6**] we obtain Table 1.

If $\mathfrak{g} = A_r$, we see that $l(w)$ is equal to the number of inversions in the lower row of the permutation $w = \begin{pmatrix} 0 & 1 & \dots & r \\ i_0 & i_1 & \dots & i_r \end{pmatrix}$.

4.13

Now we can find a convenient formula for the ν-multiplicity on any $\mathfrak{b}$-$\mathfrak{n}$-module V. Let

$$M_\nu = \{(w, j) \mid w \in W,\ j = 1, \dots, m, \quad w\rho + \beta_j = \nu\}.$$

Let L be the projection of M_ν on $\{1, \dots, m\}$.

Proposition. $r_\nu = \operatorname{card} M_\nu = \operatorname{card} L_\nu$.

Proof. The first equality immediately follows from formulas (3.5) and (4.3) because $R(x) = Q(1, x)$. Now if $w_1\rho + \beta_{j_1} = w_2\rho + \beta_{j_2}$ and $w_1 \neq w_2$, then, as we have seen, $w_1\rho \neq w_2\rho$ and so $j_1 \neq j_2$. This gives the second equality. □

Equivalently, we may describe r_ν as

$$r_\nu = \operatorname{card}\{\mu \in N_{A_V} \mid \nu - \mu = w\rho \text{ for some } w\}.$$

§5. Subquotients

5.1

We will say that a module V is a *subquotient* of U if V is isomorphic to a quotient module of a submodule of U (in particular, if V is a submodule or quotient module of U). We clearly see that if V is any $\mathfrak{b}$-module in a $\mathfrak{b}$-subquotient of some $\mathfrak{g}$-module, then V is a $\mathfrak{b}$-$\mathfrak{n}$-nilmodule.

Hereafter we will assume that all the modules are $\mathfrak{b}$-subquotients of a (usually irreducible) $\mathfrak{g}$-module. Let U be a $\mathfrak{g}$-module, V its subquotient; then we may regard $C^*(\mathfrak{g}; U)$ as a $\mathfrak{g}$-module, and $C^*(\mathfrak{n}; \tilde{U})$, $E_1^*(\mathfrak{n}; \tilde{U})$, $H^*(\mathfrak{n}; \tilde{U})$, where $\tilde{U} = U$ or V, are its subquotients.

5.2

In what follows the letter U will always denote some $\mathfrak{g}$-module and V its $\mathfrak{b}$-subquotient. Let $V_0 \supset V_1$ be $\mathfrak{b}$-submodules of U such that $V = V_0/V_1$. In U, we may find $\mathfrak{h}$-submodules V_2 and V_3, complementary to V_1 in V_0 and to V_0 in U, respectively. Evidently, $A_V(x) = A_{V_2}(x)$ and $V \cong V_2$, as $\mathfrak{h}$-modules; we will identify these modules, if necessary. We see also that the V_i are $\mathfrak{b}$-subquotients of U and $U \cong V_1 + V_2 + V_3$ as $\mathfrak{h}$-modules; so

$$A_U(x) = A_{V_1}(x) + A_{V_2}(x) + A_{V_3}(x). \tag{5.1}$$

The number $cr_\nu(V, U) = r_\nu(U) - r_\nu(V)$ is called the *ν-comultiplicity of V with respect to U*. Formula (5.1) implies that

$$E_{1\nu}^*(\mathfrak{n}; U) \cong E_{1\nu}^*(\mathfrak{n}; V_1) + E_{1\nu}^*(\mathfrak{n}; V_2) + E_{1\nu}^*(\mathfrak{n}; V_3),$$

so

$$cr_\nu(V, U) = r_\nu(V_1) + r_\nu(V_3). \tag{5.2}$$

5.3. Proposition. *Let U be an irreducible $\mathfrak{g}$-module, V its $\mathfrak{b}$-subquotient and ν some weight such that $r_\nu(U) \neq 1$. Then*

(i) $\dim H_\nu^*(\mathfrak{n}; V) \leqslant r_\nu(V)$*;*

(ii) $\dim H_\nu^*(\mathfrak{n}; V) \leqslant cr_\nu(V, U)$.

Proof. (i) is clear, because

$$\dim H^*_\nu(\mathfrak{n}; V) \leqslant \dim E^*_{1\nu}(\mathfrak{n}; V) = r_\nu(V).$$

In particular, this means that $\dim H^*_\nu(\mathfrak{n}; V_i) \leqslant r_\nu(V_i)$. Now, consider the exact sequence $0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow V \longrightarrow 0$. The corresponding exact cohomology sequence shows that

$$\dim H^*_\nu(\mathfrak{n}; V) \leqslant \dim H^*_\nu(\mathfrak{n}; V_1) + \dim H^*_\nu(\mathfrak{n}; V_0). \tag{5.3}$$

But the exact sequence $0 \longrightarrow V_0 \longrightarrow U \longrightarrow V_3 \longrightarrow 0$ shows that $H^*_\nu(\mathfrak{n}; V_0) \cong H^*_\nu(\mathfrak{n}; V_3)$, since $H^*_\nu(\mathfrak{n}; U) = 0$ by Theorem 4.11 (ii). So $\dim H^*_\nu(\mathfrak{n}; V_0) \leqslant r_\nu(V_3)$ and $\dim H^*_\nu(\mathfrak{n}; V_1) \leqslant r_\nu(V_1)$. Now it only remains to apply formulas (5.3) and (5.2). □

If $r_\nu(U) = 1$, then (ii) fails but we clearly see that in this case $\dim H^*_\nu(\mathfrak{n}; V) = r_\nu(V)$ for any V.

5.4. Proposition. *Let U, V, and ν be as in Proposition* 5.3.

(i) *If $r_\nu(V) \cdot cr(V, U) = 0$, then* $\dim H^*_\nu(\mathfrak{n}; V) = 0$.

(ii) *If $r_\nu(V)$ or $cr(V, U)$ is equal to* 1, *then* $\dim H^*_\nu(\mathfrak{n}; V) = 1$.

(iii) *V is ν-blue in all the cases mentioned in* (i), (ii).

Proof. Since $r_\nu(U)$ is even, $r_\nu(V)$ and $cr(V, U)$ are either simultaneously even or simultaneously odd and the same is true for $r_\nu(V)$ and $\dim H^*_\nu(\mathfrak{n}; V)$.

So Proposition 5.3 implies (i) and (ii). Now (iii) is a consequence of Proposition 4.3 (v). □

5.5. Proposition. *If U is an irreducible $\mathfrak{g}$-module and $r_\nu(U) \leqslant 2$, then any $\mathfrak{b}$-submodule of U is ν-blue. In particular, if $r(U) \leqslant 2$, then any $\mathfrak{b}$-submodule of U is blue.*

Proof. If $r_\nu(U) = 1$, we have to apply Proposition 4.3 (iv), otherwise we can apply Proposition 5.4 (iii), because $r_\nu(V) \leqslant 1$ or $cr_\nu(V, U) \leqslant 1$. □

We see that the proposition is true if $r_\nu(U) \leqslant 3$, but such generalization is inessential, because we know that $\dim E^*_{1\nu}(\mathfrak{n}; U)$ is either even or is equal to 1.

