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Three Order Parameters in Quantum XZ Spin-Oscillator Models with Gibbsian Ground States*

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Abstract. Quantum models on the hyper-cubic d-dimensional lattice of spin-$\frac{1}{2}$ particles interacting with linear oscillators are shown to have three ferromagnetic ground state order parameters. Two order parameters coincide with the magnetization in the first and third directions and the third one is a magnetization in a continuous oscillator variable. The proofs use a generalized Peierls argument and two Griffiths inequalities. The class of spin-oscillator Hamiltonians considered manifest maximal ordering in their ground states. The models have relevance for hydrogen-bond ferroelectrics. The simplest of these is proven to have a unique Gibbsian ground state.

Key words: order parameters; spin-boson model; Gibbsian ground state

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1 Introduction

In this paper we consider quantum lattice models of oscillators interacting with spins whose variables are indexed by the sites of a hyper-cube Λ with the finite number of sites $|\Lambda|$ in the hyper-cubic lattice $\mathbb{Z}^d$. Interaction is considered to be short-range and translation invariant. The corresponding Hamiltonian H_Λ is expressed in terms of the oscillators variables $q_\Lambda = (q_x, x \in \Lambda) \in \mathbb{R}^{|\Lambda|}$ and spin $\frac{1}{2}$ Pauli matrices $S^l_\Lambda = (S^l_x, x \in \Lambda, l = 1, 3)$, defined in the tensor product of the $2^{|\Lambda|}$-dimensional Euclidean space and the space of square integrable functions $\mathbb{L}^2_\Lambda = (\otimes\mathbb{E}^2)^{|\Lambda|} \otimes L^2(\mathbb{R}^{|\Lambda|})$,

$$H_\Lambda = \sum_{x\in\Lambda} \big[- \partial_x^2 + \mu^2(q_x + \eta\phi_x(S^3_\Lambda))^2 - \mu\big] + \sum_{A\subseteq\Lambda} J_A S^1_{[A]} + V_\Lambda, \qquad \mu \geq 0, \quad \eta \in \mathbb{R}, \tag{1.1}$$

where ∂_x is the partial derivative in q_x, J_A and V_Λ are real-valued measurable functions, the first of which depends on q_Λ and the second on S^3_Λ, q_Λ and the translation invariant ϕ_x is given by (see Remark 2 in the end of the paper)

$$\phi_x(s_\Lambda) = \sum_{A\subseteq\Lambda} J_0(x; A)s_{[A]}, \qquad J_0(x; x) = 1. \tag{1.2}$$

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For products of operators, functions and variables we use the following notation: $B_{[A]} = \prod_{x\in A} B_x$. The scalar product in $(\otimes\mathbb{E}^2)^{|\Lambda|}$, $\mathbb{L}^2_\Lambda$ will be denoted by $(\cdot,\cdot)_0$, $(\cdot,\cdot)$, respectively. The Schwartz space of test functions on $\mathbb{R}^n$ will be denoted by $\mathbb{S}(\mathbb{R}^n)$.

We require that the Hamiltonian is well defined and bounded from below on $(\otimes\mathbb{E}^2)^{|\Lambda|} \otimes C_0^\infty(\mathbb{R}^{|\Lambda|})$, i.e. the tensor product of the $2^{|\Lambda|}$-dimensional Euclidean space and the space of infinitely differentiable functions with compact supports. The ground state average for an observable (operator) F is determined by

$$\langle F\rangle_\Lambda = Z_\Lambda^{-1}(\Psi_\Lambda, F\Psi_\Lambda), \qquad Z_\Lambda = (\Psi_\Lambda, \Psi_\Lambda) = ||\Psi_\Lambda||^2,$$

where Z_Λ is a partition function. For partial cases of F we have

$$\langle \hat{q}_{[A]}\phi_{[A']}(S^3_\Lambda)\rangle_\Lambda = Z_\Lambda^{-1}\int q_{[A]}(\Psi_\Lambda(q_\Lambda), \phi_{[A']}(S^3_\Lambda)\Psi_\Lambda(q_\Lambda))_0 dq_\Lambda,$$

where the integration is performed over $\mathbb{R}^{|\Lambda|}$ and $\hat{q}_{[A]}$ is the operator of multiplication by q_x.

We will employ the orthonormal basis $\psi^0_\Lambda(s_\Lambda)$ of the Euclidean space $(\otimes\mathbb{E}^2)^{|\Lambda|}$, diagonalizing S^3_Λ, which is chosen in the following way: $\psi^0_\Lambda(s_\Lambda) = \otimes_{x\in\Lambda}\psi^0(s_x)$, $s_x = \pm 1$, $\psi^0(1) = (1,0)$, $\psi^0(-1) = (0,1)$, $S^1\psi^0(s) = \psi^0(-s)$, $S^3\psi^0(s) = s\psi^0(s)$. For $F \in \mathbb{L}^2_\Lambda$ we have the following decomposition

$$F(q_\Lambda) = \sum_{s_\Lambda} F(q_\Lambda; s_\Lambda)\psi^0_\Lambda(s_\Lambda),$$

where the summation is performed over the $|\Lambda|$-fold Cartesian product $(-1,1)^{|\Lambda|}$of the set $(-1,1)$.

The Hamiltonians in (1.1) are employed in hydrogen-bond ferroelectric crystal models, considered in [1, 2, 3], and describe interaction between heavy ions (oscillators with constant frequency) and protons (spins). The second term with $J_A = 0$, $|A| \geq 2$ corresponds to the energy of protons, tunneling along hydrogen bonds from one well to another, and J_x is associated with the tunneling frequency. The last term in the expression for our Hamiltonian $V_A = \sum_{A\subseteq\Lambda} J_1(A)S^3_{[A]}$ describes many-body interaction between protons ($J_1(A)$ is the intensity of the $|A|$-body interaction).

A rigorous analysis of a mean-field version of the Hamiltonian in (1.1) with $\phi_x(S^3_\Lambda)$ given in (1.2) that is linear in S^3, $J_x \neq 0$, $V_\Lambda = -(\mu\eta)^2\sum_{x\in\Lambda}\phi^2_x(S^3_\Lambda)$, $J_A = 0$ for $|A| \geq 2$ and $J_0(x;y)$ uniformly in lattice sites proportional to $|\Lambda|^{-1}$ was carried out in [3] in the framework of the Bogolyubov approximating Hamiltonian method [4] and occurrence of spin and oscillator orderings (the corresponding order parameters are non-zero) for two-body interaction between protons at non-zero temperatures was proved. To establish such orderings for ground states without the mean-field limit in a general case is an important task for a theory.

The oscillator and spin orderings are established if one proves the existence of ferromagnetic oscillator and spin long-range orders (lro's) in ground states for the corresponding Hamiltonians. This means that the ground state averages $\langle\hat{q}_x\hat{q}_y\rangle_\Lambda$, $\langle S^j_x S^j_y\rangle_\Lambda$, $j = 1,3$ are bounded uniformly in Λ from below by positive numbers . Occurrence of the ferromagnetic lro's implies the existence of the spin order parameters (magnetizations in the first and third directions) $M^l_\Lambda = |\Lambda|^{-1}\sum_{x\in\Lambda} S^l_x$, $l = 1,3$, and the oscillator order parameter $Q_\Lambda = |\Lambda|^{-1}\sum_{x\in\Lambda} q_x$ in the thermodynamic limit ($\Lambda \to \mathbb{Z}^d$) since the ground state averages of their squares are uniformly bounded in Λ from below by a positive number.

