Vector-Valued Polynomials and a Matrix Weight Function with B_2-Action. II

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Abstract. This is a sequel to [*SIGMA* **9** (2013), 007, 23 pages], in which there is a construction of a 2×2 positive-definite matrix function $K(x)$ on $\mathbb{R}^2$. The entries of $K(x)$ are expressed in terms of hypergeometric functions. This matrix is used in the formula for a Gaussian inner product related to the standard module of the rational Cherednik algebra for the group $W(B_2)$ (symmetry group of the square) associated to the (2-dimensional) reflection representation. The algebra has two parameters: k_0, k_1. In the previous paper K is determined up to a scalar, namely, the normalization constant. The conjecture stated there is proven in this note. An asymptotic formula for a sum of ${}_3F_2$-type is derived and used for the proof.

Key words: matrix Gaussian weight function

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1 Introduction

This is a sequel to [1] and the definitions and notations from that paper are used here. Briefly, we constructed a 2×2 positive-definite matrix function $K(x)$ on $\mathbb{R}^2$ whose entries are expressed in terms of hypergeometric functions. This matrix is used in the formula for a Gaussian inner product related to the standard module of the rational Cherednik algebra for the group $W(B_2)$ (symmetry group of the square) associated to the (2-dimensional) reflection representation. The algebra has two parameters: k_0, k_1. In [1] K is determined up to the normalization constant, henceforth denoted by $c(k_0, k_1)$. The conjecture stated there is proven in this note.

Instead of trying to integrate K directly (a problem involving squares of hypergeometric functions whose argument is x_2^2/x_1^2) we compute a sequence of integrals in two ways: asymptotically and exactly in terms of sums. Comparing the two answers will determine the value of $c(k_0, k_1)$. First the problem is reduced to a one-variable integral over the sector $\left\{(\cos\theta, \sin\theta) : 0 < \theta < \frac{\pi}{4}\right\}$ of the unit circle. With detailed information about the behavior of a function $f(\theta)$ near $\theta = 0$ one can find an asymptotic value of $\int_0^{\pi/4} \theta^n f(\theta) d\theta$. This part of the argument is described in Section 2. The other part is produced by exploiting the relationship between the Laplacian and integration over the circle. That is, the plan is to determine the result of applying appropriate powers of the Laplacian to certain polynomials behaving like θ^n near $\theta = 0$. This will be done by establishing recurrence relations; their proofs are in Section 4. The main theorem and its proof which combines the various ingredients are contained in Section 3.

Recall from [1, p. 18] that for $0 < x_2 < x_1$ and $u = \frac{x_2}{x_1}$

$$L(u)_{11} = |u|^{k_1} \left(1 - u^2\right)^{-k_0} F\left(-k_0, \frac{1}{2} - k_0 + k_1; k_1 + \frac{1}{2}; u^2\right), \tag{1}$$

$$L(u)_{12} = -\frac{k_0}{k_1+\frac{1}{2}}|u|^{k_1}\left(1-u^2\right)^{-k_0}uF\left(1-k_0,\frac{1}{2}-k_0+k_1;k_1+\frac{3}{2};u^2\right),$$
$$L(u)_{21} = -\frac{k_0}{\frac{1}{2}-k_1}|u|^{-k_1}\left(1-u^2\right)^{-k_0}uF\left(1-k_0,\frac{1}{2}-k_0-k_1;\frac{3}{2}-k_1;u^2\right),$$
$$L(u)_{22} = |u|^{-k_1}\left(1-u^2\right)^{-k_0}F\left(-k_0,\frac{1}{2}-k_0-k_1;\frac{1}{2}-k_1;u^2\right),$$

and [1, p. 20]

$$K(x) = L(u)^T\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}L(u), \tag{2}$$
$$d_1 = c(k_0,k_1)\frac{\Gamma\left(\frac{1}{2}-k_1\right)^2}{\cos\pi k_0\Gamma\left(\frac{1}{2}+k_0-k_1\right)\Gamma\left(\frac{1}{2}-k_0-k_1\right)},$$
$$d_2 = c(k_0,k_1)\frac{\Gamma\left(\frac{1}{2}+k_1\right)^2}{\cos\pi k_0\Gamma\left(\frac{1}{2}+k_0+k_1\right)\Gamma\left(\frac{1}{2}-k_0+k_1\right)}.$$

2 Integrals over the circle

First we reduce the integral to a sector of the unit circle by assuming homogeneity and an invariance property. In general, the integral of a positively homogeneous function with respect to Gaussian measure can be found by integrating over the sphere (circle). Furthermore the integral over the sphere of a polynomial p homogeneous of degree $2n$ can be found by computing $\Delta^n p$. This method applies in the present situation, when replacing Δ by Δ_κ.

As before, elements of $\mathcal{P}_V$ can be expressed as polynomials $f_1(x)t_1 + f_2(x)t_2$ or vectors $(f_1(x), f_2(x))$, as needed. The Gaussian inner product is expressed as

$$\langle f, g\rangle_G = \int_{\mathbb{R}^2} f(x)K(x)g(x)^T e^{-|x|^2/2}dx,$$

and the normalization condition is equivalent to $\langle(1,0),(1,0)\rangle_G = 1$. Recall from [1, p. 2] that the fundamental bilinear form $\langle\cdot,\cdot\rangle_\tau$ on $\mathcal{P}_V$ can be written as

$$\langle f_1(x)t_1 + f_2(x)t_2, g_1(x)t_1 + g_2(x)t_2\rangle_\tau$$
$$= \langle t_1, f_1(\mathcal{D})\{g_1(x)t_1+g_2(x)t_2\}\rangle_\tau|_{x=0} + \langle t_2, f_2(\mathcal{D})\{g_1(x)t_1+g_2(x)t_2\}\rangle_\tau|_{x=0},$$

where $f(\mathcal{D})$ and $|_{x=0}$ denote $f(\mathcal{D}_1,\mathcal{D}_2)$ and evaluation at $x=0$, respectively. The abstract Gaussian inner product is then defined as $\langle f,g\rangle_G = \langle e^{\Delta_\kappa/2}f, e^{\Delta_\kappa/2}g\rangle_\tau$ for $f,g\in\mathcal{P}_V$. The key property of this inner product is $\langle x_i f, g\rangle_G = \langle f, x_i g\rangle_G$ for $i=1,2$. Subsequently we can derive another abstract inner product which in effect acts like the use of spherical polar coordinates in computing integrals with respect to Gaussian measure. This is done in the following (S symbolizes the sphere/circle):

Definition 1. For polynomials $f\in\mathcal{P}_{V,n}, g\in\mathcal{P}_{V,m}$ let $\ell = \frac{m+n}{2}$ and

$$\langle f,g\rangle_S := \frac{1}{2^\ell \ell!}\langle f,g\rangle_G = \frac{1}{2^\ell\ell!}\left\langle e^{\Delta_\kappa/2}f, e^{\Delta_\kappa/2}g\right\rangle_\tau, \qquad m\equiv n \mod 2,$$
$$\langle f,g\rangle_S := 0, \qquad m-n\equiv 1 \mod 2.$$

This is extended to all polynomials by linearity.