5.6. Proposition. *Let U be an irreducible $\mathfrak{g}$-module, V its $\mathfrak{b}$-submodule and $\tilde{V} = U/V$. Then V is ν-blue $\Longleftrightarrow$ $\tilde{V}$ is ν-blue for any ν.*

Proof. Let $G_U(x) = \sum b_\nu x^\nu$, $G_V(x) = \sum b'_\nu x^\nu$ and $G_{\tilde{V}} = \sum b''_\nu x^\nu$ be the generating functions of the corresponding modules. As $A_U(x) = A_V(x) + A_{\tilde{V}}(x)$, we easily see that $G(x) = G_V(x) + G_{\tilde{V}}(x)$. So $b_\nu = b'_\nu + b''_\nu$. Now we have to study two cases:

(1) If $b_\nu \neq 0$, then $\dim C^*_\nu(\mathfrak{n}; U) = 1$ (Theorem 4.11 (ii)), and both V and $\tilde{V}$ are ν-blue by Proposition 4.3 (ii).

(2) If $b_\nu = 0$, then $b'_\nu = |b''_\nu|$. But in this case $H^*_\nu(\mathfrak{n}, U) = 0$, and the exact sequence $0 \longrightarrow V \longrightarrow U \longrightarrow \tilde{V} \longrightarrow 0$ shows that $H^*_\nu(\mathfrak{n}; V) \cong H^*_\nu(\mathfrak{n}; \tilde{V})$. So $b'_\nu = \dim H^*_\nu(\mathfrak{n}; V)$ if and only if $b''_\nu = \dim H^*_\nu(\mathfrak{n}, \tilde{V})$. □

5.7. Proposition. *Let U be an irreducible $\mathfrak{g}$-module, ν one of its weights, V and V_i, $i = 1, 2, 3$, as in sec. 5.2. Suppose that V is ν-blue and $\dim H^*_\nu(\mathfrak{n}; V) \geqslant \dim H^*_\nu(\mathfrak{n}; V_1) + \dim H^*_\nu(\mathfrak{n}; V_3)$. Then both V_1 and V_3 are ν-blue.*

Proof. Let G_U, G_{V_1}, G_V, and G_{V_3} be the generating functions for U, V_1, $V \cong V_2$ and V_3 respectively. Then, as earlier, $b_\nu = b'_\nu + b''_\nu + b'''_\nu = b'_\nu + b'''_\nu \pm \dim H^*_\nu(\mathfrak{n}; V)$. As the case $b_\nu \neq 0$ is trivial, we may suppose $b_\nu = 0$, so

$$\dim H^*_\nu(\mathfrak{n}; V) = |b'_\nu + b'''_\nu| \leqslant |b'_\nu| + |b'''_\nu| \leqslant \dim H^*_\nu(\mathfrak{n}; V_1) + \dim H^*_\nu(\mathfrak{n}; V_3).$$

Hence, all the inequalities are equalities and (4.1) holds for V_1, V_3. □

5.8. Proposition. *A $\mathfrak{b}$-$\mathfrak{n}$-module V is blue if and only if so is V'. Moreover, $r(V) = r(V')$.*

Proof. By formula (1.6) the generating functions corresponding to V and V' are

$$G_V(x) = P(-1, x) \cdot A_V(x) \quad \text{and} \quad G_{V'}(x) = P(-1, x) \cdot A_V(x^{-1}).$$

So we can change variables $x \mapsto x^{-1}$ and write:

$$|G_{V'}(x)| = |P(-1, x) \cdot A_V(x^{-1})| = |P(-1, x^{-1}) \cdot A_V(x)|.$$

But $P(-1, x) = G_{\mathbb{C}}(x) = \prod(x^{\alpha_i} - x^{-\alpha_i})$, so we easily see that $P(-1, x^{-1}) = \pm P(-1, x)$. Therefore $G_{V'}(x^{-1}) = \pm G_V(x)$ and so $|G_{V'}| = |G_V|$. On the other hand, it is well known that $(H^*(\mathfrak{n}; V))' \cong H^*(\mathfrak{n}; V')$ so $\dim H^*(\mathfrak{n}; V) = \dim H^*(\mathfrak{n}; V')$. The rest is clear. □

Problem. *Is it true that if V is ν-blue, then V' is ν-blue?*

In the rest of the paper we will use the methods developed above to calculate certain cohomology.

§6. The adjoint representation of $\mathfrak{n}$ is blue for $\mathfrak{g} = A_r$

This section is devoted to the proof of the following theorem:

6.1. Theorem. *Let $\mathfrak{g} = A_r$ and $\mathfrak{n}$ the nilradical of its Borel subalgebra $\mathfrak{b}$. Then $\mathfrak{n}$, the adjoint $\mathfrak{n}$-module, is blue.*

Proof will be given in sec. 6.2–6.9.

6.2

Recall that for $A_r = \mathfrak{sl}(r+1)$ we have: $\mathfrak{b}$ is the algebra of upper triangular matrices, $\mathfrak{h}$ the algebra of diagonal matrices and $\mathfrak{n}$ the algebra of strictly upper triangular matrices. We use the natural coordinates: if $h \in \mathfrak{h}$, then $h = \mathrm{diag}(h_0, \dots, h_r)$ with $\sum h_i = 0$; let $\nu = (\nu_0, \dots, \nu_r)$ be the dual coordinates on $\mathfrak{h}'$. So ν has $r + 1$ coordinates instead of r but such coordinates are more convenient.

We begin with a technical lemma. Let U_{r+1} be the standard representation of $\mathfrak{g}$ in the $(r + 1)$-dimensional space.

Lemma. $r(U_{r+1}) = 2$.

Proof. Obviously, the weights of U_{r+1} are $\beta_j = (0, \dots, 0, 1, 0, \dots, 0)$ with a 1 in the jth slot. Set $x^{\beta_j} = x_j$. Using Table 1, we obtain:

$$R_{U_{r+1}}(x) = \sum x^0_{i_0} \dots x^r_{i_r}(x_0 + x_1 + \dots + x_r). \tag{6.1}$$

Simplifying, we get

$$R_{U_{r+1}}(x) = \sum x_{i_0}^0 \dots x_{i_{r-1}}^{r-1} x_{i_r}^{r+1} + 2 \sum_{0\leqslant k\leqslant r-1;\; i_k<i_{k+1}} x_{i_0}^0 \dots x_{i_{k-1}}^{k-1} x_{i_k}^{k+1} x_{i_{k+1}}^{k+1} x_{i_{k+2}}^{k+2} \cdot \dots \cdot x_{i_r}^r.$$

So $\max_\nu r_\nu(U_{r+1}) = 2$. □

6.3

According to Propositions 5.5 and 5.4, we have:

Corollary. *Any* $\mathfrak{b}$*-subquotient* U_{r+1} *is a blue module. Moreover, if* V *is a subquotient of* U_{r+1} *and* $r_\nu(V) = 2$*, then* $H^*_\nu(V) = 0$.