In this paper we describe three classes of V_Λ for which ground states or eigenstates Ψ_Λ of the Hamiltonians in (1.1) are Gibbsian

$$\Psi_\Lambda(q_\Lambda) = \sum_{s_\Lambda} e^{-\frac{1}{2}U(s_\Lambda;q_\Lambda)}\psi^0_\Lambda(s_\Lambda)\psi_{0\Lambda}(q_\Lambda), \tag{1.3}$$

with the linear in q_Λ spin-oscillator quasi-potential energy U

$$U(s_\Lambda; q_\Lambda) = 2\mu\eta\sum_{x\in\Lambda} q_x\phi_x(s_\Lambda) + U^0(s_\Lambda), \tag{1.4}$$

where $\psi_{0\Lambda}(q_\Lambda) = \prod_{x\in\Lambda}\psi_0(q_x)$, $\psi_0(q) = (\mu\pi^{-1})^{\frac{1}{4}}\exp\{-\frac{\mu}{2}q^2\}$, is the ground state of the free oscillator Hamiltonian, that is the first term in the right-hand side of (1.1) with $\eta = 0$. The first and the second classes correspond to the choice $U^0(s_\Lambda) = \alpha U_0(s_\Lambda)$, $U^0(s_\Lambda) = \alpha U_0(s_\Lambda) + \mu\eta^2\sum_{x\in\Lambda}\phi^2_x(s_\Lambda)$, $\alpha \geq 0$ (the second choice makes V_Λ independent of oscillator variables), respectively, and the third class coincides with the set of finite-range V_Λ if the the following conditions are satisfied: $J_A = 0$ for $|A| \geq 2$ and $|J_1(A)| < 1$ is sufficiently small. For the first two of them we prove the maximal ordering in the corresponding systems (provided some simple conditions are satisfied): magnetizations M^l_Λ, $l = 1, 3$, Q_Λ are non-zero. No other ground states are known with such the property.

Gibbsian ground states were introduced by Kirkwood and Thomas in [5] in XZ spin-$\frac{1}{2}$ models with Hamiltonians (linear in S^1) that include the spin part of (1.1), i.e. the second and third terms, with only $J_x \neq 0$ and the periodic boundary condition (this boundary condition is not essential). Matsui in [6, 7] enlarged a class of spin-$\frac{1}{2}$ XZ-type models in which Gibbsian states exist. The method was further developed by Datta and Kennedy in [8]. An application of classical spin systems for constructing of quantum states was given in [9].

In [10] we showed how to find V_Λ for a given Gibbsian ground state and established existence of lro's in S^1 and S^3 for a wide class of the spin-$\frac{1}{2}$ XZ models (see also [11, 12]). This reference contains the most simple proofs of the existence of lro's in ground states of quantum many-body systems. A reader may find a review of the results concerning several quantum orders in it (see also [13, 14, 15]).

The ground state in (1.3) can be represented in the following equivalent form

$$\Psi_\Lambda(q_\Lambda) = \sum_{s_\Lambda} F_0(s_\Lambda)\psi^0_\Lambda(s_\Lambda)\psi_{0\Lambda}(q_\Lambda + \eta\phi_\Lambda(s_\Lambda)), \tag{1.5}$$

$$F_0 = \exp\left\{-\frac{1}{2}U_*\right\}, \qquad U_* = U^0 - \mu\eta^2\sum_{x\in\Lambda}\phi^2_x. \tag{1.6}$$

From (1.5), orthonormality of the basis (see the beginning of the third section) and the equalities $\int\psi^2_0(q)dq = 1$, $\int\psi^2_0(q)qdq = 0$ it follows that

$$\langle\hat{q}_x\hat{q}_y\rangle_\Lambda = \eta^2\langle\phi_x(S^3_\Lambda)\phi_y(S^3_\Lambda)\rangle_\Lambda = \eta^2\langle\phi_x(\sigma_\Lambda)\phi_y(\sigma_\Lambda)\rangle_{*\Lambda}, \qquad \langle S^3_x S^3_y\rangle_\Lambda = \langle\sigma_x\sigma_y\rangle_{*\Lambda}, \tag{1.7}$$

where $\sigma_x(s_\Lambda) = s_x$ and

$$\langle\phi_x(\sigma_\Lambda)\phi_y(\sigma_\Lambda)\rangle_{*\Lambda} = Z^{-1}_{*\Lambda}\sum_{s_\Lambda}|F_0(s_\Lambda)|^2\phi_x(s_\Lambda)\phi_y(s_\Lambda), \qquad Z_{*\Lambda} = \sum_{s_\Lambda}|F_0(s_\Lambda)|^2.$$

Equalities in (1.7) reduce a calculation of averages in our quantum systems to a calculation of averages indexed by a star in Ising models and the following statement (principle) is true.

Proposition 1. *Let $J_0 \geq 0$ in (1.1) and ferromagnetic lro occur in the Ising model with the potential energy U_* given by (1.6), that is $\langle \sigma_x \sigma_y \rangle_{*\Lambda} > 0$ uniformly in Λ, then ferromagnetic lro occurs in oscillator variables and S^3 in the quantum spin-oscillator system with the Hamiltonian (1.1).*

Another important statement is formulated as follows.

Proposition 2. *If J_A and V_Λ do not depend on oscillator variables and $\Psi_{0\Lambda} = \sum_{s_\Lambda} F_0(s_\Lambda)\psi^0_\Lambda(s_\Lambda)$, where F_0 is a complex valued function, is the ground state of the the spin part of (1.1), then the ground state Ψ_Λ of the Hamiltonian (1.1) is given by (1.5).*

The simple proof of Proposition 2 is based on the fact that the unitary operator $T^{-1}_\Lambda = \prod_{x\in\Lambda} T^{-1}_x(\phi)$ of translation of oscillator variables on each one-dimensional subspace of $(\otimes\mathbb{C}^2)^{|\Lambda|}$, where

$$(T_x(\phi)F)(q_\Lambda) = \sum_{s_\Lambda} F(q_x + \eta\phi_x(s_\Lambda), q_{\Lambda\setminus x}; s_\Lambda)\psi^0_\Lambda(s_\Lambda),$$

maps the vector in (1.5) into the tensor product of the ground states the free oscillator Hamiltonian and the spin part of the Hamiltonian (1.1). Proposition 2 guarantees that our quantum spin-oscillator system will possess a Gibbsian ground state if a pure spin system possesses such state with the Hamiltonian coinciding with the spin part $\sum_{A\subseteq\Lambda} J_A S^1_{[A]} + V_\Lambda$ of the Hamiltonian (1.1). Usefulness of Gibbsian ground states is explained by comparative simplicity of a proof of existence of lro. Our results show that Gibbsian ground states are expected to appear in many quantum spin-oscillator systems with non-trivial interactions.

Our paper is organized as follows. In the second section we formulate our main results in two theorems, a lemma and two propositions. The first and second theorems deal with the cases when the spin part of the Hamiltonian (1.1) do not depend and depend on oscillator variables, respectively. In the next sections we prove the results.

