Symmetry, Integrability and Geometry: Methods and Applications SIGMA 9 (2013), 054, 28 pages

Extended T-System of Type G_2*

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Received April 03, 2013, in final form August 16, 2013; Published online August 22, 2013
http://dx.doi.org/10.3842/SIGMA.2013.054

Abstract. We prove a family of 3-term relations in the Grothendieck ring of the category of finite-dimensional modules over the affine quantum algebra of type G_2 extending the celebrated T-system relations of type G_2. We show that these relations can be used to compute classes of certain irreducible modules, including classes of all minimal affinizations of type G_2. We use this result to obtain explicit formulas for dimensions of all participating modules.

Key words: quantum affine algebra of type G_2; minimal affinizations; extended T-systems; q-characters; Frenkel–Mukhin algorithm

2010 Mathematics Subject Classification: 17B37; 81R50; 82B23

1 Introduction

Kirillov–Reshetikhin modules are simplest examples of irreducible finite-dimensional modules over quantum affine algebras, and the T-system is a famous family of short exact sequences of tensor products of Kirillov–Reshetikhin modules, see [10, 15, 16, 20]. There are numerous applications of the T-systems in representation theory, combinatorics and integrable systems, see the survey [17].

Minimal affinizations of quantum affine algebras form an important family of irreducible modules which contains the Kirillov–Reshetikhin modules, see [3]. A procedure to extend the T-system to a larger set of relations to include the minimal affinization was described in [18], where it was conjectured to work in all types. In [18] this procedure was carried out in types A and B. In this paper, we show the existence of the extended T-system for type G_2.

We work with the quantum affine algebra $U_q\hat{\mathfrak{g}}$ of type G_2. The irreducible finite-dimensional modules of quantum affine algebras are parameterized by the highest l-weights or Drinfeld polynomials. Let $\mathcal{T}$ be an irreducible $U_q\hat{\mathfrak{g}}$-module such that zeros of all Drinfeld polynomials belong to a lattice $aq^{\mathbb{Z}}$ for some $a \in \mathbb{C}^\times$. Following [18], we define the left, right, and bottom modules, denoted by $\mathcal{L}$, $\mathcal{R}$, $\mathcal{B}$ respectively. The Drinfeld polynomials of left, right, and bottom modules are obtained by stripping the rightmost, leftmost, and both left- and rightmost zeros of the union of zeros of the Drinfeld polynomials of the top module $\mathcal{T}$.

Then the relations of the extended T-system have the form $[\mathcal{L}][\mathcal{R}] = [\mathcal{T}][\mathcal{B}] + [\mathcal{S}]$, where $[\cdot]$ denotes the equivalence class of a $U_q\hat{\mathfrak{g}}$-module in the Grothendieck ring of the category of

*This paper is a contribution to the Special Issue in honor of Anatol Kirillov and Tetsuji Miwa. The full collection is available at http://www.emis.de/journals/SIGMA/InfiniteAnalysis2013.html

finite-dimensional representations of $U_q\hat{\mathfrak{g}}$. Moreover, in all cases the modules $\mathcal{T}\otimes\mathcal{B}$ and $\mathcal{S}$ are irreducible.

We start with minimal affinizations as the top modules $\mathcal{T}$, then the left, right and bottom modules are minimal affinizations as well. We compute S and decompose it as a product of irreducible modules which we call sources. It turns out that the sources are not always minimal affinizations. Therefore, we follow up with taking the sources as top modules and compute new left, right, bottom modules, and sources. Then we use all new modules obtained on a previous step as top modules and so on.

We end up with several families of modules which we denote by $\mathcal{B}_{k,\ell}^{(s)}$, $\mathcal{C}_{k,\ell}^{(s)}$, $\mathcal{D}_{k,\ell}^{(s)}$, $\mathcal{E}_{k,\ell}^{(s)}$, $\mathcal{F}_{k,\ell}^{(s)}$, $\tilde{\mathcal{B}}_{k,\ell}^{(s)}$, $\tilde{\mathcal{C}}_{k,\ell}^{(s)}$, $\tilde{\mathcal{D}}_{k,\ell}^{(s)}$, $\tilde{\mathcal{E}}_{k,\ell}^{(s)}$, $\tilde{\mathcal{F}}_{k,\ell}^{(s)}$, where $s\in\mathbb{Z}$, $k,\ell\in\mathbb{Z}_{\geq 0}$. This is the minimal set of modules which contains all minimal affinizations $\big($these are modules $\mathcal{B}_{k,\ell}^{(s)}$, $\tilde{\mathcal{B}}_{k,\ell}^{(s)}\big)$ and which is closed under our set of relations. Namely, if any of the above modules is chosen as a top module then the left, right, bottom modules and all sources belong to this set as well, see Theorems 3.4, 7.4. The spirit of the proof Theorems 3.4, 7.4 follows the works in [10, 18, 21].

We show that the extended T-system allows us to compute the modules $\mathcal{B}_{k,\ell}^{(s)}$, $\mathcal{C}_{k,\ell}^{(s)}$, $\mathcal{D}_{k,\ell}^{(s)}$, $\mathcal{E}_{k,\ell}^{(s)}$, $\mathcal{F}_{k,\ell}^{(s)}$ recursively in terms of fundamental modules, see Proposition 3.6. We use this to compute the dimensions of all participating modules, in particular, we give explicit formulas for dimensions of all minimal affinizations of type G_2, see Theorem 8.1. We hope further, that one can use use the extended T-system to obtain the decomposition of all participating modules as the $U_q\mathfrak{g}$-modules.

Let us point out some similarities and differences with types A and B. The type A, the extended T-system is closed within the class of minimal affinizations, meaning that all sources are minimal affinizations as well. In type B, the extended T-system is not closed within the class of minimal affinizations, but it is closed in the class of so called snake modules, see [18]. For the proofs and computations it is important that all modules participating in extended T-systems of types A and B are thin and special, moreover their q-characters are known explicitly in terms of skew Young tableaux in type A, and in terms of path models in type B, see [6, 18, 19, 22].

In general the modules of the extended T-system of type G_2 are not thin and at the moment there is no combinatorial description of their q-characters. However, all modules turn out to be either special or anti-special. Therefore we are able to use the FM algorithm, see [8], to compute the sufficient information about q-characters in order to complete the proofs. Note, that a priori it is was not obvious that the extended T-system will be closed within special or anti-special modules. Moreover, since the q-characters of G_2 modules are not known explicitly, the property of being special or anti-special had to be established in each case, see Theorems 3.3, 7.2.

Note that in general the minimal affinizations of types C, D, E, F are neither special nor anti-special, therefore the methods of this paper cannot be applied in those cases.

There is a remarkable conjecture on the cluster algebra relations in the category of finite-dimensional representations of quantum affine algebras of type A, D, E, see [12]. Taking into account the work of [13, 14], one could expect that the conjecture of [12] can be formulated for other types as well, in particular for type G_2. We expect that the extended T-system is a part of cluster algebra relations.