Next we relate the Gaussian integral of certain invariant polynomials to the abstract formula for $\langle\cdot,\cdot\rangle_S$. Let $x_\theta := (\cos\theta, \sin\theta)$, a generic point on the unit circle.

Lemma 1. *Suppose $f,g\in\mathcal{P}_V$ are relative invariants of the same type, that is, for some linear character χ of W, $(wf)(x)=f(xw)w^{-1}=\chi(w)f(x)$ and $(wg)(x)=\chi(w)g(x)$ for each $w\in W$, then*

$$\int_{\mathbb{R}^2} f(x)K(x)g(x)^T e^{-|x|^2/2}dx = 8\int_0^\infty e^{-r^2/2}rdr\int_0^{\pi/4} f(rx_\theta)K(x_\theta)g(rx_\theta)^T d\theta.$$

Proof. Let $C_0=\{x:0<x_2<x_1\}$, the fundamental chamber, then

$$\begin{aligned}\int_{\mathbb{R}^2} f(x)K(x)g(x)^T e^{-|x|^2/2}dx &= \sum_{w\in W}\int_{C_0} f(xw)K(xw)g(xw)^T e^{-|x|^2/2}dx\\ &= \sum_{w\in W}\int_{C_0} f(xw)w^{-1}K(x)wg(xw)^T e^{-|x|^2/2}dx\\ &= \sum_{w\in W}\chi(w)^2\int_{C_0} f(x)K(x)g(x)^T e^{-|x|^2/2}dx.\end{aligned}$$

The statement follows from the fact $\chi(w)^2=1$ and the use of polar coordinates. Recall K is positively homogeneous of degree zero. ■

Proposition 1. *If $f,g\in\mathcal{P}_V$ are relative invariants of the same type and $f\in\mathcal{P}_{V,n}, g\in\mathcal{P}_{V,m}$ with $m\equiv n \bmod 2$ and $\ell=\frac{m+n}{2}$ then*

$$8\int_0^{\pi/4} f(x_\theta)K(x_\theta)g(x_\theta)^T d\theta = \frac{1}{2^\ell \ell!}\langle f,g\rangle_G = \langle f,g\rangle_S.$$

If further $n=2q+1$ and $m=1$ then

$$\langle f,g\rangle_S = \frac{1}{2^{2q+1}q!(q+1)!}\left\langle \Delta_\kappa^q f,g\right\rangle_\tau.$$

Proof. From Lemma 1 the factor relating $\langle\cdot,\cdot\rangle_S$ to $\langle\cdot,\cdot\rangle_G$ is $\int_0^\infty e^{-r^2/2}r^{2\ell+1}dr = 2^\ell\ell!$. Now suppose $m=1$ and $\ell=q+1$. By definition $\langle f,g\rangle_S=\frac{\langle e^{\Delta_\kappa/2}f,e^{\Delta_\kappa/2}g\rangle_\tau}{2^{q+1}(q+1)!}$, then $e^{\Delta_\kappa/2}g=g$ and the degree-1 component of $e^{\Delta_\kappa/2}f$ is $\frac{1}{2^q q!}\Delta_\kappa^q f$ (recall that $\langle f,g\rangle_\tau=0$ when f and g have different degrees of homogeneity). ■

We will find exact expressions in the form of sums for $\Delta_\kappa^q f$ for certain polynomials in Section 4.

The idea underlying the asymptotic evaluation is this: suppose that g is continuous on $[0,1]$ and satisfies $|g(t)-g(0)|\le Ct$ for some constant, then $\int_0^1 g(t)(1-t)^n dt=\frac{1}{n+1}g(0)+O\left(\frac{1}{n^2}\right)$. This formula can be adapted to the measure $t^\alpha dt$ with $\alpha>-1$. In the sequel we will use C, C' to denote constants whose values need not be explicit, as in the "big O" symbol.

Let $\phi:=x_1^2-x_2^2$; then ϕ^2 is W-invariant. Furthermore $\phi^{2n}p_{1,2}$ and $\phi^{2n+1}p_{1,4}$ are all relative invariants of the same type, that is, $\sigma_1 f=\sigma_{12}^+ f=-f$ (recall $p_{1,2}=-x_2t_1+x_1t_2$ and $p_{1,4}=-x_2t_1-x_1t_2$). We will evaluate $\langle\phi^{2n}p_{1,2},p_{1,2}\rangle_S$ and $\langle\phi^{2n+1}p_{1,4},p_{1,2}\rangle_S$. These polynomials peak at $\theta=0$ and vanish at $\theta=\frac{\pi}{4}$. The following are used in the expressions for $p_{1,4}Kp_{1,2}^T$ and $p_{1,2}Kp_{1,2}^T$. Set

$$\begin{aligned} h_1(z) &:= F\left(-k_0,\frac{1}{2}-k_0+k_1;\frac{3}{2}+k_1;z\right), \qquad (3)\\ h_2(z) &:= F\left(-k_0,-\frac{1}{2}-k_0-k_1;\frac{1}{2}-k_1;z\right),\\ h_3(z) &:= F\left(k_0,\frac{1}{2}+k_0+k_1;\frac{3}{2}+k_1;z\right),\end{aligned}$$

$$h_4(z) := F\left(k_0, -\frac{1}{2}+k_0-k_1; \frac{1}{2}-k_1; z\right).$$

Each of these hypergeometric series satisfies the criterion for absolute convergence at $z=1$ (for real $F(a,b;c;z)$ the condition is $c-a-b>0$; here $c-a-b=1\pm 2k_0$), and so each satisfies a bound of the form $|h(z)-1|\le Cz$ for $0\le z\le 1$. Recall the coordinate $u=\frac{x_2}{x_1}$; on the unit circle (in $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$)

$$x_\theta = \left(\frac{1}{\sqrt{1+u^2}}, \frac{u}{\sqrt{1+u^2}}\right), \qquad \phi = \frac{1-u^2}{1+u^2}, \qquad d\theta = \frac{du}{1+u^2}.$$