2 Main result

Proposition 2 shows that the case when J_A and V_Λ do not depend on oscillator variables is the simplest. The existence of the Gibbsian ground state will yield a proof of lro via Proposition 1 for it. We know from [10] that if

$$V_\Lambda = -\sum_{A\subseteq\Lambda} J_A e^{-\frac{\alpha}{2}W_{0A}(S^3_\Lambda)}, \qquad W_{0A}(s_\Lambda) = U_0(s_{\Lambda\setminus A}, -s_A) - U_0(s_\Lambda). \tag{2.1}$$

then the ground state of the spin part of the Hamiltonian (1.1) is Gibbsian with $F_0 = e^{-\frac{\alpha}{2}U_0}$ in Proposition 2.

Theorem 1. *Let J_A, V_Λ be independent of oscillator variables, (2.1) hold and U^0 be determined from the equality $U^0 = \alpha U_0 + \mu\eta^2 \sum_{x\in\Lambda} \phi^2_x$. Then*

I. Ψ_Λ, *given by (1.3), (1.4), is an eigenfunction of the Hamiltonian (1.1) with the zero eigenvalue and is its (unique) ground state if $J_A \leq 0$ (if the uniform bound $J_x \leq J_- < 0$ holds);*

II. *lro in S^3 and oscillator lro occurs if $J_0(x, A) \geq 0$ and lro occurs in a classical spin system with the potential energy U_0;*

III. *lro in S^1 occurs if $\lim_{\Lambda\to\mathbb{Z}^d} W_{0A}(s_\Lambda)$ exists for $|A| = 2$ and is uniformly bounded.*

The second item follows from Propositions 1 and 2. The proofs of items I and III can be recovered easily from the proof of the next theorem (see also [10]).

The following Proposition is proved without difficulty with the help of Proposition 2 and the results of [5] concerning an existence of the Gibbsian ground state for a XZ spin-$\frac{1}{2}$ model.

Proposition 3. *Let V_Λ not depend on oscillator variables, ϕ_x be finite range, $J_A = 0$ for $|A| \geq 2$, the potential $|J_1(A)| \leq 1$ be sufficiently small and the periodic boundary condition in the spin variables hold. Then there exists a function U^0 such that Ψ_Λ, given by (1.3), (1.4), is a unique ground state of the Hamiltonian (1.1).*

Items II, III of Theorem 1 for $V_\Lambda = -(\mu\eta)^2 \sum_{x\in\Lambda} \phi_x^2(S_\Lambda^3)$ do not follow from [5] since lro is proved there for nearest-neighbor interaction (note that this V_Λ determines non-nearest neighbor interaction even if ϕ_x determines such).

It turns out that Gibbsian ground states exist, also, for J_A depending on q_Λ if

$$V_\Lambda = -\sum_{A\subseteq\Lambda} J_A e^{-\frac{1}{2}W_A(S_\Lambda^3)}, \qquad W_A(S_\Lambda^3) = U(S_\Lambda^{3A}; q_\Lambda) - U(S_\Lambda^3; q_\Lambda), \tag{2.2}$$

where $S_\Lambda^{3A} = (-S_A^3, S_{\Lambda\setminus A}^3)$. The analog of Theorem 1 can be proved for such V_Λ but its proof is more involved since this operator is unbounded. Negative J_A generate positive function V_Λ in (2.2) and this enables us to prove the following lemma for the case when V_Λ depends on oscillator variables.

Lemma 1. *Let V_Λ be given by (2.2) and have a domain $D(V_\Lambda)$, J_A be bounded negative functions and U in (2.2) coincide with U in (1.4) in which $U^0 = \alpha U_0$. Then H_Λ is positive definite and essentially self-adjoint on the set $(\otimes\mathbb{C}^2)^{|\Lambda|}\otimes\mathbb{S}(\mathbb{R}^{|\Lambda|})\cap D(V_\Lambda)$ that contains $(\otimes\mathbb{C}^2)^{|\Lambda|}\otimes C_0^\infty(\mathbb{R}^{|\Lambda|})$ and item* I *of Theorem 1 is true.*

Remark 1. If the functions J_A are only negative then statement II still holds and Ψ_Λ is the ground state of the self-adjoint extension of the Hamiltonian preserving positive definiteness.

The most simple translation invariant short-range U_0 is ferromagnetic

$$U_0(s_\Lambda) = -\sum_{A\subseteq\Lambda} J_A^0 s_{[A]}, \qquad J_A^0 \geq 0, \tag{2.3}$$

where $J_A^0 = 0$ for odd $|A|$. The following theorem establishes analogs of items II–III of Theorem 1 for the case of V_Λ given by (2.2).

Theorem 2. *Let all the conditions of Lemma 1 be satisfied, $J_0 \geq 0$, $J_0(x;A) = 0$ for even $|A|$, $J_0(0;1), J_{0,1}^0 \geq \bar{J}$ and U_0 be given by (2.3). Then*

I. *For a sufficiently large $\beta = (\eta^2\mu+\alpha)\bar{J} > 1$ there exist ground state lro's in S^3 and oscillator variables for $d \geq 2$;*

II. *Let the positive constants C, B_j, $j = 0, 1, 2$, independent of Λ, exist such that $|\phi_x(s_\Lambda)| \leq C$*

$$|W_{0A}(s_\Lambda)| \leq B_0, \qquad W_A^{(j)}(s_\Lambda) \leq B_j,$$
$$W_A^{(j)}(s_\Lambda) = \sum_{x'\in\Lambda} |\phi_{x'}^j(s_\Lambda^A) - \phi_{x'}^j(s_\Lambda)|, \qquad j = 1, 2,$$

where $|A| = 2$. Then ground state ferromagnetic lro in S^1 occurs in arbitrary dimension d.

If $U_0 = 0$ then V_Λ from (2.2) is given by

$$V_\Lambda = -\sum_{A\subseteq\Lambda} J_A v_{[A]}, \qquad v_x = \cosh u_x + S_x^3 \sinh u_x, \qquad u_x = 2\eta\mu\phi_x(q_\Lambda).$$

This equality follows from the equalities

$$W_A(S_\Lambda^3) = -2\sum_{x\in A} u_x S_x^3, \qquad e^{aS^3} = \cosh a + S^3 \sinh a, \qquad (S^3)^2 = I.$$

In the simplest case the conditions of item II of Theorem 2 can be checked without difficulty.

Proposition 4. *Let ϕ_x be linear in S^3 in (1.2) and $||J_0||_1 = \sum_x |J_0(x)| < \infty$, where the summation is performed over $\mathbb{Z}^d$. Then the conditions of item* II *of Theorem 2 are satisfied.*

Note that if one uses the Pauli matrices with $\frac{1}{2}$ instead of the unity as matrix elements then V_Λ should be changed by adding to W_{0A} and W_A in (2.1) and (2.2), respectively, the number $-|A|\ln 2$.

The expression for V_Λ in (2.1) can be calculated for certain U_0 (see [10]). If one chooses the anti-ferromagnetic U_0 in (2.2), specifically,

$$U_0(s_\Lambda) = \alpha^{-1}\mu\eta^2 \sum_{x\in\Lambda} \phi_x^2(s_\Lambda) + \sum_{\langle x,y\rangle\in\Lambda} s_x s_y, \qquad \alpha > 0$$

then it can be easily proved that the spin lro in the third direction will be anti-ferromagnetic, generating a staggered magnetization (spins at the even and odd sublattices take different values).