The paper is organized as follows. In Section 2, we give some background material. In Section 3, we define the modules $\mathcal{B}_{k,\ell}^{(s)}$, $\mathcal{C}_{k,\ell}^{(s)}$, $\mathcal{D}_{k,\ell}^{(s)}$, $\mathcal{E}_{k,\ell}^{(s)}$, $\mathcal{F}_{k,\ell}^{(s)}$ and state our main result, Theorem 3.4. In Section 4, we prove that the modules $\mathcal{B}_{k,\ell}^{(s)}$, $\mathcal{C}_{k,\ell}^{(s)}$, $\mathcal{D}_{k,\ell}^{(s)}$, $\mathcal{E}_{k,\ell}^{(s)}$, $\mathcal{F}_{k,\ell}^{(s)}$ are special. In Section 5, we prove Theorem 3.4. In Section 6, we prove that the module $\mathcal{T}\otimes\mathcal{B}$ is irreducible for each relation in the extended T-system. In Section 7, we deduce the extended T-system for the modules $\tilde{\mathcal{B}}_{k,\ell}^{(s)}$, $\tilde{\mathcal{C}}_{k,\ell}^{(s)}$, $\tilde{\mathcal{D}}_{k,\ell}^{(s)}$, $\tilde{\mathcal{E}}_{k,\ell}^{(s)}$, $\tilde{\mathcal{F}}_{k,\ell}^{(s)}$. In Section 8, we compute the dimensions of the modules in the extended T-systems.

2 Background

2.1 Cartan data

Let $\mathfrak{g}$ be a complex simple Lie algebra of type G_2 and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $I = \{1, 2\}$. We choose simple roots α_1, α_2 and scalar product $(\cdot, \cdot)$ such that

$$(\alpha_1, \alpha_1) = 2, \qquad (\alpha_1, \alpha_2) = -3, \qquad (\alpha_2, \alpha_2) = 6.$$

Let $\{\alpha_1^\vee, \alpha_2^\vee\}$ and $\{\omega_1, \omega_2\}$ be the sets of simple coroots and fundamental weights respectively. Let $C = (C_{ij})_{i,j\in I}$ denote the Cartan matrix, where $C_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$. Let $r_1 = 1$, $r_2 = 3$, $D = \mathrm{diag}(r_1, r_2)$ and $B = DC$. Then

$$C = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}.$$

Let Q (resp. Q^+) and P (resp. P^+) denote the $\mathbb{Z}$-span (resp. $\mathbb{Z}_{\geq 0}$-span) of the simple roots and fundamental weights respectively. Let $\leq$ be the partial order on P in which $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda \in Q^+$.

Let $\hat{\mathfrak{g}}$ denote the untwisted affine algebra corresponding to $\mathfrak{g}$. Fix a $q \in \mathbb{C}^\times$, not a root of unity. Let $q_i = q^{r_i}$, $i = 1, 2$. Define the q-numbers, q-factorial and q-binomial:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \qquad \begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{[n]_q!}{[n-m]_q![m]_q!}.$$

2.2 Quantum affine algebra

The quantum affine algebra $U_q\hat{\mathfrak{g}}$ in Drinfeld's new realization, see [7], is generated by $x^{\pm}_{i,n}$ ($i \in I$, $n \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,n}$ ($i \in I$, $n \in \mathbb{Z}\backslash\{0\}$) and central elements $c^{\pm 1/2}$, subject to the following relations:

$$k_i k_j = k_j k_i, \qquad k_i h_{j,n} = h_{j,n} k_i, \qquad k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_i x^{\pm}_{j,n} k_i^{-1} = q^{\pm B_{ij}} x^{\pm}_{j,n},$$
$$\left[h_{i,n}, x^{\pm}_{j,m}\right] = \pm \frac{1}{n} [nB_{ij}]_q c^{\mp |n|/2} x^{\pm}_{j,n+m},$$
$$x^{\pm}_{i,n+1} x^{\pm}_{j,m} - q^{\pm B_{ij}} x^{\pm}_{j,m} x^{\pm}_{i,n+1} = q^{\pm B_{ij}} x^{\pm}_{i,n} x^{\pm}_{j,m+1} - x^{\pm}_{j,m+1} x^{\pm}_{i,n},$$
$$[h_{i,n}, h_{j,m}] = \delta_{n,-m} \frac{1}{n} [nB_{ij}]_q \frac{c^n - c^{-n}}{q - q^{-1}},$$
$$[x^+_{i,n}, x^-_{j,m}] = \delta_{ij} \frac{c^{(n-m)/2} \phi^+_{i,n+m} - c^{-(n-m)/2} \phi^-_{i,n+m}}{q_i - q_i^{-1}},$$
$$\sum_{\pi \in \Sigma_s} \sum_{k=0}^{s} (-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} x^{\pm}_{i,n_{\pi(1)}} \cdots x^{\pm}_{i,n_{\pi(k)}} x^{\pm}_{j,m} x^{\pm}_{i,n_{\pi(k+1)}} \cdots x^{\pm}_{i,n_{\pi(s)}} = 0, \qquad s = 1 - C_{ij},$$

for all sequences of integers $n_1, \ldots, n_s$, and $i \neq j$, where Σ_s is the symmetric groups on s letters, $\phi^{\pm}_{i,n} = 0$ ($n < 0$) and $\phi^{\pm}_{i,n}$'s ($n \geq 0$) are determined by the formula

$$\phi^{\pm}_i(u) := \sum_{n=0}^{\infty} \phi^{\pm}_{i,\pm n} u^{\pm n} = k_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} u^{\pm m} \right). \tag{2.1}$$

There exist a coproduct, counit and antipode making $U_q\hat{\mathfrak{g}}$ into a Hopf algebra.

The quantum affine algebra $U_q\hat{\mathfrak{g}}$ contains two standard quantum affine algebras of type A_1. The first one is $U_{q_1}(\hat{\mathfrak{sl}}_2)$ generated by $x^{\pm}_{1,n}$ ($n \in \mathbb{Z}$), $k_1^{\pm 1}$, $h_{1,n}$ ($n \in \mathbb{Z}\backslash\{0\}$) and central elements $c^{\pm 1/2}$. The second one is $U_{q_2}(\hat{\mathfrak{sl}}_2)$ generated by $x^{\pm}_{2,n}$ ($n \in \mathbb{Z}$), $k_2^{\pm 1}$, $h_{2,n}$ ($n \in \mathbb{Z}\backslash\{0\}$) and central elements $c^{\pm 1/2}$.

The subalgebra of $U_q\hat{\mathfrak{g}}$ generated by $(k_i^{\pm})_{i\in I}$, $(x^{\pm}_{i,0})_{i\in I}$ is a Hopf subalgebra of $U_q\hat{\mathfrak{g}}$ and is isomorphic as a Hopf algebra to $U_q\mathfrak{g}$, the quantized enveloping algebra of $\mathfrak{g}$. In this way, $U_q\hat{\mathfrak{g}}$-modules restrict to $U_q\mathfrak{g}$-modules.

2.3 Finite-dimensional representations and q-characters

In this section, we recall the standard facts about finite-dimensional representations of $U_q\hat{\mathfrak{g}}$ and q-characters of these representations, see [2, 4, 8, 9, 18].