By use of the identities

$$\begin{aligned} &F(a,b;c;z) - \frac{a}{c} zF(a+1,b;c+1,z) = F(a,b-1;c;z), \\ &F(a,b;c;z) - \frac{a}{c}F(a+1,b;c+1;z) = \frac{c-a}{c}F(a,b;c+1;z), \end{aligned} \tag{4}$$

we obtain for $x=x_\theta$, $0<\theta<\frac{\pi}{4}$ and $0<u<1$:

$$\begin{aligned} x_2L_{11} - x_1L_{12} &= \frac{u^{k_1+1}\left(1-u^2\right)^{-k_0}}{(1+u^2)^{1/2}}\frac{1+2k_0+2k_1}{1+2k_1}h_1\left(u^2\right), \\ x_1L_{22} - x_2L_{21} &= \frac{u^{-k_1}\left(1-u^2\right)^{-k_0}}{(1+u^2)^{1/2}}h_2\left(u^2\right); \end{aligned} \tag{5}$$

and

$$\begin{aligned} -x_2L_{11} - x_1L_{12} &= -\frac{u^{k_1+1}\left(1-u^2\right)^{k_0}}{(1+u^2)^{1/2}}\frac{1-2k_0+2k_1}{1+2k_1}h_3\left(u^2\right), \\ -x_2L_{21} - x_1L_{22} &= -\frac{u^{-k_1}\left(1-u^2\right)^{k_0}}{(1+u^2)^{1/2}}h_4\left(u^2\right). \end{aligned} \tag{6}$$

The expressions for L_{ij} (see (1)) appearing in (6) are first transformed with $F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z)$ before using identities (4). These formulae will be used to obtain asymptotic expressions for the integrals $\langle\phi^{2n}p_{1,2},p_{1,2}\rangle_S$ and $\langle\phi^{2n+1}p_{1,4},p_{1,2}\rangle_S$. The notation $a(n)\sim b(n)$ means $\lim\limits_{n\to\infty}\frac{a(n)}{b(n)}=1$.

Lemma 2. *Suppose $\alpha,\gamma>-1$, and $n=2,3,\dots$ then*

$$\int_0^1 t^\alpha(1-t)^{n+\gamma}(1+t)^{\beta-n}dt = (2n)^{-\alpha-1}\Gamma(\alpha+1)\left(1+O\left(\frac{1}{n}\right)\right).$$

Proof. The term $\left(\frac{1-t}{1+t}\right)^n$ is transformed to $(1-v)^n$ by the change of variable $t=\frac{v}{2-v}$. The integral becomes

$$2^{-\alpha-1}\int_0^1 v^\alpha(1-v)^{n+\gamma}\left(1-\frac{v}{2}\right)^{-\alpha-\beta-\gamma-2}dv = 2^{-\alpha-1}\frac{\Gamma(\alpha+1)\Gamma(n+\gamma+1)}{\Gamma(n+\alpha+\gamma+2)} + R,$$

where R is bounded by $2^{-\alpha-1}C\int_0^1 v^{\alpha+1}\left(1-v\right)^{n+\gamma}dv$ and $\left|\left(1-\frac{v}{2}\right)^{-\alpha-\beta-\gamma-2}-1\right|\le Cv$ for $0\le v\le 1$. By Stirling's formula $\frac{\Gamma(n+\gamma+1)}{\Gamma(n+\alpha+\gamma+2)}\sim n^{-\alpha-1}$ and $R\sim C'n^{-\alpha-2}$ for some constant C'. ∎

Corollary 1. *Suppose $g(t)$ is continuous and $|g(t)-g(0)| \leq Ct$ for $0 \leq t \leq 1$ then*

$$\int_0^1 t^\alpha(1-t)^{n+\gamma}(1+t)^{\beta-n}g(t)dt = (2n)^{-\alpha-1}g(0)\Gamma(\alpha+1)\left(1+O\left(\frac{1}{n}\right)\right).$$

Proof. Break up the integrand as

$$t^\alpha(1-t)^{n+\gamma}(1+t)^{\beta-n}\{g(0)+(g(t)-g(0))\};$$

by the Lemma the integral of the second part is bounded by $C(2n)^{-\alpha-2}\Gamma(\alpha+2)\left(1+O\left(\frac{1}{n}\right)\right)$. ■

Proposition 2. *Suppose $-\frac{1}{2} < k_0 \pm k_1 < \frac{1}{2}$ then*

$$\begin{aligned}\left\langle \phi^{2n}p_{1,2}, p_{1,2}\right\rangle_S &= 8\int_0^{\pi/4} \phi(x_\theta)^{2n}p_{1,2}(x_\theta)K(x_\theta)p_{1,2}(x_\theta)^T d\theta \\ &\sim \frac{2\pi c(k_0,k_1)}{\cos\pi k_0 \cos\pi k_1}\frac{2^{2k_1}n^{k_1-1/2}\Gamma\left(\frac{1}{2}+k_1\right)}{\Gamma\left(\frac{1}{2}+k_0+k_1\right)\Gamma\left(\frac{1}{2}-k_0+k_1\right)}.\end{aligned}$$

Proof. By definition

$$\begin{aligned}&\phi(x_\theta)^{2n}p_{1,2}(x_\theta)K(x_\theta)p_{1,2}(x_\theta)^T \\ &\quad = \left(x_1^2-x_2^2\right)^{2n}\left\{d_1(x_1L_{12}-x_2L_{11})^2 + d_2(x_1L_{22}-x_2L_{21})^2\right\}.\end{aligned}$$