6.4

Now, let us return to the module $\mathfrak{n}$. We will have to consider also the modules $\mathfrak{g}$ and $\mathfrak{b}$, and the modules $\mathfrak{b}_- = \mathfrak{g}/\mathfrak{n}$, $\mathfrak{n}_- = \mathfrak{g}/\mathfrak{b}$ and $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$. Obviously, $\mathfrak{g}$ is an irreducible $\mathfrak{g}$-module and all the other modules are its subquotients. We have also the isomorphisms of $\mathfrak{h}$-modules:

$$\mathfrak{g} \cong \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}, \quad \mathfrak{b}_- \cong \mathfrak{n}_- \oplus \mathfrak{h}.$$

We see from formula (1.4) that it suffices to find the generating functions for $\mathfrak{n}_-$, $\mathfrak{h}$, and $\mathfrak{n}$.

The only weight of $\mathfrak{h}$ is zero, so $x^\nu = 1$.

Now, if ν is a weight of $\mathfrak{n}$ or $\mathfrak{n}_-$, then $x^\nu = x_i x_j^{-1}$ for $i \neq j$, and ν is the weight of $\mathfrak{n}$ if and only if $i > j$, see [**7**]. Hence,

$$A_\mathfrak{n}(x) = \sum_{i>j} x_i x_j^{-1}; \tag{6.2}$$

$$A_\mathfrak{h}(x) = r; \tag{6.3}$$

$$A_{\mathfrak{n}_-}(x) = \sum_{i>j} x_i^{-1} x_j = A_\mathfrak{n}(x^{-1}). \tag{6.4}$$

Now, consider the functions $R_V(x)$ for $V \in \{\mathfrak{g}, \mathfrak{n}, \mathfrak{h}, \mathfrak{n}_-\}$. Recall that $L_\nu = \{j \mid \nu = w\rho + \beta_j\}$ for any ν (Proposition 4.13), or, equivalently, $x^{w\rho} x^{\beta_j} = x^\nu$. We know (Table 1) that $x^{w\rho} = x_{i_0}^0 \dots x_{i_r}^r$ and either $x^{\beta_j} = x_k x_1^{-1}$ or $x^{\beta_j} = 1$. Multiplying these, we obtain the following monomials:

1) $x_{i_0}^0 x_{i_1}^1 \dots x_{i_r}^r$.

2) $x_{i_0}^{-1} x_{i_1}^1 \dots x_{i_{r-1}}^{r-1} x_{i_r}^{r+1}$.

3) $x_{i_0}^0 \dots x_{i_{k-1}}^{k-1} x_{i_k}^k x_{i_{k+1}}^k x_{i_{k+2}}^{k+2} \dots x_{i_{r-1}}^{r-1} x_{i_r}^{r+1}$ for $0 \leqslant k \leqslant r-2$.

In what follows, for brevity, we will not write explicitly the "normal" factors, i.e., factors $x_{i_j}^k$ with $k = j$.

4) $x_{i_0}^{-1} \dots x_{i_k}^{k+1} x_{i_{k+1}}^{k+1} \dots$ for $1 \leqslant k \leqslant r-1$.

5) $\dots x_{i_{k-1}}^k x_{i_k}^k x_{i_{k+1}}^k \dots$ for $1 \leqslant k \leqslant r-1$.

6) $\dots x_{i_{k-1}}^k x_{i_k}^k \dots x_{i_l}^l x_{i_{l+1}}^l \dots$ for $1 \leqslant k \leqslant r$ and $0 \leqslant l \leqslant r-1$ such that $\{k-1, k\} \cap \{l, l+1\} = \emptyset$.

It is important to note that in cases **3**, **4** we can assume that $i_k < i_{k+1}$; similarly, in case **5** we will assume $i_{k-1} < i_k < i_{k+1}$ and in case **6** that $i_{k-1} < i_k$ and $l_l < i_{l+1}$. We do not assume that $k < l$.

6.5

Let us study the ν-multiplicity of the above mentioned modules for any ν of types $\mathbf{1} - \mathbf{6}$.

1) One can obtain the monomial $x_{i_0}^0 \dots x_{i_r}^r$ by multiplying the same monomial by 1 for $V = \mathfrak{h}$ or by multiplying $\dots x_{i_{k+1}}^k x_{i_k}^{k+1} \dots$ by $x_{i_{k+1}} x_{i_k}^{-1}$ (here $0 \leqslant k \leqslant r-1$ and V is isomorphic to $\mathfrak{n}$ if $i_{k+1} > i_k$ or $\mathfrak{n}_-$ otherwise). It follows that $r_\nu(\mathfrak{h}) = r$ and $r_\nu(\mathfrak{n}_-) + r_\nu(\mathfrak{n}) = r$. Since $\dim H_\nu^*(\mathfrak{n}) = 1$ and $\mathfrak{h} \simeq \mathbb{C} \oplus \dots \oplus \mathbb{C}$, as $\mathfrak{b}$-module, it follows that $\dim H_\nu^*(\mathfrak{n}; \mathfrak{h}) = r \dim H_\nu^*(\mathfrak{n}) = r$. Now Proposition 5.7 shows that both $\mathfrak{n}$ and $\mathfrak{n}_-$ are ν-blue.

Clearly, if ν is of types $\mathbf{2} - \mathbf{6}$, then $r_\nu(\mathfrak{h}) = 0$.

Now, let $X_\nu = \{x^\beta \mid \in \beta \in L_\nu\}$. Hence, we have a one-to-one correspondence between L_ν and X_ν (this does not hold for $\mathfrak{h}$, because the multiplicity of the weight zero is greater than 1), so $r_\nu(V) = \operatorname{card} X_\nu$, where ν is of types **2**-**6** and $V \in \{\mathfrak{n}, \mathfrak{n}_-, \mathfrak{g}\}$.

2)$x_{i_0}^{-1} \dots x_{i_r}^{r+1} = x_{i_0}^0 \dots x_{i_r}^r (x_{i_0}^{-1} x_{i_r})$. Obviously, this is the only way to get this monomial; so $X_\nu = \{x_{i_0}^{-1} x_{i_r}\}$, and $r_\nu(\mathfrak{g}) = 1$.

3) Clearly, $\dots x_{i_k}^k x_{i_{k+1}}^k \dots x_{i_r}^{r+1}$ can be obtained by multiplying $x^{w\rho}$ by $x_{i_k}^{-1} x_{i_r}$ or by $x_{i_{k+1}}^{-1} x_{i_r}$. So $r_\nu(\mathfrak{g}) = \operatorname{card}\{x_{i_k}^{-1} x_{i_r},\ x_{i_{k+1}}^{-1} x_{i_r}\} = 2$.

4) is similar to **3**). So $X_\nu = \{x_{i_0}^{-1} x_{i_k},\ x_{i_0}^{-1} x_{i_{k+1}}\}$, and $r_\nu(\mathfrak{g}) = 2$.

5) Here $X_\nu = \{x_{j_1}^{-1} x_{j_1}\}$, where $j_1, j_2 \in \{i_{k-1}, i_k, i_{k+1}\}$. So $r_\nu(\mathfrak{g}) = \operatorname{card} X_\nu = 6$.