The interesting and important property of the Hamiltonians with V_Λ given by (2.1), (2.2) is that they are simply related to generators of stationary Markovian processes (see, also, [16]). We believe that it is possible to apply the same mathematical technique for proving existence of order and phase transitions in equilibrium quantum systems and non-equilibrium stochastic systems (see [17]).

3 Proof of Lemma 1

For our purpose it is convenient to pass to a new representation. It is determined by the Hilbert space of sequences of functions $F(q_\Lambda; s_\Lambda)$, $s_x = \pm 1$, which are found in the expansion of the vector $F \in \mathbb{L}_\Lambda^2$ mentioned at the beginning of the introduction, with the scalar product

$$(F_1, F_2) = \sum_{s_\Lambda} \int F_1(q_\Lambda; s_\Lambda) F_2(q_\Lambda; s_\Lambda) dq_\Lambda,$$

$$(F_1(q_\Lambda), F_2(q_\Lambda))_0 = \sum_{s_\Lambda} F_1(q_\Lambda; s_\Lambda) F_2(q_\Lambda; s_\Lambda),$$

where the integration is performed over $\mathbb{R}^{|\Lambda|}$. Here we took into account the orthonormality of the basis, i.e. the equality

$$(\Psi_\Lambda^0(s_\Lambda), \Psi_\Lambda^0(s'_\Lambda))_0 = \delta(s_\Lambda; s'_\Lambda) = \prod_{x\in\Lambda} \delta_{s_x, s'_x},$$

where $\delta_{s,s'}$ is the Kronecker delta. Let

$$h_x = -\partial_x^2 + \mu^2(q_x + \eta\phi_x(S_\Lambda^3))^2 - \mu,$$

$$h_x^0 = -\partial_x^2 + \mu^2 q_x^2 - \mu = (-\partial_x + \mu q_x)(\partial_x + \mu q_x)$$

and

$$h_\Lambda = \sum_{x\in\Lambda} h_x, \qquad h_\Lambda^0 = \sum_{x\in\Lambda} h_x^0,$$

then

$$h_x = T_x(\phi) h_x^0 T_x^{-1}(\phi), \qquad h_\Lambda = T_\Lambda h_\Lambda^0 T_\Lambda^{-1}.$$

Here we took into account that differentiation commutes with T_Λ. This means that the Hamiltonian $\tilde{H}_\Lambda = T_\Lambda^{-1} H_\Lambda T_\Lambda = h_\Lambda^0 + \tilde{V}_\Lambda$ is decomposed into the sum of the free oscillator Hamiltonian and a pure spin XZ-type Hamiltonian $\tilde{V}_\Lambda$ if J_A, V_Λ do not depend on oscillator variables.

Our Hamiltonian is rewritten as follows

$$H_\Lambda = h_\Lambda + \sum_{A\subseteq\Lambda} J_A P_{[A]}, \qquad P_A = S_{[A]}^1 - e^{-\frac{1}{2}W_A(S_\Lambda^3)}.$$

The remarkable fact is that the symmetric operator P_A and the harmonic operator h_x have both common eigenvector Ψ_Λ with the zero eigenvalue (see Remark 3 in the end of the paper). Note that the space of ground states of the operator h_x (eigenfunctions with the zero eigenvalue) is $2^{|\Lambda|}$-fold degenerate since S_x^3 is diagonal and the Laplacian is translation invariant. From (1.5) and the definition of $T_x(\phi)$ if follows that $T_x^{-1}(\phi)\Psi_\Lambda$ is equal to $\psi_0(q_x)$ multiplied by a function independent of q_x

$$T_x^{-1}(\phi)\Psi_\Lambda = \sum_{s_\Lambda} e^{-\frac{1}{2}U_*(s_\Lambda)} \psi_\Lambda^0(s_\Lambda)\psi_{0\Lambda}(q_{\Lambda\setminus x} + \eta\phi_{\Lambda\setminus x}(s_{\Lambda\setminus x}))\psi_0(q_x).$$

Hence $h_x^0 T_x^{-1}(\phi)\Psi_\Lambda = 0$ and $h_x\Psi_\Lambda = T_x(\phi)h_x^0 T_x^{-1}(\phi)\Psi_\Lambda = 0$. The proof that Ψ_Λ is an eigenvector with the zero eigenvalue of P_A is inspired by our previous paper [10]. For simplicity we will omit q_Λ in the expression for U in (1.4). Taking into consideration the equalities

$$S_{[A]}^1 \psi_\Lambda^0(s_\Lambda) = \psi_\Lambda^0(s_\Lambda^A) = \psi_\Lambda^0(s_{\Lambda\setminus A}, -s_A), \qquad S_x^3 \psi_\Lambda^0(s_\Lambda) = s_x \psi_\Lambda^0(s_\Lambda),$$

we obtain

$$\begin{aligned}(\psi_{0\Lambda})^{-1} P_A \Psi_\Lambda &= \sum_{s_\Lambda} (\psi_\Lambda^0(s_{\Lambda\setminus A}, -s_A) - e^{-\frac{1}{2}W_A(s_\Lambda)}\psi_\Lambda^0(s_\Lambda)) e^{-\frac{1}{2}U(s_\Lambda)} \\ &= \sum_{s_\Lambda} (\psi_\Lambda^0(s_{\Lambda\setminus A}, -s_A) e^{-\frac{1}{2}U(s_\Lambda)} - \psi_\Lambda^0(s_\Lambda) e^{-\frac{1}{2}U(s_\Lambda^A)}) \\ &= \sum_{s_\Lambda} (e^{-\frac{1}{2}U(s_\Lambda^A)} - e^{-\frac{1}{2}U(s_\Lambda^A)})\psi_\Lambda^0(s_\Lambda) = 0.\end{aligned}$$

Here we changed signs of the spin variables s_A in the first term in the sum in s_Λ.

Positive definiteness of the Hamiltonian follows from the following proposition.

Proposition 5. *The operator $-P_A$ is positive definite on $D(V_\Lambda)$.*

Proof. V_Λ is an operator of multiplication by infinite differentiable functions on each one-dimensional spin subspace and its domain contains $(\otimes\mathbb{C}^2)^{|\Lambda|} \otimes C_0^\infty(\mathbb{R}^{|\Lambda|})$. This domain coincides with the direct sum of $2^{|\Lambda|}$ copies of of $L^2(\mathbb{R}^{|\Lambda|}, e^{|q|_0} dq_\Lambda)$, where $|q|_0 = \sum_{x\in\Lambda} |q_x||\phi_x|$.