A representation V of $U_q\hat{\mathfrak{g}}$ is of type 1 if $c^{\pm 1/2}$ acts as the identity on V and

$$V = \bigoplus_{\lambda\in P} V_\lambda, \qquad V_\lambda = \big\{v \in V : k_i v = q^{(\alpha_i,\lambda)} v\big\}. \tag{2.2}$$

In the following, all representations will be assumed to be finite-dimensional and of type 1 without further comment. The decomposition (2.2) of a finite-dimensional representation V into its $U_q\mathfrak{g}$-weight spaces can be refined by decomposing it into the Jordan subspaces of the mutually commuting operators $\phi^{\pm}_{i,\pm r}$, see [9]:

$$V = \bigoplus_{\gamma} V_\gamma, \qquad \gamma = \big(\gamma^{\pm}_{i,\pm r}\big)_{i\in I,\, r\in\mathbb{Z}_{\geq 0}}, \qquad \gamma^{\pm}_{i,\pm r} \in \mathbb{C}, \tag{2.3}$$

where

$$V_\gamma = \Big\{ v \in V : \exists\, k \in \mathbb{N}, \forall\, i \in I, m \geq 0, \big(\phi^{\pm}_{i,\pm m} - \gamma^{\pm}_{i,\pm m}\big)^k v = 0 \Big\}.$$

If $\dim(V_\gamma) > 0$, then γ is called an *l-weight* of V. For every finite dimensional representation of $U_q\hat{\mathfrak{g}}$, the l-weights are known, see [9], to be of the form

$$\gamma_i^{\pm}(u) := \sum_{r=0}^{\infty} \gamma^{\pm}_{i,\pm r} u^{\pm r} = q_i^{\deg Q_i - \deg R_i} \frac{Q_i(uq_i^{-1})R_i(uq_i)}{Q_i(uq_i)R_i(uq_i^{-1})}, \tag{2.4}$$

where the right hand side is to be treated as a formal series in positive (resp. negative) integer powers of u, and Q_i, R_i are polynomials of the form

$$Q_i(u) = \prod_{a\in\mathbb{C}^\times} (1-ua)^{w_{i,a}}, \qquad R_i(u) = \prod_{a\in\mathbb{C}^\times} (1-ua)^{x_{i,a}}, \tag{2.5}$$

for some $w_{i,a}$, $x_{i,a} \in \mathbb{Z}_{\geq 0}$, $i \in I$, $a \in \mathbb{C}^\times$. Let $\mathcal{P}$ denote the free abelian multiplicative group of monomials in infinitely many formal variables $(Y_{i,a})_{i\in I,\, a\in\mathbb{C}^\times}$. There is a bijection γ from $\mathcal{P}$ to the set of l-weights of finite-dimensional modules such that for the monomial $m = \prod_{i\in I,\, a\in\mathbb{C}^\times} Y_{i,a}^{w_{i,a}-x_{i,a}}$, the l-weight $\gamma(m)$ is given by (2.4), (2.5).

Let $\mathbb{Z}\mathcal{P} = \mathbb{Z}\big[Y_{i,a}^{\pm 1}\big]_{i\in I,\, a\in\mathbb{C}^\times}$ be the group ring of $\mathcal{P}$. For $\chi \in \mathbb{Z}\mathcal{P}$, we write $m \in \mathcal{P}$ if the coefficient of m in χ is non-zero.

The q-character of a $U_q\hat{\mathfrak{g}}$-module V is given by

$$\chi_q(V) = \sum_{m\in\mathcal{P}} \dim(V_m) m \in \mathbb{Z}\mathcal{P},$$

where $V_m = V_{\gamma(m)}$.

Let $\mathrm{Rep}(U_q\hat{\mathfrak{g}})$ be the Grothendieck ring of finite-dimensional representations of $U_q\hat{\mathfrak{g}}$ and $[V] \in \mathrm{Rep}(U_q\hat{\mathfrak{g}})$ the class of a finite-dimensional $U_q\hat{\mathfrak{g}}$-module V. The q-character map defines an injective ring homomorphism, see [9],

$$\chi_q : \ \mathrm{Rep}(U_q\hat{\mathfrak{g}}) \to \mathbb{Z}\mathcal{P}.$$

For any finite-dimensional representation V of $U_q\hat{\mathfrak{g}}$, denote by $\mathscr{M}(V)$ the set of all monomials in $\chi_q(V)$. For each $j \in I$, a monomial $m = \prod_{i\in I,\, a\in\mathbb{C}^\times} Y_{i,a}^{u_{i,a}}$, where $u_{i,a}$ are some integers, is said to be *j-dominant* (resp. *j-anti-dominant*) if and only if $u_{j,a} \geq 0$ (resp. $u_{j,a} \leq 0$) for all $a \in \mathbb{C}^\times$. A monomial is called *dominant* (resp. *anti-dominant*) if and only if it is j-dominant (resp. j-anti-dominant) for all $j \in I$. Let $\mathcal{P}^+ \subset \mathcal{P}$ denote the set of all dominant monomials.

Let V be a representation of $U_q\hat{\mathfrak{g}}$ and $m \in \mathscr{M}(V)$ a monomial. A non-zero vector $v \in V_m$ is called a *highest l-weight vector* with *highest l-weight* $\gamma(m)$ if

$$x^+_{i,r} \cdot v = 0, \qquad \phi^\pm_{i,\pm t} \cdot v = \gamma(m)^\pm_{i,\pm t} v, \qquad \forall\, i \in I,\ r \in \mathbb{Z},\ t \in \mathbb{Z}_{\geq 0}.$$

The module V is called a *highest l-weight representation* if $V = U_q\hat{\mathfrak{g}} \cdot v$ for some highest l-weight vector $v \in V$.

It is known, see [2, 4], that for each $m_+ \in \mathcal{P}^+$ there is a unique finite-dimensional irreducible representation, denoted $L(m_+)$, of $U_q\hat{\mathfrak{g}}$ that is highest l-weight representation with highest l-weight $\gamma(m_+)$, and moreover every finite-dimensional irreducible $U_q\hat{\mathfrak{g}}$-module is of this form for some $m_+ \in \mathcal{P}^+$. Also, if $m_+, m'_+ \in \mathcal{P}^+$ and $m_+ \neq m'_+$, then $L(m_+) \ncong L(m'_+)$. For $m_+ \in \mathcal{P}^+$, we use $\chi_q(m_+)$ to denote $\chi_q(L(m_+))$.

The following lemma is well-known.

Lemma 2.1. *Let m_1, m_2 be two monomials. Then $L(m_1m_2)$ is a sub-quotient of $L(m_1)\otimes L(m_2)$. In particular, $\mathscr{M}(L(m_1m_2)) \subseteq \mathscr{M}(L(m_1))\mathscr{M}(L(m_2))$.*

For $b \in \mathbb{C}^\times$, define the shift of spectral parameter map $\tau_b : \mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{P}$ to be a homomorphism of rings sending $Y^{\pm1}_{i,a}$ to $Y^{\pm1}_{i,ab}$. Let m_1, $m_2 \in \mathcal{P}^+$. If $\tau_b(m_1) = m_2$, then

$$\tau_b\chi_q(m_1) = \chi_q(m_2). \tag{2.6}$$

Let m_+ be a dominant l-weight. We call the polynomial $\chi_q(m_+)$ special if it contains exactly one dominant monomial.

A finite-dimensional $U_q\hat{\mathfrak{g}}$-module V is said to be *special* if and only if $\mathscr{M}(V)$ contains exactly one dominant monomial. It is called *anti-special* if and only if $\mathscr{M}(V)$ contains exactly one anti-dominant monomial. It is called *thin* if and only if no l-weight space of V has dimension greater than 1. We also call a polynomial in $\mathbb{Z}\mathcal{P}$ *special*, *antispecial*, or *thin* if this polynomial contains a unique dominant monomial, a unique anti-dominant monomial, or if all coefficients are zero and one respectively. A finite-dimensional $U_q\hat{\mathfrak{g}}$-module is said to be *prime* if and only if it is not isomorphic to a tensor product of two non-trivial $U_q\hat{\mathfrak{g}}$-modules, see [5]. Clearly, if a module is special or anti-special, then it is irreducible.

Define $A_{i,a} \in \mathcal{P}$, $i \in I$, $a \in \mathbb{C}^\times$, by

$$A_{1,a} = Y_{1,aq}Y_{1,aq^{-1}}Y^{-1}_{2,a}, \qquad A_{2,a} = Y_{2,aq^3}Y_{2,aq^{-3}}Y^{-1}_{1,aq^{-2}}Y^{-1}_{1,a}Y^{-1}_{1,aq^2}.$$

Let $\mathcal{Q}$ be the subgroup of $\mathcal{P}$ generated by $A_{i,a}$, $i \in I$, $a \in \mathbb{C}^\times$. Let $\mathcal{Q}^\pm$ be the monoids generated by $A^{\pm1}_{i,a}$, $i \in I$, $a \in \mathbb{C}^\times$. There is a partial order $\leq$ on $\mathcal{P}$ in which

$$m \leq m' \ \text{ if and only if } \ m'm^{-1} \in \mathcal{Q}^+. \tag{2.7}$$

For all $m_+ \in \mathcal{P}^+$, $\mathscr{M}(L(m_+)) \subset m_+\mathcal{Q}^-$, see [8].