6)$X_\nu = \{x_{j_1}^{-1} x_{j_1}\}$, where $j_2 \in \{i_{k-1}, i_k\}$ and $j_1 \in \{i_l, i_{l+1}\}$. So $r_\nu(\mathfrak{g}) = 4$.

6.6. Lemma. *$\mathfrak{n}$ is ν-blue if ν is of types* $\mathbf{1} - \mathbf{4}$.

Proof. Lemma was already proved for ν of type **1**. For the types $\mathbf{2} - \mathbf{4}$ it is a corollary of Proposition 5.5. □

6.7

We have not yet proved the theorem for the types **5** and **6**. Here we will deal with type **5**.

Let $x^\nu = \dots x_{i_{k-1}}^k x_{i_k}^k x_{i_{k+1}}^k \dots$, where $i_k < i_k < i_{k+1}$. If $x^\nu = x^{w\rho} x^\beta$, then there are three weights β which belong to $\mathfrak{n}$:

1) $x^\beta = x_{i_{k-1}}^{-1} x_{i_k}$;

2) $x^\beta = x_{i_k}^{-1} x_{i_{k+1}}$;

3) $x^\beta = x_{i_{k-1}}^{-1} x_{i_{k+1}}$.

Thus, $x^{w\rho}$ is, respectively,

$$\dots x_{i_k}^{k-1} x_{i_{k+1}}^k x_{i_{k-1}}^{k+1} \dots \tag{6.5}$$

$$\dots x_{i_{k+1}}^{k-1} x_{i_{k-1}}^k x_{i_k}^{k+1} \dots \tag{6.6}$$

$$\dots x_{i_{k+1}}^{k-1} x_{i_k}^k x_{i_{k-1}}^{k+1} \dots \tag{6.7}$$

Let s be the number of inversions in the transposition $(i_0 \dots i_r)$. Then $(i_0 \dots i_k i_{k+1} i_{k-1} \dots i_r)$ has $s+2$ inversions. So if $x^{w\rho}$ is given by (6.5), then $l(w) = s+2$. Similarly, $l(w) = s+2$ for (6.6) and $l(w) = s+3$ for (6.7). This means that $\dim E_{1\nu}^{s+2,*}(\mathfrak{n};\mathfrak{n}) = 2$ and $\dim\ E_{1\nu}^{s+3,*}(\mathfrak{n};\mathfrak{n}) = 1$. So there are two possibilities:

either $\dim\ H_\nu^{s+2}(\mathfrak{n};\mathfrak{n}) = 2$ and $\dim\ H_\nu^{s+3}(\mathfrak{n};\mathfrak{n}) = 1$;
or $\dim\ H_\nu^{s+2}(\mathfrak{n};\mathfrak{n}) = 1$ and $\dim\ H_\nu^{s+3}(\mathfrak{n};\mathfrak{n}) = 0$.

But similar considerations show that $\dim\ E_{1\nu}^{1,*}(\mathfrak{n};\mathfrak{n}_-) = \begin{cases} 2 & \text{if } l = s+1, \\ 1 & \text{if } l = s, \\ 0 & \text{if } l \neq s, s+1. \end{cases}$

In particular, $E_{1\nu}^{s+2,*}(\mathfrak{n};\mathfrak{n}_-) = 0$ and so $H_\nu^{s+2}(\mathfrak{n};\mathfrak{n}_-) = 0$. Since $E_{1\nu}^{*}(\mathfrak{n};\mathfrak{h}) = 0$, we have $H_\nu^{s+2}(\mathfrak{n};\mathfrak{b}_-) = H_\nu^{s+2}(\mathfrak{n};\mathfrak{n}_-) = 0$.

Now it remains to use the exact sequence

$$0 \longrightarrow \mathfrak{n} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{b}_- \longrightarrow 0.$$

We see that $H_\nu^*(\mathfrak{n};\mathfrak{g}) = 0$ because $r_\nu(\mathfrak{g}) = 6 \neq 1$, and the corresponding cohomology sequence

$$\dots \longrightarrow H^{s+2}(\mathfrak{n};\mathfrak{b}_-) \longrightarrow H_\nu^{s+3}(\mathfrak{n};\mathfrak{n}) \longrightarrow H_\nu^{s+3}(\mathfrak{n};\mathfrak{g}) \longrightarrow \dots$$

shows that $H_\nu^{s+3}(\mathfrak{n};\mathfrak{n}) = 0$. Thus, the case 2) takes place and

$$\dim\ H_\nu^*(\mathfrak{n};\mathfrak{n}) = \dim\ H_\nu^{s+2}(\mathfrak{n};\mathfrak{n}) = 1. \tag{6.8}$$

Hence, $\mathfrak{n}$ is ν-blue by Proposition 4.3 (v).

6.8

Finally, let ν be of type $\mathbf{6}, x = \dots x_{i_{k-1}}^k x_{i_k}^k \dots x_{i_l}^l x_{i_{l+1}}^l \dots$. We have to study 6 subcases:

(a) $i_{k-1} < i_k < i_l < i_{l+1}$ (d) $i_l < i_{k-1} < i_k < i_{l+1}$
(b) $i_{k-1} < i_l < i_k < i_{l+1}$ (e) $i_l < i_{k-1} < i_{l+1} < i_k$
(c) $i_{k-1} < i_l < i_{l+1} < i_k$ (f) $i_l < i_{l+1} < i_{k-1} < i_k$

Recall that we have assumed that $i_l < i_{l+1}$, $i_{k-1} < i_k$. The elements of

$$X_\nu = \{x_{i_{k-1}} x_{i_l}^{-1}, x_{i_k} x_{i_l}^{-1}, x_{i_{k-1}} x_{i_{l+1}}^{-1}, x_{i_k} x_{i_{l+1}}^{-1}\}$$

belong partly to $X_\nu(\mathfrak{n})$ and partly to $X_\nu(\mathfrak{n}_-)$, namely:

(a) all the weights belong to $\mathfrak{n}_-$, so $r_\nu(\mathfrak{n}) = 0$, and $r_\nu(\mathfrak{n}_-) = 4$.

(b) ν belongs to $\mathfrak{n}$ if and only if $x = x_{i_k} x_{i_l}^{-1}$; the remaining three weights belong to $\mathfrak{n}_-$ and so $r_\nu(\mathfrak{n}) = 1$, while $r_\nu(\mathfrak{n}_-) = 3$.

(c) $X_\nu(\mathfrak{n}_-) = \{x_{i_k} x_{i_l}^{-1}$, and $x_{i_k} x_{i_{l+1}}^{-1}\}$; hence, $r_\nu(\mathfrak{n}) = 2$, and $r_\nu(\mathfrak{n}_-) = 2$.

(d) $X_\nu(\mathfrak{n}_-) = \{x_{i_k} x_{i_l}^{-1}$, and $x_{i_{k-1}} x_{i_{l+1}}^{-1}\}$, hence, $r_\nu(\mathfrak{n}) = r_\nu(\mathfrak{n}_-) = 2$.