The scalar product in the Hilbert space $\mathbb{L}^2_\Lambda$ is given by $(F_1, F_2) = \int (F_1(q_\Lambda), F_2(q_\Lambda))\, dq_\Lambda$, where the integration is performed over $\mathbb{R}^{|\Lambda|}$. We have to show that $-(P_A F(q_\Lambda), F(q_\Lambda))_0 \geq 0$. Let us define the operator

$$P_A^+ = e^{\frac{1}{2}U(S^3_\Lambda)} P_A e^{-\frac{1}{2}U(S^3_\Lambda)}.$$

It is not difficult to check on the basis ψ^0_Λ that

$$P_A^+ = e^{-\frac{1}{2}W_A(S^3_\Lambda)}(S^1_{[A]} - I),$$

where I is the unit operator. Here we used the following equality

$$e^{-\frac{1}{2}U(S^{3A}_\Lambda)} S^1_{[A]} = S^1_{[A]} e^{-\frac{1}{2}U(S^3_\Lambda)}.$$

For the operator P_A^+ we have

$$P_A^+ F = \sum_{s_\Lambda} (P_A^+ F)(q_\Lambda; s_\Lambda) \psi^0_\Lambda(s_\Lambda)$$

and

$$(P_A^+ F)(q_\Lambda; s_\Lambda) = -e^{-\frac{1}{2}W_A(s_\Lambda)}(F(q_\Lambda; s_\Lambda) - F(q_\Lambda; s^A_\Lambda)).$$

It is convenient to introduce the new scalar product

$$\begin{aligned}(F_1, F_2)_U &= (e^{-U(S^3_\Lambda)} F_1, F_2) = \sum_{s_\Lambda} \int F_1(q_\Lambda, s_\Lambda) F_2(q_\Lambda, s_\Lambda) e^{-U(s_\Lambda)} dq_\Lambda \\ &= \int (F_1(q_\Lambda), F_2(q_\Lambda))^0_U dq_\Lambda = \int (e^{-U(S^3_\Lambda)} F_1(q_\Lambda), F_2(q_\Lambda))_0 dq_\Lambda.\end{aligned}$$

The operator P_A^+ is symmetric with respect to the new scalar product since

$$(P_A^+ F_1(q_\Lambda), F_2(q_\Lambda))^0_U = (P_A e^{-\frac{1}{2}U(S^3_\Lambda)} F_1(q_\Lambda), e^{-\frac{1}{2}U(S^3_\Lambda)} F_2(q_\Lambda))_0. \tag{3.1}$$

It is not difficult to check that

$$\begin{aligned}-(P_A^+ F(q_\Lambda), F(q_\Lambda))^0_U &= \sum_{s_\Lambda} e^{-\frac{1}{2}[U(s_\Lambda)+U(s^A_\Lambda)]}(F(q_\Lambda; s_\Lambda) - F(q_\Lambda; s^A_\Lambda))F(q_\Lambda; s_\Lambda) \\ &= \frac{1}{2}\sum_{s_\Lambda} e^{-\frac{1}{2}[U(s_\Lambda)+U(s^A_\Lambda)]}(F(q_\Lambda; s_\Lambda) - F(q_\Lambda; s^A_\Lambda))^2 \geq 0.\end{aligned} \tag{3.2}$$

Here we took into account that the function under the sign of exponent is invariant under changing signs of the spin variables s_A. From (3.1), (3.2) it follows that $-P_A \geq 0$. ■

From Proposition 5 it follows that $H^1_\Lambda = \sum_{A \subseteq \Lambda} J_A P_A$ is positive definite since for the negative J_A we have

$$(H^1_\Lambda F, F) = \sum_{A \subseteq \Lambda} \int J_A(q_\Lambda)(P_A F(q_\Lambda), F(q_\Lambda))_0 dq_\Lambda \geq 0.$$

The fact that the Hamiltonian is essentially self-adjoint is derived from the following proposition (see example X.9.3 in [18]).

Proposition 6. *The operator $-\Delta+\mu^2x^2+V+(y,x)$, where (y,x) is the Euclidean scalar product of $x=x_j$, $j=1,\ldots,n$ with the constant vector y, $\Delta=\sum_{j=1}^n\frac{\partial}{\partial x_j}$, $\mu\neq 0$ if $y\neq 0$, V is the operator of multiplication by a non-negative function $V(x)\in L^2(\mathbb{R}^n,e^{-x^2}dx)$, $x^2=(x,x)$, is essentially self-adjoint on $\mathbb{S}(\mathbb{R}^n)\cap D(V)$, i.e. the intersection of the Schwartz space and the domain of V.*

Proof. The following two inequalities are valid

$$\mu^2||x^2\psi||\leq||(-\Delta+V+\mu^2x^2)\psi||+2\mu^2n||\psi||,\qquad (y,x)\leq\epsilon x^2+\frac{n}{4\epsilon}|y|^2, \tag{3.3}$$

where μ is a real number, $|y|=\max_j|y|_j$, $||\cdot||$ is the scalar product in the Hilbert space of square integrable functions. We tacitly assume that x^2 means the operator of multiplication of a squared variable. The first inequality follows from the inequalities ($\partial_j=\frac{\partial}{\partial x_j}$)

$$\begin{aligned}&(-\Delta+V+\mu^2x^2)^2\geq(\mu^2x^2)^2-\mu^2(\Delta x^2+x^2\Delta),\\&-(\partial_j^2x_k^2+x_k^2\partial_j^2)=-(2x_k\partial_j^2x_k+2\delta_{j,k})\geq-2\delta_{j,k},\\&-(\Delta x^2+x^2\Delta)\geq-\sum_{j=1}^n(\partial_j^2x_j^2+x_j^2\partial_j^2).\end{aligned}$$

Here we took into account that V is positive and $[\partial_j,x_k]=\partial_jx_k-x_k\partial_j=\delta_{j,k}$. As a result

$$||(y,x)\psi||\leq a||(-\Delta+V+\mu^2x^2)\psi||+b||\psi||,\qquad a=\mu^{-2}\epsilon,\qquad b=n(2+\frac{|y|^2}{4\epsilon}).$$

For $\mu\neq 0$ number a can be arbitrary small and from the Kato–Rellich theorem [18] it follows that the essential domain of $-\Delta+V+\mu^2x^2+(y,x)$ coincides with the essential domain of $-\Delta+V+\mu^2x^2$. But the last one coincides with $\mathbb{S}(\mathbb{R}^n)\cap D(V)$ [18, Theorem X.59]. Inequality (3.3) is sufficient, also, for the proof of the proposition in the case $\mu=y=0$ (this is explained in Example X.3 in [18]). ■

Since S_Λ^3 is a diagonal operator on $(\otimes\mathbb{C}^2)^{|\Lambda|}$ the operator

$$-\sum_{x\in\Lambda}\partial_x^2+\mu^2\sum_{x\in\Lambda}q_x^2+V_\Lambda+2\eta\mu^2\sum_{x\in\Lambda}q_x\phi_x(S_\Lambda^3)$$

is the direct sum of $2^{|\Lambda|}$ copies of the minus $|\Lambda|$-dimensional Laplacian plus the three functions coinciding with the three functions in Proposition 6. From Proposition 6 it follows that this operator is essentially self-adjoint on the set $(\otimes\mathbb{C}^2)^{|\Lambda|}\otimes\mathbb{S}(\mathbb{R}^{|\Lambda|})\cap D(V_\Lambda)$ that contains $(\otimes\mathbb{C}^2)^{|\Lambda|}\otimes\mathbb{C}_0^\infty(\mathbb{R}^{|\Lambda|})$. The same is true for the Hamiltonian since operator $(\eta\mu)^2\sum_{x\in\Lambda}\phi_x^2(S_\Lambda^3)$ and the operator depending on S_Λ^1 in its expression are bounded.