A monomial m is called *right negative* if and only if $\forall a \in \mathbb{C}^\times$ and $\forall i \in I$, we have the following property: if the power of $Y_{i,a}$ is non-zero and the power of Y_{j,aq^k} is zero for all $j \in I$, $k \in \mathbb{Z}_{>0}$, then the power of $Y_{i,a}$ is negative. For $i \in I$, $a \in \mathbb{C}^\times$, $A_{i,a}^{-1}$ is right-negative. A product of right-negative monomials is right-negative. If m is right-negative, then $m' \leq m$ implies that m' is right-negative.

2.4 Minimal affinizations of $U_q\mathfrak{g}$-modules

Let $\lambda = k\omega_1 + \ell\omega_2$. A simple $U_q\hat{\mathfrak{g}}$-module $L(m_+)$ is called a *minimal affinization* of $V(\lambda)$ if and only if m_+ is one of the following monomials

$$\left(\prod_{i=0}^{\ell-1} Y_{2,aq^{6i}}\right)\left(\prod_{i=0}^{k-1} Y_{1,aq^{6\ell+2i+1}}\right), \qquad \left(\prod_{i=0}^{k-1} Y_{1,aq^{2i}}\right)\left(\prod_{i=0}^{\ell-1} Y_{2,aq^{2k+6i+5}}\right),$$

for some $a \in \mathbb{C}^\times$, see [3]. In particular, when $k = 0$ or $\ell = 0$, the minimal affinization $L(m_+)$ is called a *Kirillov–Reshetikhin module*.

Let $L(m_+)$ be a Kirillov–Reshetikhin module. It is shown in [10] that any non-highest monomial in $\mathscr{M}(L(m_+))$ is right-negative and in particular $L(m_+)$ is special.

2.5 q-characters of $U_q\hat{\mathfrak{sl}}_2$-modules and the FM algorithm

The q-characters of $U_q\hat{\mathfrak{sl}}_2$-modules are well-understood, see [1, 9]. We recall the results here.

Let $W_k^{(a)}$ be the irreducible representation $U_q\hat{\mathfrak{sl}}_2$ with highest weight monomial

$$X_k^{(a)} = \prod_{i=0}^{k-1} Y_{aq^{k-2i-1}},$$

where $Y_a = Y_{1,a}$. Then the q-character of $W_k^{(a)}$ is given by

$$\chi_q\big(W_k^{(a)}\big) = X_k^{(a)} \sum_{i=0}^{k} \prod_{j=0}^{i-1} A_{aq^{k-2j}}^{-1},$$

where $A_a = Y_{aq^{-1}} Y_{aq}$.

For $a \in \mathbb{C}^\times$, $k \in \mathbb{Z}_{\geq 1}$, the set $\Sigma_k^{(a)} = \{aq^{k-2i-1}\}_{i=0,\dots,k-1}$ is called a *string*. Two strings $\Sigma_k^{(a)}$ and $\Sigma_{k'}^{(a')}$ are said to be in *general position* if the union $\Sigma_k^{(a)} \cup \Sigma_{k'}^{(a')}$ is not a string or $\Sigma_k^{(a)} \subset \Sigma_{k'}^{(a')}$ or $\Sigma_{k'}^{(a')} \subset \Sigma_k^{(a)}$.

Denote by $L(m_+)$ the irreducible $U_q\hat{\mathfrak{sl}}_2$-module with highest weight monomial m_+. Let $m_+ \neq 1$ and $\in \mathbb{Z}[Y_a]_{a\in\mathbb{C}^\times}$ be a dominant monomial. Then m_+ can be uniquely (up to permutation) written in the form

$$m_+ = \prod_{i=1}^{s}\left(\prod_{b\in\Sigma_{k_i}^{(a_i)}} Y_b\right),$$

where s is an integer, $\Sigma_{k_i}^{(a_i)}$, $i = 1, \dots, s$, are strings which are pairwise in general position and

$$L(m_+) = \bigotimes_{i=1}^{s} W_{k_i}^{(a_i)}, \qquad \chi_q(m_+) = \prod_{i=1}^{s} \chi_q\Big(W_{k_i}^{(a_i)}\Big).$$

For $j \in I$, let

$$\beta_j : \mathbb{Z}\big[Y_{i,a}^{\pm 1}\big]_{i\in I; a\in\mathbb{C}^\times} \to \mathbb{Z}\big[Y_a^{\pm 1}\big]_{a\in\mathbb{C}^\times}$$

be the ring homomorphism which sends, for all $a \in \mathbb{C}^\times$, $Y_{k,a} \mapsto 1$ for $k \neq j$ and $Y_{j,a} \mapsto Y_a$.

Let V be a $U_q\hat{\mathfrak{g}}$-module. Then $\beta_i(\chi_q(V))$, $i = 1, 2$, is the q-character of V considered as a $U_{q_i}(\hat{\mathfrak{sl}}_2)$-module.

In some situation, we can use the q-characters of $U_q\hat{\mathfrak{sl}}_2$-modules to compute the q-characters of $U_q\hat{\mathfrak{g}}$-modules for arbitrary $\mathfrak{g}$, see [8, Section 5]. The corresponding algorithm is called the FM algorithm. The FM algorithm recursively computes the minimal possible q-character which contains m_+ and is consistent when restricted to $U_{q_i}(\hat{\mathfrak{sl}}_2)$, $i = 1, 2$.

Although the FM algorithm does not give the q-character of a $U_q\hat{\mathfrak{g}}$-module in general, the FM algorithm works for a large family of $U_q\hat{\mathfrak{g}}$-modules. For example, if a module $L(m_+)$ is special, then the FM algorithm applied to m_+, produces the correct q-character $\chi_q(m_+)$, see [8].

2.6 Truncated q-characters

We use the truncated q-characters [12, 18]. Given a set of monomials $\mathcal{R} \subset \mathcal{P}$, let $\mathbb{Z}\mathcal{R} \subset \mathbb{Z}\mathcal{P}$ denote the $\mathbb{Z}$-module of formal linear combinations of elements of $\mathcal{R}$ with integer coefficients. Define

$$\mathrm{trunc}_{\mathcal{R}} : \mathcal{P} \to \mathcal{R}; \qquad m \mapsto \begin{cases} m & \text{if} \quad m \in \mathcal{R}, \\ 0 & \text{if} \quad m \notin \mathcal{R}, \end{cases}$$

and extend $\mathrm{trunc}_{\mathcal{R}}$ as a $\mathbb{Z}$-module map $\mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{R}$.

Given a subset $U \subset I \times \mathbb{C}^\times$, let $\mathcal{Q}_U$ be the subgroups of $\mathcal{Q}$ generated by $A_{i,a}$ with $(i,a) \in U$. Let $\mathcal{Q}_U^{\pm}$ be the monoid generated by $A_{i,a}^{\pm 1}$ with $(i,a) \in U$. We call $\mathrm{trunc}_{m_+\mathcal{Q}_U^-}\chi_q(m_+)$ *the q-character of $L(m_+)$ truncated to U*.