The cases (e) and (f) are opposite to (b) and (a), respectively, so we have:

(e) $r_\nu(\mathfrak{n}) = 3$ and $r_\nu(\mathfrak{n}_-) = 1$ (f) $r_\nu(\mathfrak{n}) = 4$ and $r_\nu(\mathfrak{n}_-) = 0$.

Clearly, $r_\nu(\mathfrak{n}_-) = cr_\nu(\mathfrak{n};\mathfrak{g})$. So, by Proposition 5.4, $\mathfrak{n}$ is ν-blue in cases (a), (b), (e) and (f).

6.9

It remains to study cases **6**(c), **6**(d). Let ν be of type **6**(c). The corresponding to $X_\nu(\mathfrak{n})$ eigenvectors in $\mathfrak{n}$ are the elementary matrices $E_{i_k i_l}$ and $E_{i_k i_{l+1}}$.

Now let V_j be the $\mathfrak{b}$-submodule of $\mathfrak{n}$ which consists of matrices with first j rows zero and $\tilde{V}_j = V_{j-1}/V_j$. We may identify this subquotient with the linear space of matrices with all the rows zero, except the jth row. So this $\mathfrak{n}$-module is isomorphic to U'_{r+1}, cf. sec. 6.2. To demonstrate the $\mathfrak{b}$-isomorphism of these modules, one has to add the constant weight λ such that $x^\lambda = x_j$.

We see that $r_\nu(\mathfrak{n}) = r_\nu(\tilde{V}_{i_k}) = 2$. Obviously, Lemma 6.2 holds for the dual module U'_{r+1}, so $H^*_\nu(\mathfrak{n}; \tilde{V}_{i_k}) = 0$. Since $\dim E^*_{1\nu}(\mathfrak{n}; \mathfrak{n}) = \dim E^*_{1\nu}(\mathfrak{n}; \tilde{V}_{i_k})$, they are isomorphic. So $H^*_\nu(\mathfrak{n}; \mathfrak{n}) = H^*_\nu(\mathfrak{n}; \tilde{V}_{i_k}) = 0$, and $\mathfrak{n}$ is ν-blue.

Similar arguments (take columns instead of rows) show that $\mathfrak{n}$ is ν-blue if ν is of type **6**(d).

Thus, we proved Theorem 6.1 for all the ν. □

6.10. Corollary. *$\mathfrak{b}$-$\mathfrak{n}$-modules $\mathfrak{b}$, $\mathfrak{n}_-$, and $\mathfrak{b}_-$ are blue.*

Proof. As follows from (6.4), $\mathfrak{n}_-$ is isomorphic to $\mathfrak{n}'$, $\mathfrak{b}_-$ is isomorphic to $\mathfrak{g}/\mathfrak{n}$, and $\mathfrak{n}_- \cong \mathfrak{g}/\mathfrak{b}$. Now apply Propositions 5.6, 5.8. □

§7. Dimensions of the cohomology with the coefficients in the adjoint module

7.1

In §§7 and 8 we still assume that $\mathfrak{g} = A_r$ and preserve the notations of the previous sections. We will also need many new notations: for any $w = \begin{pmatrix} 0 & \dots & r \\ i_0 & \dots & i_r \end{pmatrix}$ set

$$\begin{aligned}
&h_{\mathbf{1}}(w) = |\{j \mid i_j < i_{j+1}\}|;\\
&h_{\mathbf{2}}(w) = \begin{cases} 1 & \text{if } i_0 < i_r, \\ 0 & \text{if } i_0 > i_r; \end{cases}\\
&h_{\mathbf{3}}(w) = |\{j \mid i_j < i_r < i_{j+1}\}|;\\
&h_{\mathbf{4}}(w) = |\{j \mid i_j < i_0 < i_{j+1}\}|;\\
&h_{\mathbf{5}}(w) = |\{j \mid i_{j-1} < i_j < i_{j+1}\}|;\\
&h_{\mathbf{6}}(w) = |\{(k,l) \mid i_{k-1} < i_l < i_k < i_{l+1}\}| = |\{(k,l) \mid i_l < i_{k-1} < i_{l+1} < i_k\}|.
\end{aligned} \tag{7.1}$$

Further, for any $\mathbf{N} \in \{\mathbf{1}, \dots, \mathbf{6}\}$ set $h_{\mathbf{N}} = \sum_{w \in W} h_{\mathbf{N}}(w)$.

Finally, let $W_l = \{w \in W \mid l(w) = l\}$, and denote: $h_{\mathbf{N}}(l) = \sum_{N \in W_l} h_{\mathbf{N}}(w)$. For $l < 0$ set $h_{\mathbf{N}}(l) = 0$.

7.2. Theorem. $\dim H^*(\mathfrak{n}; \mathfrak{n}) = h_{\mathbf{1}} + h_{\mathbf{2}} + h_{\mathbf{3}} + h_{\mathbf{4}} + h_{\mathbf{5}} + 2h_{\mathbf{6}}$.

7.3. Theorem.

$$\begin{aligned}
\dim H^k(\mathfrak{n}; \mathfrak{n}) = h_{\mathbf{1}}(k-1) + h_{\mathbf{2}}(k) + h_{\mathbf{3}}(k-1) + \\
h_{\mathbf{4}}(k-1) + h_{\mathbf{5}}(k-2) + h_{\mathbf{6}}(k-2) + h_{\mathbf{6}}(k-1).
\end{aligned}$$

7.4. Proof of Theorems 7.2, 7.3

Consider, for example, type **1**. If

$$x^\nu = x^0_{i_0} \dots x^r_{i_r} = x_j \cdot x_k^{-1} \cdot x^{w\rho}, \text{ where } x^{\beta j} = x_j \cdot x_k^{-1} \in \mathfrak{n},$$

then, as we have seen, $j = i_{s+1}$, $k = i_s$ and $j > k$. So $w \in W_{l+1}$ for all j, k, and the number of admissible pairs (j, k) is equal to the number of indices s such that $i_{s+1} > i_s$, i.e., to $h_{\mathbf{1}}(w)$. So we see that $\sum\limits_\nu \dim H^*_\nu$, where ν runs over type **1** vectors, is equal to $h_{\mathbf{1}}$. If we want to find $\sum\limits_\nu \dim H^k_\nu$, we must take $l+1 = k$, so $w \in W_{k-1}$.

Similar considerations for other types lead us to Table 2. Theorems 7.2, 7.3 are its direct consequences. □

7.5

We have subdivided the set of weights into several subsets according to their types. Set $H^*_{\mathbf{N}} = \bigoplus\limits_{\nu \in \mathbf{N}} H^*_\nu$ for any type **N** from Table 2. Let us find $\dim\ H^*_{\mathbf{N}}$.