Thus equation (5) implies

$$\begin{aligned}&8\int_0^{\pi/4} \phi(x_\theta)^{2n}p_{1,2}(x_\theta)K(x_\theta)p_{1,2}(x_\theta)^T d\theta \\ &\quad = 8d_1\left(\frac{1+2k_0+2k_1}{1+2k_1}\right)^2\int_0^1\left(\frac{1-u^2}{1+u^2}\right)^{2n}\frac{\left(1-u^2\right)^{-2k_0}}{\left(1+u^2\right)^2}u^{2k_1+2}h_1\left(u^2\right)^2 du \\ &\qquad + 8d_2\int_0^1\left(\frac{1-u^2}{1+u^2}\right)^{2n}\frac{\left(1-u^2\right)^{-2k_0}}{\left(1+u^2\right)^2}u^{-2k_1}h_2\left(u^2\right)^2 du,\end{aligned}$$

where h_1 and h_2 are from equation (3). The key fact is that $h_i(0)=1$ and $\left|h_i\left(u^2\right)-1\right| \leq Cu^2$ for $0 \leq u \leq 1$ with some constant C, $i=1,2$. In each integral change the variable $u = v^{1/2}$; the first integral equals

$$\frac{1}{2}(4n)^{-k_1-3/2}\Gamma\left(k_1+\frac{3}{2}\right)\left(1+O\left(\frac{1}{n}\right)\right) \tag{7}$$

and the second integral equals

$$\frac{1}{2}(4n)^{k_1-1/2}\Gamma\left(\frac{1}{2}-k_1\right)\left(1+O\left(\frac{1}{n}\right)\right). \tag{8}$$

This is the dominant term in the sum because $k_1-\frac{1}{2} > -k_1-\frac{3}{2}$ (that is, $2k_1+1>0$). Using the value of d_2 in equation (2) and the identity $\Gamma\left(\frac{1}{2}-k_1\right)\Gamma\left(\frac{1}{2}+k_1\right) = \frac{\pi}{\cos\pi k_1}$ we find

$$\begin{aligned}\left\langle \phi^{2n}p_{1,2}, p_{1,2}\right\rangle_S &\sim 4c(k_0,k_1)(4n)^{k_1-1/2}\frac{\Gamma\left(\frac{1}{2}-k_1\right)\Gamma\left(\frac{1}{2}+k_1\right)^2}{\cos\pi k_0\Gamma\left(\frac{1}{2}+k_0+k_1\right)\Gamma\left(\frac{1}{2}-k_0+k_1\right)} \\ &= \frac{2\pi c(k_0,k_1)}{\cos\pi k_0\cos\pi k_1}\frac{2^{2k_1}n^{k_1-1/2}\Gamma\left(\frac{1}{2}+k_1\right)}{\Gamma\left(\frac{1}{2}+k_0+k_1\right)\Gamma\left(\frac{1}{2}-k_0+k_1\right)}.\end{aligned}$$

■

Proposition 3. *Suppose* $-\frac{1}{2} < k_0 \pm k_1 < \frac{1}{2}$ *then*

$$\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S = 8 \int_0^{\pi/4} \phi(x_\theta)^{2n+1} p_{1,4}(x_\theta) K(x_\theta) p_{1,2}(x_\theta)^T d\theta \\ \sim \frac{-2\pi c(k_0,k_1)}{\cos \pi k_0 \cos \pi k_1} \frac{2^{2k_1} n^{k_1-1/2} \Gamma\left(\frac{1}{2}+k_1\right)}{\Gamma\left(\frac{1}{2}+k_0+k_1\right)\Gamma\left(\frac{1}{2}-k_0+k_1\right)}.$$

Proof. By definition

$$\phi(x_\theta)^{2n+1} p_{1,4}(x_\theta) K(x_\theta) p_{1,2}(x_\theta)^T = -\left(x_1^2 - x_2^2\right)^{2n+1} \\ \times \left\{ d_1 (x_2 L_{11} + x_1 L_{12})(x_2 L_{11} - x_1 L_{12}) + d_2 (x_2 L_{21} + x_1 L_{22})(x_2 L_{21} - x_1 L_{22}) \right\}.$$

Thus equations (5) and (6) imply

$$8 \int_0^{\pi/4} \phi(x_\theta)^{2n+1} p_{1,4}(x_\theta) K(x_\theta) p_{1,2}(x_\theta)^T d\theta = -8 d_1 \frac{(1-2k_0+2k_1)(1+2k_0+2k_1)}{(1+2k_1)^2} \\ \times \int_0^1 \left(\frac{1-u^2}{1+u^2}\right)^{2n+1} \frac{u^{2k_1+2}}{(1+u^2)^2} h_1\left(u^2\right) h_3\left(u^2\right) du \\ - 8 d_2 \int_0^1 \left(\frac{1-u^2}{1+u^2}\right)^{2n+1} \frac{u^{-2k_1}}{(1+u^2)^2} h_2\left(u^2\right) h_4\left(u^2\right) du.$$

Arguing as in the previous proof, one obtains the same expressions (7) and (8) for the first and second integrals respectively. ■

3 Evaluation of the normalizing constant

The proof of the following appears in Section 4. (Recall $(a)_n := \prod_{i=1}^{n} (a+i-1)$.)

Theorem 1. *For arbitrary* k_0, k_1 *and* $n \geq 0$

$$\left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S = \frac{1}{n! \left(\frac{1}{2}\right)_{n+1}} \sum_{j=0}^{n} \frac{(-n)_j^2}{j!} (-k_1)_j \left(\frac{1}{2}+k_1+k_0\right)_{n+1-j} \left(\frac{1}{2}+k_1-k_0\right)_{n-j},$$

$$\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S = -\frac{1}{(n+1)! \left(\frac{1}{2}\right)_{n+1}} \\ \times \sum_{j=0}^{n} \frac{(-n)_j (-n-1)_j}{j!} (-k_1)_j \left(\frac{1}{2}+k_1+k_0\right)_{n+1-j} \left(\frac{1}{2}+k_1-k_0\right)_{n+1-j}.$$

Corollary 2. *Suppose* $-k_1 \pm k_0 \notin \frac{1}{2} + \mathbb{N}_0$ *then*

$$\left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S = \frac{\left(\frac{1}{2}+k_1+k_0\right)_{n+1} \left(\frac{1}{2}+k_1-k_0\right)_n}{\left(\frac{1}{2}\right)_{n+1} n!} \\ \times {}_3F_2\left(\begin{matrix} -n, -n, -k_1 \\ -n-\frac{1}{2}-k_1-k_0, -n+\frac{1}{2}-k_1+k_0 \end{matrix}; 1 \right),$$

$$\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S = -\frac{\left(\frac{1}{2}+k_1+k_0\right)_{n+1} \left(\frac{1}{2}+k_1-k_0\right)_{n+1}}{\left(\frac{1}{2}\right)_{n+1} (n+1)!} \\ \times {}_3F_2\left(\begin{matrix} -n, -n-1, -k_1 \\ -n-\frac{1}{2}-k_1-k_0, -n-\frac{1}{2}-k_1+k_0 \end{matrix}; 1 \right).$$

The next step is to compare the two hypergeometric series to ${}_2F_1\left(\begin{smallmatrix}-n,-k_1\\ -n-2k_1\end{smallmatrix};1\right)$ which equals $\frac{(1+k_1)_n}{(1+2k_1)_n}$ (Chu–Vandermonde sum). The following lemma will be used with $a=\frac{1}{2}+k_1+k_0$, $b=-\frac{1}{2}+k_1-k_0$, and $c=-k_1$.