If $U = I \times \mathbb{C}^\times$, then $\mathrm{trunc}_{m_+\mathcal{Q}_U^-}\chi_q(m_+)$ is the ordinary q-character of $L(m_+)$.

The main idea of using the truncated q-characters is the following. Given m^+, one chooses $\mathcal{R}$ in such a way that the dropped monomials are all right-negative and the truncated q-character is much smaller than the full q-character. The advantage is that the truncated q-character is much easier to compute and to describe in a combinatorial way. At the same time, if the truncating set $\mathcal{R}$ can be used for both m_1^+ and m_2^+, then the same $\mathcal{R}$ works for the tensor product $L(m_1^+) \otimes L(m_2^+)$. Moreover, the product of truncated characters of $L(m_1^+)$ and $L(m_2^+)$ contains all dominant monomials of the tensor product $L(m_1^+)\otimes L(m_2^+)$ and can be used to find the decomposition of it into irreducible components in the Grothendieck ring. We compute the truncated q-characters using the following theorem.

Theorem 2.2 ([18, Theorem 2.1])**.** *Let $U \subset I \times \mathbb{C}^\times$ and $m_+ \in \mathcal{P}^+$. Suppose that $\mathcal{M} \subset \mathcal{P}$ is a finite set of distinct monomials such that*

(i) $\mathcal{M} \subset m_+\mathcal{Q}_U^-$,

(ii) $\mathcal{P}^+ \cap \mathcal{M} = \{m_+\}$,

(iii) for all $m \in \mathcal{M}$ and all $(i,a) \in U$, if $mA_{i,a}^{-1} \notin \mathcal{M}$, then $mA_{i,a}^{-1}A_{j,b} \notin \mathcal{M}$ unless $(j,b) = (i,a)$,

(iv) for all $m \in \mathcal{M}$ and all $i \in I$, there exists a unique i-dominant monomial $M \in \mathcal{M}$ such that

$$\mathrm{trunc}_{\beta_i(M\mathcal{Q}_U^-)}\chi_q(\beta_i(M)) = \sum_{m' \in m\mathcal{Q}_{\{i\}\times\mathbb{C}^\times}} \beta_i(m').$$

Then

$$\operatorname{trunc}_{m_+\mathcal{Q}_U^-}\chi_q(m_+) = \sum_{m\in\mathcal{M}} m.$$

Here by $\chi_q(\beta_i(M))$ we mean the q-character of the irreducible $U_{q_i}(\hat{\mathfrak{sl}}_2)$-module with highest weight monomial $\beta_i(M)$ and by $\operatorname{trunc}_{\beta_i}(M\mathcal{Q}_U^-)$ we mean keeping only the monomials of $\chi_q(\beta_i(M))$ in the set $\beta_i(M\mathcal{Q}_U^-)$.

3 Main results

3.1 First examples

Without loss of generality, we fix $a\in\mathbb{C}^\times$ and consider modules V with $\mathscr{M}(V)\subset\mathbb{Z}[Y_{i,aq^k}]_{i\in I,\,k\in\mathbb{Z}}$. In the following, for simplicity we write i_s, i_s^{-1} ($s\in\mathbb{Z}$) instead of Y_{i,aq^s}, Y_{i,aq^s}^{-1} respectively. The q-characters of fundamental modules are easy to compute by using the FM algorithm.

Lemma 3.1. *The fundamental q-characters for $U_q\hat{\mathfrak{g}}$ of type G_2 are given by*

$$\begin{aligned}
\chi_q(1_0) &= 1_0 + 1_2^{-1}2_1 + 1_41_62_7^{-1} + 1_41_8^{-1} + 1_6^{-1}1_8^{-1}2_5 + 1_{10}2_{11}^{-1} + 1_{12}^{-1},\\
\chi_q(2_0) &= 2_0 + 1_11_31_52_6^{-1} + 1_11_31_7^{-1} + 1_11_5^{-1}1_7^{-1}2_4 + 1_3^{-1}1_5^{-1}1_7^{-1}2_22_4\\
&\quad + 1_11_92_{10}^{-1} + 2_42_8^{-1} + 1_3^{-1}1_92_22_{10}^{-1} + 1_51_71_92_8^{-1}2_{10}^{-1} + 1_11_{11}^{-1}\\
&\quad + 1_3^{-1}1_{11}^{-1}2_2 + 1_51_71_{11}^{-1}2_8^{-1} + 1_51_9^{-1}1_{11}^{-1} + 1_7^{-1}1_9^{-1}1_{11}^{-1}2_6 + 2_{12}^{-1}.
\end{aligned}$$

For $s\in\mathbb{Z}$, $\chi_q(1_s)$ and $\chi_q(2_s)$ are obtained by shift all indices by s in $\chi_q(1_0)$ and $\chi_q(2_0)$ respectively.

It is convenient to keep in mind the following lemma.

Lemma 3.2. *If $b\in\mathbb{Z}\backslash\{\pm2,\pm8,\pm12\}$, then*

$$L(1_01_b) = L(1_0)\otimes L(1_b), \qquad \dim L(1_01_b) = 49.$$

If $b\in\mathbb{Z}\backslash\{\pm6,\pm8,\pm10,\pm12\}$, then

$$L(2_02_b) = L(2_0)\otimes L(2_b), \qquad \dim L(2_02_b) = 225.$$

If $b\in\mathbb{Z}\backslash\{\pm7,\pm11\}$, then

$$\begin{aligned}
&L(1_02_b) = L(1_0)\otimes L(2_b), \qquad L(2_01_b) = L(2_0)\otimes L(1_b),\\
&\dim L(1_02_b) = \dim L(2_01_b) = 105.
\end{aligned}$$

In addition, we have

$$\begin{aligned}
&\dim L(1_01_2) = 34, \qquad \dim L(1_01_8) = 42, \qquad \dim L(1_01_{12}) = 48,\\
&\dim L(2_02_6) = 92, \qquad \dim L(2_02_8) = 210, \qquad \dim L(2_02_{10}) = 183, \qquad \dim L(2_02_{12}) = 224,\\
&\dim L(1_02_7) = \dim L(2_01_7) = 71, \qquad \dim L(1_02_{11}) = \dim L(2_01_{11}) = 98.
\end{aligned}$$

Proof. By Lemma 3.1, the tensor products in the first three cases of the lemma are special. Therefore the tensor products are irreducible. Hence the first three cases of the lemma are true. The last part of the lemma can be proved using the methods of Section 5. In fact some of the dimensions follow from Theorem 8.1. We do not use this lemma in the proofs. Therefore we omit the details of the proof. ∎

3.2 Definition of the modules $\mathcal{B}_{k,\ell}^{(s)}$, $\mathcal{C}_{k,\ell}^{(s)}$, $\mathcal{D}_{k,\ell}^{(s)}$, $\mathcal{E}_{k,\ell}^{(s)}$, $\mathcal{F}_{k,\ell}^{(s)}$

For $s \in \mathbb{Z}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, define the following monomials.

$$B_{k,\ell}^{(s)} = \left(\prod_{i=0}^{k-1} 2_{s+6i}\right)\left(\prod_{i=0}^{\ell-1} 1_{s+6k+2i+1}\right), \qquad C_{k,\ell}^{(s)} = \left(\prod_{i=0}^{k-1} 2_{s+6i}\right)\left(\prod_{i=0}^{\ell-1} 2_{s+6k+6i+4}\right),$$

$$D_{k,\ell}^{(s)} = \left(\prod_{i=0}^{k-1} 2_{s+6i}\right) 1_{s+6k+1}\left(\prod_{i=0}^{\ell-1} 2_{s+6k+6i+8}\right), \qquad F_{k,\ell}^{(s)} = \left(\prod_{i=0}^{k-1} 1_{s+2i}\right)\left(\prod_{i=0}^{\ell-1} 1_{s+2k+2i+6}\right),$$

$$E_{k,\ell}^{(s)} = \left(\prod_{i=0}^{k-1} 1_{s+2i}\right)\left(\prod_{i=0}^{\lfloor\frac{\ell-1}{2}\rfloor} 2_{s+2k+6i+3}\right)\left(\prod_{i=0}^{\lfloor\frac{\ell-2}{2}\rfloor} 2_{s+2k+6i+5}\right).$$