The operators h_Λ, H_Λ^1 are positive definite on the dense set $(\otimes\mathbb{C}^2)^{|\Lambda|}\otimes\mathbb{S}(\mathbb{R}^{|\Lambda|})\cap D(V_\Lambda)$ which is the essential set for H_Λ. This implies that H_Λ is positive definite on its domain $D(H_\Lambda)$ and Ψ_Λ is its ground state.

Proof of uniqueness. We have to establish that the symmetric semigroup P_Λ^t, generated by $-H_\Lambda$, maps non-negative functions into (strictly) positive functions (increases positivity) and this will imply that the ground state is unique (see Theorem XIII.44 in [18]). We will establish this property with the help of a perturbation expansion. The kernel of the semigroup P_1^t, generated by $h_\Lambda+V_\Lambda$, is expressed in terms of the Feynman–Kac (FK) formula [18, 19]

$$P_1^t(q_\Lambda,s_\Lambda;q'_\Lambda,s'_\Lambda)=\delta_{s_\Lambda,s'_\Lambda}\int P^t_{q_\Lambda,q'_\Lambda}(dw_\Lambda)\exp\left\{-\int_0^t V_\Lambda^+(w_\Lambda(\tau),s_\Lambda)d\tau\right\},$$

where $V_\Lambda^+(q_\Lambda;s_\Lambda) = V_\Lambda(q_\Lambda;s_\Lambda) + \sum_{x\in\Lambda}[\mu^2(q_x+\eta\phi_x(s_\Lambda))^2 - \mu]$, $P^t_{q_\Lambda,q'_\Lambda}(dw_\Lambda) = \prod_{x\in\Lambda} P^t_{q_x,q'_x}(dw_x)$ is the conditional Wiener measure and $w_\Lambda(t)$ is the sequence of continuous paths. The semigroup P^t is represented as a perturbation series in powers of V_0

$$V_0 = -\sum_{A\subseteq\Lambda} J_A S^1_{[A]}.$$

This series is convergent in the uniform operator norm [20] since V_0 is a bounded operator. Its perturbation expansion is given by

$$P^t = \sum_{n\geq 0} P^t_n, \qquad P^t_n = \int\limits_{0\leq\tau_1\leq\tau_2\leq\cdots\leq\tau_n\leq t} d\tau_1\cdots d\tau_n P_1^{\tau_1}\prod_{j=2}^{n+1}(V_0 P_1^{\tau_j-\tau_{j-1}}),$$

where $\tau_{n+1} = t$. We now use the following simple inequality

$$\int_0^t |V_\Lambda^+(w_\Lambda(\tau), s_\Lambda)| d\tau$$
$$\leq \bar{V}(w_\Lambda) = |J|_\Lambda \int_0^t \left[\exp\left\{\alpha\bar{U}_0 + 2\eta\mu\sum_{x\in\Lambda}\bar{\phi}_x|w_x(\tau)|\right\} + \sum_{x\in\Lambda}(\mu^2(|w_x(\tau)| + \eta\bar{\phi}_x)^2 - \mu)\right] d\tau,$$

where $\bar{U}_0 = \max_{s_\Lambda} U_0(s_\Lambda)$, $|J|_\Lambda = \sup_{q_\Lambda} \sum_{A\subseteq\Lambda} |J_A|$, $\bar{\phi}_x = \max_{s_\Lambda}|\phi_x(s_\Lambda)|$. Let

$$\bar{P}_1^t(q_\Lambda;q_\Lambda) = \int P^t_{q_\Lambda,q'_\Lambda}(dw_\Lambda)e^{-\bar{V}(w_\Lambda)}, \qquad V_- = -J_-\sum_{x\in\Lambda} S^1_x$$

then it follows from the positivity of the the kernel P_1^t and V_0 that

$$P^t(q_\Lambda, s_\Lambda; q'_\Lambda, s'_\Lambda) \geq \bar{P}_1^t(q_\Lambda;q_\Lambda)\sum_{n\geq 0}\frac{t^n}{n!}V_-^n(s_\Lambda;s'_\Lambda). \tag{3.4}$$

Here we utilized the semigroup property of $\bar{P}^t$, the inequalities $e^a \geq e^{-|a|}$, $V_0 \geq V_-$,

$$(V_0 P_1^{\tau_j-\tau_{j-1}})(q_\Lambda, s_\Lambda; q'_\Lambda, s'_\Lambda) \geq \bar{P}^{\tau_j-\tau_{j-1}}(q_\Lambda;q'_\Lambda)V_-(s_\Lambda;,s'_\Lambda).$$

Now, it is easily proved as in [10] that the matrix V_- is irreducible. As a result there exists a positive integer n such that and that V_-^n, has positive non-diagonal elements [21, 22]. Hence the kernel in the left-hand side of (3.4) is positive.

4 Order parameters

In this section we will prove Theorem 2, i.e. occurrence of different types of lro in the considered quantum systems. In the proof we will rely on the following basic theorem.

Theorem 3. *Let the ferromagnetic short-range potential energy U of the classical spin-*1 *Ising model on the hyper-cubic lattice $\mathbb{Z}^d$ with the partition function $Z_\Lambda = \sum_{s_\Lambda}\exp\{-\beta U(s_\Lambda)\}$ be given by*

$$U(s_\Lambda) = -\sum_{A\subseteq\Lambda}\varphi(A)s_{[A]}, \tag{4.1}$$

where $\varphi(A) \geq 0$. Let, also, the uniform bound $\varphi(x,y) \geq \bar{\varphi} > 0$ for nearest neighbors x, y hold and $\varphi(A) = 0$ for A with odd number of sites. Then for a sufficiently large $\bar{\varphi}\beta > 1$ and the dimension $d \geq 2$ there is the ferromagnetic lro, that is, for the Gibbsian two point spin average the uniform in Λ bound holds

$$\langle \sigma_x \sigma_y \rangle_\Lambda > 0,$$

where $\sigma_x(s_\Lambda) = s_x$ and the magnetization (an order parameter) $M_\Lambda = |\Lambda|^{-1} \sum_{x\in\Lambda} s_x$ is non-zero in the thermodynamic limit $\Lambda \to \mathbb{Z}^d$.

The proof of this theorem is based on an application of the generalized Peierls principle (argument). It will be given in the end of this section (see, also, [17, 26]. The next theorem is the consequence of the basic theorem.