Lemma 3. *Suppose* $0<a<1$, $-1<b<0$ *and* $c>-1$ *then*

$$ {}_3F_2\left(\begin{matrix}-n,-n-1,c\\ -n-a,-n-b-1\end{matrix};1\right)\leq\frac{(1+a+b+c)_n}{(1+a+b)_n}\leq {}_3F_2\left(\begin{matrix}-n,-n,c\\ -n-a,-n-b\end{matrix};1\right),\qquad c\geq 0; $$

$$ {}_3F_2\left(\begin{matrix}-n,-n,c\\ -n-a,-n-b\end{matrix};1\right)\leq\frac{(1+a+b+c)_n}{(1+a+b)_n}\leq {}_3F_2\left(\begin{matrix}-n,-n-1,c\\ -n-a,-n-b-1\end{matrix};1\right),\qquad c\leq 0. $$

Proof. If $c=0$ then each expression equals 1. The middle term equals ${}_2F_1\left(\begin{smallmatrix}-n,c\\ -n-a-b\end{smallmatrix};1\right)$. For $0\leq i\leq n$ set

$$ s_i:=\frac{(-n)_i^2(c)_i}{i!(-n-a)_i(-n-b)_i},\qquad t_i:=\frac{(-n)_i(c)_i}{i!(-n-a-b)_i},\qquad u_i:=\frac{(-n)_i(-n-1)_i(c)_i}{i!(-n-a)_i(-n-b-1)_i}. $$

Note $s_0=t_0=u_0=1$. From the relation $(-n-d)_i=(-1)^i(n-i+1+d)_i$ it follows that $\operatorname{sign}(s_i)=\operatorname{sign}(t_i)=\operatorname{sign}(u_i)=\operatorname{sign}((c)_i)$ for each i with $1\leq i\leq n$ (by hypothesis $a+1>2$ and $b+1>0$). If $c>0$ then $\operatorname{sign}((c)_i)=1$ for all i and if $-1<c<0$ then $\operatorname{sign}((c)_i)=-1$ for $i\geq 1$. We find

$$ \frac{s_i}{t_i}=\frac{(-n)_i(-n-a-b)_i}{(-n-a)_i(-n-b)_i}=\frac{(n-i+1)(n-i+1+a+b)}{(n-i+1+a)(n-i+1+b)}\frac{s_{i-1}}{t_{i-1}},\qquad i\geq 1, $$

and

$$ \frac{m(m+a+b)}{(m+a)(m+b)}=1+\frac{-ab}{(m+a)(m+b)}>1,\qquad m\geq 1, $$

because $-ab>0$ and $b>-1$ (setting $m=n-i+1$). This shows the sequence $\frac{s_i}{t_i}$ is positive and increasing. Also

$$ \frac{u_i}{t_i}=\frac{(-n-1)_i(-n-a-b)_i}{(-n-a)_i\,(-n-b-1)_i}=\frac{(n-i+2)\,(n-i+1+a+b)}{(n-i+1+a)(n-i+2+b)}\frac{u_{i-1}}{t_{i-1}},\qquad i\geq 1, $$

$$ \frac{(m+1)(m+a+b)}{(m+a)(m+b+1)}=1-\frac{b(a-1)}{(m+a)(m+b+1)}<1,\qquad m\geq 1, $$

because $a<1$ and $b<0$ (and $a+b>-1$). The sequence $\frac{u_i}{t_i}$ is positive and decreasing. If $c>0$ then $s_i,t_i,u_i>0$ for $1\leq i\leq n$ and $\frac{s_i}{t_i}>1>\frac{u_i}{t_i}$ implies $s_i>t_i>u_i$. This proves the first inequalities. If $-1<c<0$ then $s_i,t_i,u_i<0$ for $1\leq i\leq n$ and thus $\frac{s_i}{t_i}>1>\frac{u_i}{t_i}$ implies $s_i<t_i<u_i$. This proves the second inequalities. ■

We will use a version of Stirling's formula to exploit the lemma:

$$ \frac{(a)_n}{(b)_n}=\frac{\Gamma(b)}{\Gamma(a)}\frac{\Gamma(a+n)}{\Gamma(b+n)}\sim\frac{\Gamma(b)}{\Gamma(a)}n^{a-b}. $$

Theorem 2. *Suppose* $-\frac{1}{2}<k_0\pm k_1<\frac{1}{2}$ *then the normalizing constant*

$$ c(k_0,k_1)=\frac{1}{2\pi}\cos\pi k_0\cos\pi k_1. $$

Proof. Denote the ${}_3F_2$-sums in Corollary 2 by $f_1(n)$ and $f_2(n)$ respectively, then by Stirling's formula we obtain

$$\left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S \sim \frac{\Gamma\left(\frac{1}{2}\right) f_1(n)}{\Gamma\left(\frac{1}{2}+k_1+k_0\right)\Gamma\left(\frac{1}{2}+k_1-k_0\right)} n^{2k_1-\frac{1}{2}},$$

$$\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S \sim -\frac{\Gamma\left(\frac{1}{2}\right) f_2(n)}{\Gamma\left(\frac{1}{2}+k_1+k_0\right)\Gamma\left(\frac{1}{2}+k_1-k_0\right)} n^{2k_1-\frac{1}{2}}.$$