1) for $w = \begin{pmatrix} 0 & \dots & r \\ i_0 & \dots & i_r \end{pmatrix}$, set $\bar{w} = \begin{pmatrix} 0 & 1 & \dots & r \\ i_r & i_{r-1} & \dots & i_0 \end{pmatrix}$. Then, clearly, $h_{\mathbf{1}}(\bar{w}) = r - h_{\mathbf{1}}(w)$. Since the map $w \mapsto \bar{w}$ is a one-to-one,

$$h_{\mathbf{1}} = \sum_{w \in W} h_{\mathbf{1}}(w) = \sum_{w \in W} h_{\mathbf{1}}(\bar{w}) = \sum_{w \in W} (r - h_{\mathbf{1}}(w)) = r \cdot |W| - h_{\mathbf{1}};$$

hence,

$$\dim H^*_{\mathbf{1}}(\mathfrak{n}; \mathfrak{n}) = h_{\mathbf{1}} = \frac{1}{2} r \cdot (r+1)! \tag{7.2}$$

2) Clearly, $i_0 < i_r$ in the half of the cases, hence,

$$\dim\ H^*_{\mathbf{2}}(\mathfrak{n}; \mathfrak{n}) = h_{\mathbf{2}} = \frac{1}{2}|W| = \frac{(r+1)!}{2} \tag{7.3}$$

3) Let us fix j. Then $|\{w \mid i_j < i_r < i_{j+1}\}| = \frac{1}{6}|W|$. So $\dim\ H^*_{\mathbf{3}} = h_{\mathbf{3}}$ is equal to the sum of such numbers for all j such that $1 \leqslant j \leqslant r-1$. Hence,

$$\dim H^*_{\mathbf{3}}(\mathfrak{n}; \mathfrak{n}) = h_{\mathbf{3}} = \sum_{1 \leqslant j \leqslant r-1} \frac{|W|}{6} = \frac{(r-1)}{6} \cdot (r+1)! \tag{7.4}$$

4), **5**) Obviously $h_{\mathbf{3}} = h_{\mathbf{4}}$ (moreover, $h_{\mathbf{3}}(k) = h_{\mathbf{4}}(k)$ for any k), hence,

$$\dim\ H^*_{\mathbf{4}}(\mathfrak{n}; \mathfrak{n}) = h_{\mathbf{4}} = \frac{(r-1)}{6} \cdot (r+1)! \tag{7.5}$$

The same formula is true for $H_{\mathbf{5}}$:

$$\dim\ H^*_{\mathbf{5}}(\mathfrak{n}; \mathfrak{n}) = h_{\mathbf{5}} = \frac{(r-1)}{6} \cdot (r+1)! \tag{7.6}$$

because in all the cases $\mathbf{3} - \mathbf{5}$ we have to order 3 indices in the transposition.

6) Similarly, fix k and l. Then there are $\frac{1}{24}|W|$ transpositions which satisfy the condition $i_{k-1} < i_l < i_k < i_{l+1}$. So

$$\dim\ H^*_{\mathbf{6}(b)} = \sum_{(k,l)} \frac{|W|}{24} = \dim\ H^*_{\mathbf{6}(e)}.$$

There are $(r-1)\cdot(r-2)$ pairs (k,l) such that $\{k-1,k\}\cap\{l,l-1\}=\emptyset$. So

$$\dim\ H^*_{\mathbf{6}}(\mathfrak{n};\mathfrak{n}) = 2h_{\mathbf{6}} = \frac{(r-1)(r-2)}{12}(r+1)! \tag{7.7}$$

7.6. Theorem. $\dim\ H^*(\mathfrak{n};\mathfrak{n}) = \frac{1}{12}(r^2+9r+2)(r+1)!$.

Proof: formulas (7.2)–(7.7). □

7.7

From Theorem 7.3 one can deduce $\dim\ H^k(\mathfrak{n};\mathfrak{n})$ for small k:

Theorem.

$$\dim\ H^k(\mathfrak{n};\mathfrak{n}) = \begin{cases} 1 & \textit{if } k=0 \textit{ and } r>0 \\ 2r & \textit{if } k=1 \textit{ and } r>1 \\ (3r^2+3r-4)/2 & \textit{if } k=2 \textit{ and } r>2 \\ \frac{2}{3}r^3+2r^2-\frac{11}{3}r-1 & \textit{if } k=3 \textit{ and } r>3 \end{cases}$$

If $r=2$, *then* $\dim\ H^2=5$, $\dim\ H^3=2$. *If* $r=3$, *then* $\dim\ H^3=21$.

Proof of this theorem also consists in studying different types of transpositions with 1, 2 or 3 inversions. It is trivial, but cumbersome, so we omit it. □

§8. The behavior of $h(r,k)=\dim\ H^k(\mathfrak{n};\mathfrak{n})$ as a function of k and $r=\mathrm{rk}\mathfrak{g}$

8.1. Theorem. *If k is fixed and r is sufficiently great, then for some rational in k functions $c_l(k)$ we have*

$$h(r,k) = \sum_{0\leqslant l\leqslant k} c_l(k)\cdot r^{k-1}. \tag{8.1}$$

Proof will be given in sec. 8.2–8.8. In sec. 8.2–8.6 we fix a permutation $w = \begin{pmatrix} 0 & \dots & r \\ i_0 & \dots & i_r \end{pmatrix}$. Let $l=l(w)$ and $X=\{0,1,\dots,r\}$.

8.2

Let $\sigma_i = \begin{pmatrix} 0 & \dots & i-1 & i & \dots & r \\ 0 & \dots & i & i-1 & \dots & r \end{pmatrix}$, where $i=1,\dots,r$, be the standard generators of W. By definition of length, w is a product of l generators: $w=\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_l}$. Let $K=\{i_1,\dots,i_l\}$. Let us subdivide K into subsets $K_1,\dots,K_m$ so that

(i) if $|i_a-i_b|\leqslant 1$ (in particular, if $i_a=i_b$), then i_a,i_b belong to the same K_j;

(ii) if on the contrary, there is an integer i such that $i_a < i < i_b$ and none of i_j is equal to i, then i_a and i_b belong to distinct sets K_j.

Clearly, (i) and (ii) define the subsets K_t uniquely, up to permutation. It is well known that if i and j belong to distinct sets K_t, then $\sigma_i\sigma_j = \sigma_j\sigma_i$. Thus we can write

$$w = w_1 w_2 \dots w_m, \tag{8.2}$$

where w_j is the product of the generators which belong to K_j. Such a representation of w is a canonical one, and $w_i w_j = w_j w_i$ for all i, j.

8.3

Let us fix j and consider K_j and the corresponding permutation w_j. If follows from the definition of K_j that there exist integers $k < l$ such that K_j contains σ_i (perhaps, more than once) if and only if $k < i \leqslant l$ and the segments $[k, l]$ have no common elements for distinct sets K_j. We will call $B_j = \{k, k+1, \dots, l\}$ the *block corresponding to* K_j. One easily sees that B_j is w-invariant and that

$$w_j(i) = \begin{cases} w(i) & \text{if } i \in B_j \\ i & \text{if } i \notin B_j \end{cases}. \tag{8.3}$$

So w and w_j give the same permutation of $\{k, \dots, l\}$. Let us denote by w_B the corresponding permutation of $\{0, \dots, l-k\}$; so

$$w_B = \begin{pmatrix} 0 & \dots & l-k \\ i_k - k & \dots & i_l - k \end{pmatrix}. \tag{8.4}$$

8.4. Lemma. *Let $B_1, \dots, B_m$ be the blocks of w, then*

i) $l(w_{B_j}) \geqslant \frac{1}{2}\operatorname{card} B_j$ *for any j.*

ii) $l(w) = \sum\limits_{1 \leqslant j \leqslant m} l(w_{B_j})$.