Note that, in particular, for $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$, we have the following trivial relations

$$\mathcal{B}_{k,0}^{(s)} = \mathcal{C}_{k,0}^{(s)} = \mathcal{C}_{0,k}^{(s-4)}, \qquad \mathcal{D}_{k,0}^{(s)} = \mathcal{B}_{k,1}^{(s)}, \qquad \mathcal{E}_{k,0}^{(s)} = \mathcal{B}_{0,k}^{(s-1)} = \mathcal{F}_{0,k}^{(s-6)} = \mathcal{F}_{k,0}^{(s)}. \tag{3.1}$$

Denote by $\mathcal{B}_{k,\ell}^{(s)}$, $\mathcal{C}_{k,\ell}^{(s)}$, $\mathcal{D}_{k,\ell}^{(s)}$, $\mathcal{E}_{k,\ell}^{(s)}$, $\mathcal{F}_{k,\ell}^{(s)}$ the irreducible finite-dimensional highest l-weight $U_q\hat{\mathfrak{g}}$-modules with highest l-weight $B_{k,\ell}^{(s)}, C_{k,\ell}^{(s)}, D_{k,\ell}^{(s)}, E_{k,\ell}^{(s)}, F_{k,\ell}^{(s)}$ respectively.

Note that $\mathcal{B}_{k,\ell}^{(s)}$, $\mathcal{D}_{0,\ell}^{(s)}$, $\mathcal{D}_{k,0}^{(s)}$ are minimal affinizations. The modules $\mathcal{B}_{0,\ell}^{(s)}$, $\mathcal{C}_{0,\ell}^{(s)}$, $\mathcal{F}_{0,\ell}^{(s)}$, $\mathcal{B}_{k,0}^{(s)}$, $\mathcal{C}_{k,0}^{(s)}$, $\mathcal{E}_{k,0}^{(s)}$, $\mathcal{F}_{k,0}^{(s)}$ are Kirillov–Reshetikhin modules.

Our first result is

Theorem 3.3. *The modules $\mathcal{B}_{k,\ell}^{(s)}$, $\mathcal{C}_{k,\ell}^{(s)}$, $\mathcal{D}_{k,\ell}^{(s)}$, $\mathcal{E}_{k,\ell}^{(s)}$, $\mathcal{F}_{k,\ell}^{(s)}$, $s \in \mathbb{Z}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, are special. In particular, the FM algorithm works for all these modules.*

We prove Theorem 3.3 in Section 4. Note that the case of $\mathcal{B}_{k,\ell}^{(s)}$ has been proved in Theorem 3.8 of [11]. In general, the modules in Theorem 3.3 are not thin. For example, $\chi_q(1_0 1_2)$ has monomial $1_4 1_6 1_8^{-1} 1_{10}^{-1}$ with coefficient 2.

3.3 Extended T-system

It is known that Kirillov–Reshetikhin modules $\mathcal{B}_{k,0}^{(s)}$, $\mathcal{B}_{0,\ell}^{(s)}$ satisfy the following T-system relations, see [16],

$$\left[\mathcal{B}_{0,\ell}^{(s)}\right]\left[\mathcal{B}_{0,\ell}^{(s+2)}\right] = \left[\mathcal{B}_{0,\ell+1}^{(s)}\right]\left[\mathcal{B}_{0,\ell-1}^{(s+2)}\right] + \left[\mathcal{B}_{\lfloor\frac{\ell+2}{3}\rfloor,0}^{(s+1)}\right]\left[\mathcal{B}_{\lfloor\frac{\ell+1}{3}\rfloor,0}^{(s+3)}\right]\left[\mathcal{B}_{\lfloor\frac{\ell}{3}\rfloor,0}^{(s+5)}\right], \tag{3.2}$$

$$\left[\mathcal{B}_{k,0}^{(s)}\right]\left[\mathcal{B}_{k,0}^{(s+6)}\right] = \left[\mathcal{B}_{k+1,0}^{(s)}\right]\left[\mathcal{B}_{k-1,0}^{(s+6)}\right] + \left[\mathcal{B}_{0,3k}^{(s+1)}\right], \tag{3.3}$$

where $s \in \mathbb{Z}$, $k, \ell \in \mathbb{Z}_{\geq 1}$.

Our main result is

Theorem 3.4. *For $s \in \mathbb{Z}$ and $k, \ell \in \mathbb{Z}_{\geq 1}$, $t \in \mathbb{Z}_{\geq 2}$, we have the following relations in* $\mathrm{Rep}(U_q\hat{\mathfrak{g}})$*:*

$$\left[\mathcal{B}_{k,\ell-1}^{(s)}\right]\left[\mathcal{B}_{k-1,\ell}^{(s+6)}\right] = \left[\mathcal{B}_{k,\ell}^{(s)}\right]\left[\mathcal{B}_{k-1,\ell-1}^{(s+6)}\right] + \left[\mathcal{E}_{3k-1,\lceil\frac{2\ell-2}{3}\rceil}^{(s+1)}\right]\left[\mathcal{B}_{\lfloor\frac{\ell-1}{3}\rfloor,0}^{(s+6k+6)}\right], \tag{3.4}$$

$$\left[\mathcal{E}_{0,\ell}^{(s)}\right] = \left[\mathcal{B}_{\lfloor\frac{\ell+1}{2}\rfloor,0}^{(s+3)}\right]\left[\mathcal{B}_{\lfloor\frac{\ell}{2}\rfloor,0}^{(s+5)}\right], \tag{3.5}$$

$$\left[\mathcal{E}_{1,\ell}^{(s)}\right] = \left[\mathcal{D}_{0,\lfloor\frac{\ell}{2}\rfloor}^{(s-1)}\right]\left[\mathcal{B}_{\lfloor\frac{\ell+1}{2}\rfloor,0}^{(s+5)}\right], \tag{3.6}$$

$$\left[\mathcal{E}_{t,\ell-1}^{(s)}\right]\left[\mathcal{E}_{t-1,\ell}^{(s+2)}\right] = \left[\mathcal{E}_{t,\ell}^{(s)}\right]\left[\mathcal{E}_{t-1,\ell-1}^{(s+2)}\right]$$

$$+\begin{cases} \left[\mathcal{D}^{(s+1)}_{r,p-1}\right]\left[\mathcal{B}^{(s+3)}_{r+p,0}\right]\left[\mathcal{B}^{(s+5)}_{r,3p-2}\right], & \textit{if} \quad t=3r+2,\ \ell=2p-1, \\ \left[\mathcal{B}^{(s+1)}_{r+p+1,0}\right]\left[\mathcal{C}^{(s+3)}_{r,p}\right]\left[\mathcal{B}^{(s+5)}_{r,3p-1}\right], & \textit{if} \quad t=3r+2,\ \ell=2p, \\ \left[\mathcal{B}^{(s+1)}_{r+1,3p-2}\right]\left[\mathcal{D}^{(s+3)}_{r,p-1}\right]\left[\mathcal{B}^{(s+5)}_{r+p,0}\right], & \textit{if} \quad t=3r+3,\ \ell=2p-1, \\ \left[\mathcal{B}^{(s+1)}_{r+1,3p-1}\right]\left[\mathcal{B}^{(s+3)}_{r+p+1,0}\right]\left[\mathcal{C}^{(s+5)}_{r,p}\right], & \textit{if} \quad t=3r+3,\ \ell=2p, \\ \left[\mathcal{B}^{(s+1)}_{r+p+1,0}\right]\left[\mathcal{B}^{(s+3)}_{r+1,3p-2}\right]\left[\mathcal{D}^{(s+5)}_{r,p-1}\right], & \textit{if} \quad t=3r+4,\ \ell=2p-1, \\ \left[\mathcal{C}^{(s+1)}_{r+1,p}\right]\left[\mathcal{B}^{(s+3)}_{r+1,3p-1}\right]\left[\mathcal{B}^{(s+5)}_{r+p+1,0}\right], & \textit{if} \quad t=3r+4,\ \ell=2p, \end{cases} \tag{3.7}$$