By Propositions 2 and 3 these imply for $i = 1, 2$

$$f_i(n) \sim \frac{2\pi c(k_0,k_1)}{\cos\pi k_0 \cos\pi k_1} \times 2^{2k_1} n^{-k_1} \frac{\Gamma\left(\frac{1}{2}+k_1\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{2\pi c(k_0,k_1)}{\cos\pi k_0 \cos\pi k_1} n^{-k_1} \frac{\Gamma(1+2k_1)}{\Gamma(1+k_1)},$$

by the duplication formula. By Lemma 3 $f_1(n) \leq \frac{(1+k_1)_n}{(1+2k_1)_n} \leq f_2(n)$ for $0 \leq k_1 < \frac{1}{2}$, and the reverse inequality holds for $-\frac{1}{2} < k_1 < 0$. The fact that $\frac{(1+k_1)_n}{(1+2k_1)_n} \sim \frac{\Gamma(1+2k_1)}{\Gamma(1+k_1)} n^{-k_1}$ completes the proof. ∎

The weight function K is integrable if the inequalities $-\frac{1}{2} < k_0, k_1 < \frac{1}{2}$ are satisfied (and the same value of $c(k_0,k_1)$ applies). However K is not positive-definite and integrable unless $-\frac{1}{2} < k_0 \pm k_1 < \frac{1}{2}$. It was shown in [1, p. 21] that $\det K = d_1 d_2$. By using the (now-known) value of $c(k_0,k_1)$ and the values of d_1 and d_2 (see (2)) we find $\det K = \frac{1}{4\pi^2} \cos\pi(k_0+k_1)\cos\pi(k_0-k_1)$.

4 Formulae for $\left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S$ and $\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S$

The inner products are evaluated by computing $\Delta_\kappa^{2n}\left(\phi^{2n} p_{1,2}\right)$ and $\Delta_\kappa^{2n+1}\left(\phi^{2n+1} p_{1,4}\right)$ by means of recurrence relations. The start is a product formula for Δ_κ (using ∂_i to denote $\frac{\partial}{\partial x_i}$):

Lemma 4. *Suppose $f(x)$ is a W-invariant polynomial and $g(x,t) \in \mathcal{P}_V$ then*

$$\begin{aligned} \Delta_\kappa(fg) - f\Delta_\kappa(g) &= g\Delta f + 2\langle \nabla f, \nabla g\rangle \\ &\quad + 2k_1\left(g(x,-t_1,t_2)\frac{\partial_1 f}{x_1} + g(x,t_1,-t_2)\frac{\partial_2 f}{x_2}\right) \\ &\quad + 2k_0\left(g(x,t_2,t_1)\frac{\partial_1 f - \partial_2 f}{x_1 - x_2} + g(x,-t_2,-t_1)\frac{\partial_1 f + \partial_2 f}{x_1 + x_2}\right). \end{aligned} \tag{9}$$

Lemma 5. *Suppose $f \in \mathcal{P}_{V,2n+1}$ then $\Delta_\kappa^{n+1}|x|^2 f = 4(n+1)(n+2)\Delta_\kappa^n f$.*

Proof. If $g \in \mathcal{P}_{V,m}$ then $\Delta_\kappa |x|^2 g = 4(m+1)g + |x|^2\Delta_\kappa g$ by [1, p. 4, equation (4)]. Apply Δ_κ repeatedly to this expression, and by induction obtain $\Delta_\kappa^\ell |x|^2 g = 4\ell(m-\ell+2)\Delta_\kappa^{\ell-1} g + |x|^2\Delta_\kappa^\ell g$ for $\ell = 1,2,3,\ldots$. Set $g = f$, $m = 2n+1$ and $\ell = n+1$ then $\Delta_\kappa^{n+1} f = 0$. ∎

Recall that $\phi := x_1^2 - x_2^2$; and $\phi^{2n} p_{1,2}$ and $\phi^{2n+1} p_{1,4}$ are all relative invariants of the same type as $p_{1,2}$, that is, $\sigma_1 f = \sigma_{12}^+ f = -f$. Thus $\Delta_\kappa^{2n}\left(\phi^{2n} p_{1,2}\right)$ and $\Delta_\kappa^{2n+1}\left(\phi^{2n+1} p_{1,4}\right)$ are both scalar multiples of $p_{1,2}$, because Δ_κ commutes with the action of the group and $p_{1,2}$ is the unique degree-1 relative invariant of this type.

Proposition 4. *For $n = 0,1,2,3,\ldots$*

$$\begin{aligned} \Delta_\kappa \phi^{2n} p_{1,2} &= -8n(1+2k_1+2k_0)\phi^{2n-1} p_{1,4} \\ &\quad + 8n(2n-1-2k_0)|x|^2\phi^{2n-2} p_{1,2}, \end{aligned} \tag{10}$$

$$\begin{aligned} \Delta_\kappa \phi^{2n+1} p_{1,4} &= -4(2n+1)(1+2k_1-2k_0)\phi^{2n} p_{1,2} \\ &\quad + 8n(2n+1+2k_0)|x|^2\phi^{2n-1} p_{1,4}. \end{aligned} \tag{11}$$

Proof. We use Lemma 4 with $f=\phi^{2n}$ and $g=p_{1,2}$ or $g=\phi p_{1,4}$. Simple computation shows that

$$\Delta\phi^{2n}=8n(2n-1)|x|^2\phi^{2n-2},\qquad \nabla\phi^{2n}=4n\phi^{2n-1}\left(x_1,-x_2\right),$$
$$\Delta_\kappa\left(\phi p_{1,4}\right)=-4(1+2k_1-2k_0)p_{1,2}.$$

For $g=p_{1,2}$ we find $2\left\langle\nabla\phi^{2n},\nabla g\right\rangle=-8n\phi^{2n-1}p_{1,4}$, the coefficient of k_1 in (9) is $-16n\phi^{2n-1}p_{1,4}$ and the coefficient of k_0 is $32n\phi^{2n-2}x_1x_2\left(x_1t_1-x_2t_2\right)$. The first formula now follows from

$$x_1x_2\left(x_1t_1-x_2t_2\right)=-\frac{1}{2}\left(\phi p_{1,4}+|x|^2p_{1,2}\right).$$

For $g=\phi p_{1,4}$ we obtain $2\left\langle\nabla\phi^{2n},\nabla g\right\rangle=8n\phi^{2n-1}\left(2|x|^2p_{1,4}-\phi p_{1,2}\right)$. The coefficient of k_1 in (9) is $-16n\phi^{2n}p_{1,2}$ and the coefficient of k_0 is $-32n\phi^{2n-1}x_1x_2\left(x_1t_1+x_2t_2\right)$. Similarly to the previous case

$$x_1x_2\left(x_1t_1+x_2t_2\right)=-\frac{1}{2}\left(\phi p_{1,2}+|x|^2p_{1,4}\right).$$

The proof of the second formula is completed by adding up the parts, including $\phi^{2n}\Delta_\kappa\phi p_{1,4}$. ∎

We use this to set up a recurrence relation.