Proof of item I of Theorem 2. Condition (1.2) shows that

$$\sum_{x\in\Lambda} \phi_x^2(s_\Lambda) = |\Lambda| + 2\sum_{A\subseteq\Lambda} J_2(A)s_{[A]} + \sum_{x\in\Lambda}\left(\sum_{x\notin A\subseteq\Lambda} J_0(x;A)s_{[A]}\right)^2.$$

where $J_2(x\cup A) = J_0(x;A)$ and $J_2(A) = 0$ for odd $|A|$. The last term is equal to

$$\sum_{x\in\Lambda}\sum_{x\notin A\subseteq\Lambda} J_0^2(x;A) + 2\sum_{x\in\Lambda}\sum_{x\notin A_1,A_2\subseteq\Lambda} J_0(x;A_1)J_0(x;A_2)s_{[A_1\Delta A_2]},$$

where $s_\varnothing = 1$ and $A_1\Delta A_2 = (A_1\cup A_2)\backslash(A_1\cap A_2)$. Due to translation invariance of interaction the first term is bounded by $|\Lambda|\sum_A J_0^2(0;A)$, where the summation is performed over $\mathbb{Z}^d$, and this expression is finite since the interaction is short-range. Hence subtracting a finite constant proportional to $|\Lambda|$ from U_* one sees that the result admits representation (4.1) with positive J_* instead of φ such that $J_*(A) = 0$ for odd $|A|$ and for nearest neighbors x, y the inequality $J_*(x,y) \geq (2\eta^2\mu + \alpha)\bar{J}$. The first Griffiths inequality and (1.2) with $J_0 \geq 0$ imply that

$$\eta^{-2}\langle \hat{q}_x\hat{q}_y\rangle_\Lambda = \langle \phi_x(\sigma_\Lambda)\phi_y(\sigma_\Lambda))\rangle_{*\Lambda} \geq \langle\sigma_x\sigma_y\rangle_{*\Lambda},$$

where $\sigma_x(s_\Lambda) = s_x$. It is obvious that the following equality also holds

$$\langle S_x^3 S_y^3\rangle_\Lambda = \langle\sigma_x\sigma_y\rangle_{*\Lambda}.$$

The basic theorem and Proposition 1 imply occurrence of ferromagnetic lro in S^3 and oscillator lro. ■

Proof of item II of Theorem 2. Let $\psi_\Lambda(q_\Lambda; s_\Lambda) = \psi_\Lambda^0(s_\Lambda)\psi_{0\Lambda}(q_\Lambda)$. Then (1.3) and the definition of S^1 imply that

$$S_x^1 S_y^1 \Psi_\Lambda(q_\Lambda) = \sum_{s_\Lambda} e^{-\frac{1}{2}U(s_\Lambda;q_\Lambda)}\psi_\Lambda(q_\Lambda; s_\Lambda^{x,y}) = \sum_{s_\Lambda} e^{-\frac{1}{2}U(s_\Lambda^{x,y};q_\Lambda)}\psi_\Lambda(q_\Lambda; s_\Lambda).$$

Taking into account also the orthonormality of the basis one obtains

$$\begin{aligned}\langle S_x^1 S_y^1\rangle_\Lambda &= Z_\Lambda^{-1}\sum_{s_\Lambda}\int e^{-\frac{1}{2}[U(s_\Lambda^{x,y};q_\Lambda)+U(s_\Lambda;q_\Lambda)]}\psi_{0\Lambda}^2(q_\Lambda)dq_\Lambda \\ &= Z_\Lambda^{-1}\sum_{s_\Lambda} e^{\frac{\eta^2\mu}{4}\sum_{x'\in\Lambda}(\phi_{x'}(s_\Lambda)+\phi_{x'}(s_\Lambda^{x,y}))^2} e^{-\frac{\alpha}{2}[U_0(s_\Lambda^{x,y})+U_0(s_\Lambda)]}\end{aligned}$$

$$\geq e^{-\frac{\alpha}{2}B_0} Z_\Lambda^{-1} \sum_{s_\Lambda} e^{\frac{\eta^2\mu}{4}\sum\limits_{x'\in\Lambda}(\phi_{x'}(s_\Lambda)+\phi_{x'}(s_\Lambda^{x,y}))^2} e^{-\alpha U_0(s_\Lambda)}.$$

We also have

$$\begin{aligned}
&\sum_{x'\in\Lambda}(\phi_{x'}(s_\Lambda)+\phi_{x'}(s_\Lambda^{x,y}))^2 \\
&\quad= \sum_{x'\in\Lambda}[3\phi^2_{x'}(s_\Lambda)+\phi^2_{x'}(s_\Lambda^{x,y})] + 2\sum_{x'\in\Lambda}\phi_{x'}(s_\Lambda)[-\phi_{x'}(s_\Lambda)+\phi_{x'}(s_\Lambda^{x,y})] \\
&\quad= \sum_{x'\in\Lambda}4\phi^2_{x'}(s_\Lambda) + \sum_{x'\in\Lambda}[-\phi^2_{x'}(s_\Lambda)+\phi^2_{x'}(s_\Lambda^{x,y})] + 2\sum_{x'\in\Lambda}\phi_{x'}(s_\Lambda)[-\phi_{x'}(s_\Lambda)+\phi_{x'}(s_\Lambda^{x,y})] \\
&\quad\geq 4\sum_{x'\in\Lambda}\phi^2_{x'}(s_\Lambda) - B_2 - 2CB_1.
\end{aligned}$$

This yields

$$\begin{aligned}
\langle S_x^1 S_y^1\rangle_\Lambda &\geq e^{-\frac{\eta^2\mu}{4}(B_2+2B_1)} e^{-\frac{\alpha}{2}B_0} Z_\Lambda^{-1}\sum_{s_\Lambda} e^{\eta^2\mu\sum\limits_{x'\in\Lambda}\phi^2_{x'}(s_\Lambda)} e^{-\alpha U_0(s_\Lambda)} \\
&= e^{-\frac{\eta^2\mu}{4}(B_2+2CB_1)} e^{-\frac{\alpha}{2}B_0}. \qquad \blacksquare
\end{aligned}$$

Proof of Proposition 4. It is obvious that $|\phi_x(s_\Lambda)| \leq ||J_0||_1$ and that

$$|\phi_{x'}(s_\Lambda^{x,y}) - \phi_{x'}(s_\Lambda)| = |-2[s_y J_0(y-x') + s_x J_0(x-x')]| \leq 2[|J_0(y-x')| + |J_0(x-x')|].$$

As a result $W_{x,y}^{(1)}(s_\Lambda) \leq 4||J_0||_1$. Further

$$\begin{aligned}
|\phi^2_{x'}(s_\Lambda^{x,y}) - \phi^2_{x'}(s_\Lambda)| &= \Bigg|\Bigg[\sum_{z\in\Lambda\backslash(x,y)} J_0(z-x')s_z - s_y J_0(y-x') - s_x J_0(x-x')\Bigg]^2 \\
&\quad - \Bigg[\sum_{z\in\Lambda\backslash(x,y)} J_0(z-x')s_z + s_y J_0(y-x') + s_x J_0(x-x')\Bigg]^2\Bigg| \\
&= \Bigg| -4\sum_{z\in\Lambda\backslash(x,y)} J_0(z-x')s_z(s_y J_0(y-x') + s_x J_0(x-x'))\Bigg| \\
&\leq 4\sum_{z\in\Lambda}|J_0(z-x')|(|J_0(y-x')| + |J_0(x-x')|) \leq 4||J_0||_1(|J_0(y-x')| + |J_0(x-x')|).
\end{aligned}$$

Hence $W_{x,y}^{(2)}(s_\Lambda) \leq 8||J_0||_1^2$. $\blacksquare$

Proof of Theorem 3. Let $\chi_x^\pm = \frac{1}{2}(1\pm\sigma_x)$ then one obtains

$$4\langle\chi_x^+\chi_y^-\rangle_\Lambda = 1 + \langle\sigma_x\rangle_\Lambda - \langle\sigma_y\rangle_\Lambda - \langle\sigma_x\sigma_y\rangle_\Lambda.$$

Since the systems are invariant under the transformation of changing signs of spins the third and the second terms in the right-hand side of last equality are equal to zero and

$$\langle\sigma_x\sigma_y\rangle_\Lambda = 1 - 4\langle\chi_x^+\chi_y^-\rangle_\Lambda.$$