$$\left[\mathcal{C}^{(s)}_{k,\ell-1}\right]\left[\mathcal{C}^{(s+6)}_{k-1,\ell}\right] = \left[\mathcal{C}^{(s)}_{k,\ell}\right]\left[\mathcal{C}^{(s+6)}_{k-1,\ell-1}\right] + \left[\mathcal{F}^{(s+1)}_{3k-2,3\ell-2}\right], \tag{3.8}$$

$$\left[\mathcal{D}^{(s)}_{0,\ell-1}\right]\left[\mathcal{B}^{(s+8)}_{\ell,0}\right] = \left[\mathcal{D}^{(s)}_{0,\ell}\right]\left[\mathcal{B}^{(s+8)}_{\ell-1,0}\right] + \left[\mathcal{B}^{(s+4)}_{0,3\ell-1}\right], \tag{3.9}$$

$$\left[\mathcal{D}^{(s)}_{k,\ell-1}\right]\left[\mathcal{D}^{(s+6)}_{k-1,\ell}\right] = \left[\mathcal{D}^{(s)}_{k,\ell}\right]\left[\mathcal{D}^{(s+6)}_{k-1,\ell-1}\right] + \left[\mathcal{F}^{(s+1)}_{3k-1,3\ell-1}\right], \tag{3.10}$$

$$\left[\mathcal{F}^{(s)}_{k,\ell-1}\right]\left[\mathcal{F}^{(s+2)}_{k-1,\ell}\right] = \left[\mathcal{F}^{(s)}_{k,\ell}\right]\left[\mathcal{F}^{(s+2)}_{k-1,\ell-1}\right] \tag{3.11}$$

$$+\begin{cases} \left[\mathcal{B}^{(s+1)}_{r,0}\right]\left[\mathcal{D}^{(s+3)}_{r,\lfloor\frac{\ell}{3}\rfloor}\right]\left[\mathcal{C}^{(s+5)}_{r,\lfloor\frac{\ell+1}{3}\rfloor}\right]\left[\mathcal{B}^{(s+2k+11)}_{\lfloor\frac{\ell-1}{3}\rfloor,0}\right], & \textit{if} \quad k=3r+1, \\ \left[\mathcal{C}^{(s+1)}_{r+1,\lfloor\frac{\ell+1}{3}\rfloor}\right]\left[\mathcal{B}^{(s+3)}_{r,0}\right]\left[\mathcal{D}^{(s+5)}_{r,\lfloor\frac{\ell}{3}\rfloor}\right]\left[\mathcal{B}^{(s+2k+11)}_{\lfloor\frac{\ell-1}{3}\rfloor,0}\right], & \textit{if} \quad k=3r+2, \\ \left[\mathcal{D}^{(s+1)}_{r+1,\lfloor\frac{\ell}{3}\rfloor}\right]\left[\mathcal{C}^{(s+3)}_{r+1,\lfloor\frac{\ell+1}{3}\rfloor}\right]\left[\mathcal{B}^{(s+5)}_{r,0}\right]\left[\mathcal{B}^{(s+2k+11)}_{\lfloor\frac{\ell-1}{3}\rfloor,0}\right], & \textit{if} \quad k=3r+3. \end{cases}$$

We prove Theorem 3.4 in Section 5.

Note that since $D^{(s)}_{k,0} = B^{(s)}_{k,1}$, equations for $\mathcal{D}^{(s)}_{k,0}$ are included in the equations for $\mathcal{B}^{(s)}_{k,1}$.

All relations except (3.5), (3.6) in Theorem 3.4 are written in the form $[\mathcal{L}][\mathcal{R}] = [\mathcal{T}][\mathcal{B}] + [\mathcal{S}]$, where $\mathcal{L}$, $\mathcal{R}$, $\mathcal{T}$, $\mathcal{B}$ are irreducible modules which we call *left, right, top and bottom modules* and $\mathcal{S}$ is a tensor product of some irreducible modules. We call the factors of $\mathcal{S}$ *sources*. Moreover, we have the following theorem.

Theorem 3.5. *For each relation in Theorem 3.4, all summands on the right hand side, $\mathcal{T}\otimes\mathcal{B}$ and $\mathcal{S}$, are irreducible.*

We will prove Theorem 3.5 in Section 6.

Recall that the q-characters of modules for different s are related by the simple shift of indexes, see (2.6).

We have the following proposition.

Proposition 3.6. *Given $\chi_q(1_s)$, $\chi_q(2_s)$, one can obtain the q-characters of $\mathcal{B}^{(s)}_{k,\ell}$, $\mathcal{C}^{(s)}_{k,\ell}$, $\mathcal{D}^{(s)}_{k,\ell}$, $\mathcal{E}^{(s)}_{k,\ell}$, $\mathcal{F}^{(s)}_{k,\ell}$, $s\in\mathbb{Z}$, $k,\ell\in\mathbb{Z}_{\geq 0}$, recursively, by using (3.1), and computing the q-character of the top module through the q-characters of other modules in relations in Theorem 3.4.*

Proof. Claim 1. Let n, m be positive integers. Then the q-characters

$$\chi_q\left(\mathcal{B}^{(s)}_{k,\ell}\right), \quad k\leq n, \quad \ell\leq m, \qquad \chi_q\left(\mathcal{C}^{(s)}_{k,\ell}\right), \quad k\leq n-1, \quad \ell\leq\left\lceil\frac{2m+1}{6}\right\rceil,$$
$$\chi_q\left(\mathcal{D}^{(s)}_{k,\ell}\right), \quad k\leq n-1, \quad \ell\leq\left\lceil\frac{2m+1}{6}\right\rceil, \qquad \chi_q\left(\mathcal{E}^{(s)}_{k,\ell}\right), \quad k\leq 3n-1, \quad \ell\leq\left\lceil\frac{2m-2}{3}\right\rceil,$$
$$\chi_q\left(\mathcal{F}^{(s)}_{k,\ell}\right), \quad k\leq 3n-4, \quad \ell\leq m+2,$$

can be computed recursively starting from $\chi_q(1_0)$, $\chi_q(2_0)$.

We use induction on n, m to prove Claim 1. For simplicity, we do not write the upper-subscripts "(s)" in the remaining part of the proof. We know that, see [10], the q-characters of Kirillov–Reshetikhin modules can be computed from $\chi_q(1_0)$, $\chi_q(2_0)$.