Proof. Since $w = \sigma_{i_1} \dots \sigma_{i_l}$ is a minimal representation of w as the product of generators, we easily deduce that $i \in X$ belongs to some block if and only if i forms an inversion with some i_1 from the same block. This gives us (i).

Now, the elements of distinct blocks do not form inversions, and the number of inversions in the restriction of w onto B and in w_B coincide. This implies (ii). □

8.5. Lemma. *Let $B_1, \dots, B_m$ be the blocks of w, and $r_j = \operatorname{card} B_j$. Then*

$$h_{\mathbf{1}}(w) = r + m - \sum_{1 \leqslant j \leqslant m} r_j + \sum_{1 \leqslant j \leqslant m} h_{\mathbf{1}}(w_{B_j}). \tag{8.5}$$

Proof. Let $Z = \{j \mid i_j < i_{j+1}\}$, so $h_{\mathbf{1}}(w) = |Z|$. Let $Z_1 = X \setminus \cup B_j$ and let Z_2 be the set of last elements of blocks: $i \in Z_2$ if and only if there exists a B_j equal to $\{k, \dots, i\}$. Then, clearly

$$|Z_1| = r + 1 - \sum_{1 \leqslant j \leqslant m} r_j, \quad |Z_2| = m, \quad Z_1 \cap (Z \cup Z_2) = (Z_1 \cup Z_2) \setminus \{r\} \tag{8.6}$$

and it is always true that $r \in Z_1 \cup Z_2$. Finally, let $i \in Z$, $i \notin Z_1 \cup Z_2$. Then i and $i+1$ belong to some block $B = \{k, \ldots, l\}$. So

$$|Z_1 \backslash (Z \cup Z_2)| = \sum_B h_{\mathbf{1}}(w_B). \tag{8.7}$$

Now (8.5) is a direct consequence of (8.6) and (8.7). □

8.6

We can get similar formulas for functions $h_{\mathbf{2}}(w)$, ..., $h_{\mathbf{6}}(w)$; for example, we see that if $r > 2l$, then $h_{\mathbf{2}}(w) = 1$. We won't need the exact formulas, so we shall formulate the corresponding result as follows.

Call B a *block of type* 1 if $B \ni 0$, *of type* 2 if $B \ni r$ or *of type* 3 if $0, r \notin B$, respectively. (Lemma 8.4 (i) implies that if $r > 2l$, then B cannot simultaneously contain 0 and r.) Let Ω be the set of all the w_B for given types of blocks. We will say that Ω is a *structure on the Weyl group* W, and that w is a *permutation of the structure* Ω.

Lemma. *In what follows the* c_i *depend only on* Ω *and do not depend on* w *or* r*:*

$$h_{\mathbf{2}}(w) = c_2 = 1, \quad h_{\mathbf{3}}(w) = c_3, \quad h_{\mathbf{4}}(w) = c_4, \quad h_{\mathbf{5}}(w) = r + c_5, \quad h_{\mathbf{6}}(w) = c_6. \tag{8.8}$$

Proof is similar to that of Lemma 8.5, and we omit it. □

Note, however, the difference between $h_{\mathbf{1}}, h_{\mathbf{5}}$ and the other four functions. This difference is due to the fact that if $i \notin \cup B_j$, then the inequalities $i_i < i_{i+1}$ and $i_{i-1} < i_i < i_{i+1}$ hold, and other inequalities in formulas (7.1) do not hold.

8.7

Let Ω be a structure on W, and $W(r, \Omega)$ the set of all permutations of $\{0, \ldots, r\}$ with the given structure Ω. We will consider $f(r, \Omega) = |W(r, \Omega)|$ as a function of r, Ω for a fixed Ω.

Lemma. $f(r, \Omega) = \sum_{1 \leqslant k \leqslant N} \alpha_k r^k$*, where* $\alpha_N \neq 0$ *and* N *is equal to the number of blocks of type* 3 *in* Ω*.*

Proof. First, suppose, that $\Omega = \{w_1, \ldots, w_m\}$, all the w_i are of type 3, and no two pairs of them coincide. Then there is a one-to-one correspondence between the permutations of the structure Ω and the arrangements of the sets $B_1, \ldots, B_m$ of given lengths $a_1, \ldots, a_m$ from $\{1, \ldots, r-1\}$. It is well-known that the number of such arrangements is equal to

$$\frac{(r - 1 + m - \sum_j a_j)!}{(r - 1 - \sum_j a_j)!}. \tag{8.9}$$

So it is a polynomial in r of degree m.

If we have blocks of type 1 or 2, then their places are fixed, and we have to place the remaining blocks. So $f(r,\Omega)$ is still a polynomial in r whose degree is equal to the number of blocks of type 3.

Finally, if $w_1 = w_2 = \ldots = w_{b_1}$, $w_{b_1+1} = \ldots = w_{b_1+b_2}$, and so on, we have to divide (8.9) by $\prod_i (b_i)!$. □

8.8

Now we are ready to finish the proof of the theorem. Let Ψ_l be the set of all structures with l inversions. Lemma 8.4 shows that Ψ_l is finite and does not depend on r if r is sufficiently large. We know (Lemmas 8.5, 8.6) that the functions $h_i(w)$ depend only on Ω, i.e., $h_i(w) = h_i(\Omega)$.

So we may rewrite Theorem 7.3 so that every summand in it takes the form

$$h_i(l) = \sum_{\Omega\in\Psi_l} h_i(\Omega) f(t,\Omega). \tag{8.10}$$

Every summand is a polynomial and their number does not depend on r. Applying Theorem 7.3 we complete the proof. □

8.9. Theorem. *Let $c_l(k)$ be the coefficients in* (8.1). *Then $(k-l)!\cdot c_l(k)$ is a polynomial in k.*

Proof. It is clear from formulas (8.10), (8.9), that $k!\cdot c_l(k)$ is a polynomial. But if $l > k$ then the term of degree $k-l$ vanishes, so $c_l(k) = 0$. □

8.10

Let us find $c_0(k)$. Clearly, we must consider only two terms in Theorem 7.3:

$$\dim H^k(\mathfrak{n};\mathfrak{n}) = \sum_{\Omega\in\Psi_l} f(r,\Omega)\cdot h_{\mathbf{2}}(\Omega) + \sum_{\Omega\in\Psi_l} f(r,\Omega)\cdot h_{\mathbf{1}}(\Omega) + \ldots \tag{8.11}$$

because the degrees of the terms denoted by dots are $< k$. We know, that $h_{\mathbf{2}}(\Omega) = 1$ and $h_{\mathbf{1}}(\Omega) = r + C$. Hence, the terms in (8.11) have degree k if and only if Ω consists of the maximal number of blocks (k for the first term and $k-1$ for the second one, respectively).