Hence if

$$\langle\chi_x^+\chi_y^-\rangle_\Lambda < \frac{1}{4} \tag{4.2}$$

then the ferromagnetic lro occurs, i.e.

$$\langle \sigma_x \sigma_y \rangle_\Lambda \geq a > 0, \tag{4.3}$$

where a is independent of Λ. If one succeeds in proving that there exists a positive function $E_0(\beta)$ and positive constants a, a' independent of Λ such that

$$\langle \chi_x^+ \chi_y^- \rangle_\Lambda \leq a' e^{E_0(\beta)} \tag{4.4}$$

and proves that E_0 is increasing at infinity then (4.2), (4.3) will hold for a sufficiently large inverse temperature β. The Peierls principle reduces the derivation of (4.4) to the derivation of the contour bound. ■

Peierls principle. *Let the contour bound hold*

$$\langle \prod_{\langle x,y\rangle \in \Gamma} \chi_x^+ \chi_y^- \rangle_\Lambda \leq e^{-|\Gamma| E}, \tag{4.5}$$

where $\langle \cdot \rangle_\Lambda$ denotes the Gibbs average for the spin system confined to a compact domain Λ, Γ is a set of the nearest neighbors, adjacent to the (connected) contour, i.e. a boundary of the connected set of unit hypercubes centered at lattice sites. Then (4.4) is valid with $E_0 = a''E$, where a'' is a positive constant independent of Λ.

Proof of contour bound. Bricmont and Fontain derived the contour bound for the spin systems with the potential energy (4.1) with the help of the second Griffiths [23] and Jensen inequalities [24] (see also [25, 26])

$$\langle \sigma_{[A]} \sigma_{[B]} \rangle_{\Lambda[\Gamma]} - \langle \sigma_{[A]} \rangle_{\Lambda[\Gamma]} \langle \sigma_{[B]} \rangle_{\Lambda[\Gamma]} \geq 0, \qquad \int e^f d\mu \geq \exp\left\{ \int f d\mu \right\},$$

where $d\mu$ is a probability measure on a measurable space. Their proof starts form the inequality

$$\chi_x^+ \chi_y^- = e^{-\frac{\beta}{2}\sigma_x\sigma_y} e^{\frac{\beta}{2}\sigma_x\sigma_y} \chi_x^+ \chi_y^- \leq e^{-\frac{\beta}{2}\sigma_x\sigma_y} \chi_x^+ \chi_y^- \leq e^{-\frac{\beta}{2}\sigma_x\sigma_y}.$$

As a result ($\beta' = \bar{\varphi}\beta$)

$$\langle \prod_{\langle x,y\rangle \in \Gamma} \chi_x^+ \chi_y^- \rangle_\Lambda \leq \langle e^{-\frac{\beta'}{2} \sum\limits_{\langle x,y\rangle \in \Gamma} \sigma_x \sigma_y} \rangle_\Lambda = \langle e^{\frac{\beta'}{2} \sum\limits_{\langle x,y\rangle \in \Gamma} \sigma_x \sigma_y} \rangle^{-1}_{\Lambda[\Gamma]}$$

$$\leq e^{-\frac{\beta'}{2} \sum\limits_{\langle x,y\rangle \in \Gamma} \langle \sigma_x \sigma_y \rangle_{\Lambda[\Gamma]}} = e^{-E_\Gamma},$$

where $\langle \cdot, \cdot \rangle_{\Lambda[\Gamma]}$ is the average corresponding to the potential energy

$$U_\Gamma(q_\Lambda) = U(s_\Lambda) + \frac{\bar{\varphi}}{2} \sum_{\langle x,y\rangle \in \Gamma} s_x s_y.$$

In the last line we applied the Jensen inequality. From the second Griffiths inequality it follows that the average $\langle \sigma_x \sigma_y \rangle_{\Lambda[\Gamma]}$ is a monotone increasing function in φ_A. So, in the potential energy determining this average we can put $\varphi_A = 0$, except $A = \langle x, y \rangle$ and leave the coefficient $\bar{\varphi}$ in front of the bilinear nearest-neighbor pair potential in (4.1) without increasing the average. This leads to

$$\langle \sigma_x \sigma_y \rangle_{\Lambda[\Gamma]} \geq \langle \sigma \sigma' \rangle = Z_2^{-1}\left(\frac{\beta'}{2}\right) \sum_{s_1, s_2 = \pm 1} s_1 s_2 e^{\frac{\beta'}{2} s_1 s_2}, \qquad Z_2(\beta) = \sum_{s_1, s_2 = \pm 1} e^{\frac{\beta'}{2} s_1 s_2}.$$

That is,

$$E_\Gamma \geq |\Gamma| E, \qquad E = 2^{-1}\beta'\langle\sigma\sigma'\rangle$$

or

$$E = \beta(e^{2^{-1}\beta'} - e^{-2^{-1}\beta'})(e^{2^{-1}\beta'} + e^{-2^{-1}\beta'})^{-1} \geq 2^{-1}\beta'(1 - e^{-\beta'}).$$

Here we used in the denominator the inequality $e^{-2^{-1}\beta'} \leq e^{2^{-1}\beta'}$. Obviously, E tends to infinity if β' tends to infinity. This implies (4.5). ■

5 Discussion

We showed that in the considered lattice spin-boson models with $J_A \leq 0$ ground states are Gibbsian and the ground state averages for special observables are reduced to averages in classical Ising models. This means that existence of ground states order parameters is connected with existence of order parameters in the associated Ising models and that a breakdown of symmetries in the quantum systems is determined by a breakdown of symmetries in Ising models. We considered the free boundary conditions implying that for the cases of the perturbation V_Λ, considered in the two theorems, the ground state averages of q_x, S_x^3 are zero, that is $\langle\hat{q}_x\rangle_\Lambda = 0$, $\langle S_x^3\rangle_\Lambda = 0$ if the associated Ising potential energy is an even function. In order to make such the averages non-zero (explicit symmetry breaking) one has to introduce special boundary conditions (quasi-averages) which have to single out pure Gibbsian states in the associated Ising models. It is known [28] that for the two-dimensional ferromagnetic Ising nearest-neighbor model there are two boundary conditions which generate pure states and that every other state is a convex linear combination of these two states. A discussion of a construction of ground states in lattice spin and fermion quantum systems with an explicit symmetry breaking a reader may find in [29].

Remark 2. Translation invariance means that

$$J_{x_1,\dots,x_n} = J_{0,x_2-x_1,\dots,x_n-x_1}, \qquad J_0(x;x_1,\dots,x_n) = J_0(0;x_1-x,\dots,x_n-x)$$

where J, J_0 are symmetric functions. The short-range character of interaction means that

$$\max_x \sum_A |J_{x,A}| < \infty, \qquad \max_x \sum_A |J_0(x;A)| < \infty.$$

Remark 3. If only one-point sets are left in the sum for V_Λ then the expression for H_Λ can be rewritten in the following way

$$H_\Lambda = \sum_{x\in\Lambda} H_x.$$

The property of the ground state Ψ_Λ to be a ground state with the zero eigenvalue of a local Hamiltonian H_x was found earlier for special isotropic anti-ferromagnetic Heisenberg chains with valence bond ground state in [27].

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