Definition 2. For $n=0,1,2,\dots$ the constants α_n, β_n implicitly depending on k_0, k_1 are defined by

$$\left\langle\phi^{2n}p_{1,2},p_{1,2}\right\rangle_S=\alpha_n\left\langle p_{1,2},p_{1,2}\right\rangle_S=\alpha_n(1+2k_1+2k_0),$$
$$\left\langle\phi^{2n+1}p_{1,4},p_{1,2}\right\rangle_S=\beta_n\left\langle p_{1,2},p_{1,2}\right\rangle_S=\beta_n(1+2k_1+2k_0).$$

Also $\alpha_n':=2^{4n}(2n)!(2n+1)!\alpha_n$ and $\beta_n':=2^{4n+2}(2n+1)!(2n+2)!\beta_n$.

Proposition 5. *Suppose* $n=0,1,2,\dots$ *then* $\Delta_\kappa^{2n}\left(\phi^{2n}p_{1,2}\right)=\alpha_n'p_{1,2}$, $\Delta_\kappa^{2n+1}\left(\phi^{2n+1}p_{1,4}\right)=\beta_n'p_{1,2}$ *and* α_n', β_n' *satisfy the recurrence* (*with* $\alpha_0'=1,\beta_{-1}'=0$)

$$\beta_n'=-4(2n+1)(1+2k_1-2k_0)\alpha_n'+64n^2(2n+1)(2n+1+2k_0)\beta_{n-1}',$$
$$\alpha_n'=-8n(1+2k_1+2k_0)\beta_{n-1}'+64n^2(2n-1)(2n-1-2k_0)\alpha_{n-1}',\qquad n\geq 1.$$

Proof. By Proposition 1

$$2^{4n+1}(2n)!(2n+1)!\left\langle\phi^{2n}p_{1,2},p_{1,2}\right\rangle_S$$
$$=\left\langle\Delta_\kappa^{2n}\left(\phi^{2n}p_{1,2}\right),p_{1,2}\right\rangle_\tau=\alpha_n\langle p_{1,2},p_{1,2}\rangle_\tau=2\alpha_n\langle p_{1,2},p_{1,2}\rangle_S.$$

Similarly

$$2^{4n+3}(2n+1)!(2n+2)!\left\langle\phi^{2n+1}p_{1,4},p_{1,2}\right\rangle_S$$
$$=\left\langle\Delta_\kappa^{2n+1}\left(\phi^{2n+1}p_{1,4}\right),p_{1,2}\right\rangle_\tau=\beta_n\langle p_{1,2},p_{1,2}\rangle_\tau=2\beta_n\langle p_{1,2},p_{1,2}\rangle_S.$$

Apply Δ_κ^{2n-1} to both sides of equation (10) to obtain

$$\alpha_n'p_{1,2}=-8n(1+2k_1+2k_0)\beta_{n-1}'p_{1,2}+8n(2n-1-2k_0)\Delta_\kappa^{2n-1}\left(|x|^2\phi^{2n-2}p_{1,2}\right).$$

By Lemma 5

$$\Delta_\kappa^{2n-1}\left(|x|^2\phi^{2n-2}p_{1,2}\right)=8n(2n-1)\Delta_\kappa^{2n-2}\left(\phi^{2n-2}p_{1,2}\right)=8n(2n-1)\alpha_{n-1}'p_{1,2}.$$

Apply Δ_κ^{2n} to both sides of equation (11) to obtain

$$\beta_n'p_{1,2}=-4(2n+1)(1+2k_1-2k_0)\alpha_n'p_{1,2}+8n(2n+1+2k_0)\Delta_\kappa^{2n}\left(|x|^2\phi^{2n-1}p_{1,4}\right),$$

and by the same lemma

$$\Delta_\kappa^{2n}\left(|x|^2\phi^{2n-1}p_{1,4}\right)=8n(2n+1)\Delta_\kappa^{2n-1}\left(\phi^{2n-1}p_{1,4}\right)=8n(2n+1)\beta_{n-1}'p_{1,2}.$$ ∎

By a simple computation with the change of scale for α_n, β_n we obtain the following:

Corollary 3. *α_n, β_n satisfy the recurrence*

$$\beta_n = -\frac{1+2k_1-2k_0}{2(n+1)}\alpha_n + \frac{n(2n+1+2k_0)}{(n+1)(2n+1)}\beta_{n-1},$$
$$\alpha_n = -\frac{1+2k_1+2k_0}{2n+1}\beta_{n-1} + \frac{2n-1-2k_0}{2n+1}\alpha_{n-1}.$$

The following formulae arose from examining values of α_n, β_n for some small n, calculated by using the recurrence and symbolic computation. In the following we use relations like $(a)_m(a+m) = (a)_{m+1}$ and $\frac{1}{(m-1)!} = \frac{m}{m!}$.

Theorem 3. *Suppose $n = 0, 1, 2, \ldots$ then*

$$\alpha_n = \sum_{j=0}^{n} \frac{(-n)_j^2}{n!\left(\frac{3}{2}\right)_n j!}(-k_1)_j\left(\frac{3}{2}+k_1+k_0\right)_{n-j}\left(\frac{1}{2}+k_1-k_0\right)_{n-j},$$
$$\beta_n = -\sum_{j=0}^{n} \frac{(-n)_j(-1-n)_j}{(n+1)!\left(\frac{3}{2}\right)_n j!}(-k_1)_j\left(\frac{3}{2}+k_1+k_0\right)_{n-j}\left(\frac{1}{2}+k_1-k_0\right)_{n+1-j}.$$