But then every block has only one inversion, so $\Omega = \left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \ldots, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)$ and $|\Omega|$ is equal to k or $k-1$. Then

$$f(r,\Omega) = \binom{r-|\Omega|+1}{|\Omega|} = \frac{r^{|\Omega|}}{|\Omega|!} + \ldots. \tag{8.12}$$

So

$$c_0(k) = \frac{1}{k!} + \frac{1}{(k-1)!} = \frac{1}{k!}\cdot(k+1). \tag{8.13}$$

8.11

Using the same method, one can prove that

$$c_1(k) = \frac{1}{(k-1)!}\cdot\frac{k^2-1}{2}; \quad c_2(k) = \frac{1}{(k-2)!}\cdot\frac{3k^3-k^2-92k+116}{24}. \tag{8.14}$$

For example,

$$\dim\ H^3(\mathfrak{n};\mathfrak{n}) = \frac{4}{3!}r^3 + \frac{4}{2!}r^2 - \frac{88}{24}r + \dots$$

which coincides with Theorem 7.7;

$$\dim\ H^4(\mathfrak{n};\mathfrak{n}) = \frac{5}{24}r^4 + \frac{5}{4}r^3 - \frac{19}{6}r^2 + \dots$$

§9. Some other results and problems

9.1

The most natural generalization of the problems solved in this paper is to study $\mathfrak{g} \neq A_r$. I also studied the other classical algebras and G_2. Their study is much more complicated then that of A_r because one has to consider more cases. The values of $|G_\mathfrak{n}|$ are given in Table 3.

Moreover, I know, that $\dim\ H^*(\mathfrak{n};\mathfrak{n}) = |G_\mathfrak{n}|$ in all these cases except $\mathfrak{g} = D_r$. The main difficulty is that a straightforward analogue of Lemma 6.2 fails for D_r (see below).

Problem. *Calculate* $\dim\ H^*(\mathfrak{n};\mathfrak{n})$ *for* $\mathfrak{g} = D_r$ *as well as for the exceptional algebras.*

9.2

Now let us discuss other modules. The classical Lie algebras A_r, B_r, C_r, D_r have standard (identity) representations, whose dimensions are $r+1, 2r+1, 2r$ and $2r$, respectively. Their multiplicities (see 3.9) are equal to 2 for the types A, B, C and 4 for the type D. We will analyze these representations elsewhere; here I only give the main result. For simplicity we assume $\mathfrak{g} = A_r$, the same results hold for the types B and C.

Let $v_0, \dots, v_r$ be the standard basis of U_{r+1}. Then all the $\mathfrak{b}$-submodules are generated by $V_i = Span(v_i, v_{i+1}, \dots, v_r)$ for $0 \leqslant i \leqslant r$. Let $V_{ij} = V_i/V_{j+1}$; clearly, $\dim V_{ij} = j + 1 - i$. Set $l = \dim V_{ij}$. Then

$$\dim\ H^*(\mathfrak{n}; V_{ij}) = |W| \cdot \frac{l(r+2-l)}{r+1} \tag{9.1}$$

and $\dim\ H^k(\mathfrak{n}; V_{ij})$ for a fixed k does not depend on i, j, l if $i > k$, $l > k$, $r - j > k$. This dimension is a polynomial in r and its degree is equal to k.

It is worth to note that $\dim\ H^*$ grows as $|W| \cdot \dim\ V$ as $r \longrightarrow \infty$ and the growth of $\dim\ H^k$ is a polynomial one. Moreover, $\dim\ H^*$ has a single formula for all r, great and small, whereas $\dim\ H^k$ has such formula only for sufficiently large r.

9.3

Finally, Table 4 gives $\dim\ H(\mathfrak{n}; \Lambda^*\mathfrak{n})$ for different types of $\mathfrak{g}$.

Problem. *Study the growth of* $\dim\ H(\mathfrak{n}; \Lambda^*\mathfrak{n})$ *as* $r \longrightarrow \infty$ *for various series of matrix algebras* $\mathfrak{g}$*; first, for* $\mathfrak{g} = A_r$*.*

Table 1

$\mathfrak{g}$	ρ	$P(t,x)$
A_r	$(1\ 2\ \dots\ r)$	$\sum x_{i_0}^0 x_{i_1}^1 \dots x_{i_r}^r t^{l(w)}$
B_r	$(\frac{1}{2}\ \frac{3}{2}\ \frac{5}{2} \dots \frac{2r-1}{2})$	$\sum x_{i_1}^{\pm 1/2} \dots x_{i_r}^{\pm(2r-1)/2} t^{l(w)}$
C_r	$(1\ 2\ 3\ \dots\ r)$	$\sum x_{i_1}^{\pm 1} \dots x_{i_r}^{\pm r} t^{l(w)}$
D_r	$(0\ 1\ 2\ \dots\ (r-1))$	$\sum x_{i_1}^0 x_{i_2}^{\pm 1} \dots x_{i_r}^{\pm(r-1)} t^{l(w)}$

Table 2

ν		Dim$E_{1\nu}^{k*}$ for different k				H^*
		$k=l$	$k=l+1$	$k=l+2$	$k=l+3$	
I		0	$h_I(w)$	0	0	$\dim H^{l+1} = h_I(w)$
II	(a) $i_0 < i_r$	1	0	0	0	$\dim H^l = 1$
	(b) $i_0 > i_r$	0	0	0	0	$H^* = 0$
III	(a) $i_j < i_{j+1} < i_r$	1	1	0	0	$H^* = 0$
	(b) $i_j < i_r < i_{j+1}$	0	1	0	0	$\dim H^{l+1} = 1$
	(c) $i_r < i_j < i_{j+1}$	0	0	0	0	$H^* = 0$
IV	(a) $i_0 < i_j < i_{j+1}$	1	1	0	0	$H^* = 0$
	(b) $i_0 < i_j < i_{j+1}$	0	1	0	0	$\dim H^{l+1} = 1$
	(c) $i_0 < i_j < i_{j+1}$	0	0	0	0	$H^* = 0$
V		0	0	2	1	$\dim H^{l+2} = 1$
VI	(a)	0	0	0	0	$H^* = 0$
	(b)	0	0	1	0	$\dim H^{l+2} = 1$
	(c)	0	1	1	0	$H^* = 0$
	(d)	0	1	1	0	$H^* = 0$
	(e)	0	2	1	0	$\dim H^{l+1} = 1$
	(f)	1	2	1	0	$H^* = 0$

Table 3

$\mathfrak{g}$	A_r	B_r	C_r	D_r	G_2
$\frac{1}{\lvert W\rvert}\lvert G_{\mathfrak{n}}\rvert$	$\frac{r^2+9r+2}{12}$	$\frac{2r^2+10r-1}{12}$	$\frac{2r^2+8r+3}{12}$	$\frac{2r^2+7r+3}{12}$	34

Table 4

$\mathfrak{g}$	A_1	A_2	A_3	B_2	G_2
$\dim H^*(\mathfrak{n};\Lambda^*\mathfrak{n})$	4	36	600	68	220

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