For brevity $k_+ := k_1 + k_0$ and $k_- := k_1 - k_0$, as previously. We use induction on the sequence $\alpha_0 \to \beta_0 \to \alpha_1 \to \beta_1 \to \alpha_2 \to \cdots$. The formulae are clearly valid for $n = 0$. Suppose they are valid for some $n-1$. Consider $-\frac{1+2k_1+2k_0}{2n+1}\beta_{n-1} + \frac{2n-1-2k_0}{2n+1}\alpha_{n-1}$; split up the j-term in α_{n-1} by writing $1 = \frac{j-k_1}{n-\frac{1}{2}-k_0} + \frac{n-\frac{1}{2}+k_1-k_0-j}{n-\frac{1}{2}-k_0}$, then

$$\frac{2n-1-2k_0}{2n+1}\alpha_{n-1} = \frac{1}{(n-1)!\left(\frac{3}{2}\right)_n}\sum_{j=0}^{n-1}\frac{1}{j!}(1-n)_j^2\left(\frac{3}{2}+k_+\right)_{n-1-j} \times\left\{(-k_1)_j\left(\frac{1}{2}+k_-\right)_{n-j} + (-k_1)_{j+1}\left(\frac{1}{2}+k_-\right)_{n-1-j}\right\}, \tag{12}$$

$$-\frac{1+2k_1+2k_0}{2n+1}\beta_{n-1} = \frac{\frac{1}{2}+k_+}{n!\left(\frac{3}{2}\right)_n}\sum_{j=0}^{n-1}\frac{1}{j!}(1-n)_j(-n)_j(-k_1)_j\left(\frac{3}{2}+k_+\right)_{n-1-j}\left(\frac{1}{2}+k_-\right)_{n-j}. \tag{13}$$

The coefficient of $(-k_1)_j$ in the sum of the first part of $\{\ \}$ in (12) and (13) is

$$\frac{1}{n!\left(\frac{3}{2}\right)_n j!}(1-n)_j\left(\frac{3}{2}+k_+\right)_{n-1-j}\left(\frac{1}{2}+k_-\right)_{n-j}\left[n(1-n)_j + \left(\frac{1}{2}+k_+\right)(-n)_j\right]$$
$$= \frac{1}{n!\left(\frac{3}{2}\right)_n j!}(1-n)_j\left(\frac{3}{2}+k_+\right)_{n-1-j}\left(\frac{1}{2}+k_-\right)_{n-j}(-n)_j\left(n-j+\frac{1}{2}+k_+\right)$$
$$= \frac{1}{n!\left(\frac{3}{2}\right)_n j!}(1-n)_j\left(\frac{3}{2}+k_+\right)_{n-j}\left(\frac{1}{2}+k_-\right)_{n-j}(-n)_j.$$

For $j = 0$ this establishes the validity of the $j = 0$ term in α_n. For $1 \le j \le n$ replace j by $j-1$ in the second part of $\{\ \}$ in (12) and obtain

$$\frac{jn}{n!\left(\frac{3}{2}\right)_n j!}(1-n)_{j-1}^2\left(\frac{3}{2}+k_+\right)_{n-j}\left(\frac{1}{2}+k_-\right)_{n-j}(-k_1)_j;$$

adding all up leads to

$$\frac{1}{n!\left(\frac{3}{2}\right)_n j!}\left(\frac{3}{2}+k_+\right)_{n-j}\left(\frac{1}{2}+k_-\right)_{n-j}\left\{(1-n)_j(-n)_j+nj(1-n)_{j-1}^2\right\},$$

and the expression in { } evaluates to $(-n)_j^2$. This proves the validity of the formula for α_n.

To prove the formula for β_n consider $-\frac{1+2k_1-2k_0}{2(n+1)}\alpha_n+\frac{n(2n+1+2k_0)}{(n+1)(2n+1)}\beta_{n-1}$ and as before split up the j-term in β_{n-1} by writing $1=\frac{j-k_1}{n+\frac{1}{2}+k_0}+\frac{n+\frac{1}{2}+k_1+k_0-j}{n+\frac{1}{2}+k_0}$. Then

$$-\frac{1+2k_1-2k_0}{2(n+1)}\alpha_n=-\frac{\frac{1}{2}+k_1-k_0}{(n+1)!\left(\frac{3}{2}\right)_n}\sum_{j=0}^{n}\frac{1}{j!}(-n)_j^2(-k_1)_j\left(\frac{3}{2}+k_+\right)_{n-j}\left(\frac{1}{2}+k_-\right)_{n-j}, \tag{14}$$

$$\frac{n(2n+1+2k_0)}{(n+1)(2n+1)}\beta_{n-1}=\frac{1}{(n+1)!\left(\frac{3}{2}\right)_n}\sum_{j=0}^{n-1}\frac{1}{j!}(-n)_{j+1}(-n)_j\left(\frac{1}{2}+k_-\right)_{n-j} \tag{15}$$
$$\times\left\{(-k_1)_j\left(\frac{3}{2}+k_+\right)_{n-j}+(-k_1)_{j+1}\left(\frac{3}{2}+k_+\right)_{n-j-1}\right\}.$$

The coefficient of $(-k_1)_j$ in the sum of (14) and the first part of { } in (15) is

$$\frac{-1}{(n+1)!\left(\frac{3}{2}\right)_n j!}(-n)_j^2\left(\frac{3}{2}+k_+\right)_{n-j}\left(\frac{1}{2}+k_-\right)_{n-j}\left(\frac{1}{2}+k_-+n-j\right)$$
$$=\frac{-1}{(n+1)!\left(\frac{3}{2}\right)_n j!}(-n)_j^2\left(\frac{3}{2}+k_+\right)_{n-j}\left(\frac{1}{2}+k_-\right)_{n+1-j}.$$

For $j=0$ this establishes the validity of the $j=0$ term in β_n. For $1\le j\le n$ replace j by $j-1$ in the second part of { } in (15) and obtain

$$\frac{j}{(n+1)!\left(\frac{3}{2}\right)_n j!}(-n)_j(-n)_{j-1}\left(\frac{1}{2}+k_-\right)_{n-j+1}\left(\frac{3}{2}+k_+\right)_{n-j};$$

adding all up leads to

$$\frac{-1}{(n+1)!\left(\frac{3}{2}\right)_n j!}(-n)_j\left(\frac{1}{2}+k_-\right)_{n-j+1}\left(\frac{3}{2}+k_+\right)_{n-j}\{(-n)_j-j(-n)_{j-1}\}$$

and the expression in { } evaluates to $(-1-n)_j$. This proves the validity of the formula for β_n and completes the induction.

This completes the proof of Theorem 1.

References

[1] Dunkl C.F., Vector-valued polynomials and a matrix weight function with B_2-action, *SIGMA* **9** (2013), 007, 23 pages, arXiv:1210.1177.