When $n=0$, $m=1$, Claim 1 is clearly true. It is clear that $\chi_q(\mathcal{D}_{0,1})$ can be computed using (3.9). Therefore Claim 1 holds for $n=1$, $m=0$,

Suppose that for $n\le n_1$ and $m\le m_1$, Claim 1 is true. Let $n=n_1+1$, $m=m_1$. We need to show that Claim 1 is true. Then we need to show that

$$\begin{aligned}
&\chi_q(\mathcal{B}_{n_1+1,\ell}),\quad \ell\le m_1,\qquad \chi_q(\mathcal{C}_{n_1,\ell}),\quad \ell\le\left\lceil\frac{2m_1+1}{6}\right\rceil,\qquad \chi_q(\mathcal{D}_{n_1,\ell}),\quad \ell\le\left\lceil\frac{2m_1+1}{6}\right\rceil,\\
&\chi_q(\mathcal{E}_{k,\ell}),\quad k=3n_1,\quad 3n_1+1,\quad 3n_1+2,\quad \ell\le\left\lceil\frac{2m_1-2}{3}\right\rceil,\\
&\chi_q(\mathcal{F}_{k,\ell}),\quad k=3n_1-3,\quad 3n_1-2,\quad 3n_1-1,\quad \ell\le m_1+2,
\end{aligned}$$

can be computed.

We compute the following modules

$$\begin{aligned}
&\chi_q(\mathcal{F}_{3n_1-3,\ell}),\quad \ell\le m_1+2,\qquad \chi_q(\mathcal{F}_{3n_1-2,\ell}),\quad \ell\le m_1+2,\\
&\chi_q(\mathcal{C}_{n_1,\ell}),\quad \ell\le\left\lfloor\frac{m_1+3}{3}\right\rfloor,\qquad \chi_q(\mathcal{F}_{3n_1-1,\ell}),\quad \ell\le m_1+2,\\
&\chi_q(\mathcal{D}_{n_1,\ell}),\quad \ell\le\left\lceil\frac{2m_1+1}{6}\right\rceil,\qquad \chi_q(\mathcal{C}_{n_1,\ell}),\quad \ell\le\left\lceil\frac{2m_1+1}{6}\right\rceil,\\
&\chi_q(\mathcal{E}_{3n_1,\ell}),\quad \ell\le\left\lceil\frac{2m_1-2}{3}\right\rceil,\qquad \chi_q(\mathcal{E}_{3n_1,\ell}),\quad \ell\le\left\lceil\frac{2m_1-2}{3}\right\rceil,\\
&\chi_q(\mathcal{E}_{3n_1+1,\ell}),\quad \ell\le\left\lceil\frac{2m_1-2}{3}\right\rceil,\qquad \chi_q(\mathcal{E}_{3n_1+2,\ell}),\quad \ell\le\left\lceil\frac{2m_1-2}{3}\right\rceil,\\
&\chi_q(\mathcal{B}_{n_1+1,\ell}),\quad \ell\le m_1
\end{aligned}$$

in the order as shown. At each step, we consider the module that we want to compute as a top module and use the corresponding relation in Theorem 3.4 and known q-characters. For example, we consider the first set of modules $\chi_q(\mathcal{F}_{3n_1-3,\ell})$, $\ell\le m_1+2$. Since $\lfloor\frac{m_1+3}{3}\rfloor\le\lceil\frac{2m_1+1}{6}\rceil$, $\chi_q(\mathcal{C}_{n_1-1,\ell})$, $\ell\le\lfloor\frac{m_1+3}{3}\rfloor$, is known by induction hypothesis. Similarly, $\chi_q(\mathcal{D}_{n_1-1,\ell})$, $\ell\le\lfloor\frac{m_1+2}{3}\rfloor$ is known. Therefore $\chi_q(\mathcal{F}_{3n_1-3,\ell})$, $\ell\le m_1+2$, is computed using the last equation of (3.11).

Similarly, we show that Claim 1 holds for $n=n_1$, $m=m_1+1$. Therefore Claim 1 is true for all $n\ge 1$, $m\ge 1$. ■

4 Proof of Theorem 3.3

In this section, we will show that the modules $\mathcal{B}^{(s)}_{k,\ell}$, $\mathcal{C}^{(s)}_{k,\ell}$, $\mathcal{D}^{(s)}_{k,\ell}$, $\mathcal{E}^{(s)}_{k,\ell}$, $\mathcal{F}^{(s)}_{k,\ell}$ are special.

Since $\mathcal{B}^{(s)}_{0,\ell}$, $\mathcal{C}^{(s)}_{0,\ell}$, $\mathcal{F}^{(s)}_{0,\ell}$, $\mathcal{B}^{(s)}_{k,0}$, $\mathcal{C}^{(s)}_{k,0}$, $\mathcal{E}^{(s)}_{k,0}$, $\mathcal{F}^{(s)}_{k,0}$ are Kirillov–Reshetikhin modules, they are special.

4.1 The case of $\mathcal{C}^{(s)}_{k,\ell}$

Let $m_+=C^{(s)}_{k,\ell}$ with $k,\ell\in\mathbb{Z}_{\ge1}$. Without loss of generality, we can assume that $s=6$. Then

$$m_+=(2_6 2_{12}\cdots 2_{6k})(2_{6k+10}2_{6k+16}\cdots 2_{6k+6\ell+4}).$$

Case 1. $k=1$. Let $U=I\times\{aq^s: s\in\mathbb{Z}, s<6\ell+13\}$. Clearly, all monomials in $\chi_q(m_+)-\operatorname{trunc}_{m_+\mathcal{Q}_U^-}\chi_q(m_+)$ are right-negative. Therefore it is sufficient to show that $\operatorname{trunc}_{m_+\mathcal{Q}_U^-}\chi_q(m_+)$ is special.

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+, \qquad m_1 = m_0 A_{2,9}^{-1}, \qquad m_2 = m_1 A_{1,12}^{-1},$$
$$m_3 = m_2 A_{1,10}^{-1}, \qquad m_4 = m_3 A_{1,8}^{-1}, \qquad m_5 = m_4 A_{2,11}^{-1}.$$

It is clear that $\mathcal{M}$ satisfies the conditions in Theorem 2.2. Therefore

$$\mathrm{trunc}_{m_+\mathcal{Q}_U^-}\, \chi_q(m_+) = \sum_{m\in\mathcal{M}} m$$

and $\mathrm{trunc}_{m_+\mathcal{Q}_U^-}\, \chi_q(m_+)$ is special.

Case 2. $k > 1$. Since the conditions of Theorem 2.2 do not apply to this case, we use another technique to show that $L(m_+)$ is special. We embed $L(m_+)$ into two different tensor products. In both tensor products, each factor is special. Therefore we can use the FM algorithm to compute the q-characters of the factors. We classify the dominant monomials in the first tensor product and show that the only dominant monomial in the first tensor product which occurs in the second tensor product is m_+ which proves that $L(m_+)$ is special.

The first tensor product is $L(m_1') \otimes L(m_2')$, where

$$m_1' = 2_6 2_{12} \cdots 2_{6k}, \qquad m_2' = 2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell+4}.$$

We use the FM algorithm to compute $\chi_q(m_1'), \chi_q(m_2')$ and classify all dominant monomials in $\chi_q(m_1')\chi_q(m_2')$. Let $m = m_1 m_2$ be a dominant monomial, where $m_i \in \chi_q(m_i')$, $i = 1, 2$. If $m_2 \neq m_2'$, then m is a right negative monomial therefore m is not dominant. Hence $m_2 = m_2'$.

If $m_1 \neq m_1'$, then m_1 is right negative. Since m is dominant, each factor with a negative power in m_1 needs to be canceled by a factor in m_2'. All possible cancellations cancel 2_{6k+10} in m_2'. We have $\mathcal{M}(L(m_1')) \subset \mathcal{M}(\chi_q(2_6 2_{12} \cdots 2_{6k-6})\chi_q(2_{6k}))$. Only monomials in $\chi_q(2_{6k})$ can cancel 2_{6k+10}. These monomials are $1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}$, $1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1}$, and $1_{6k+5} 1_{6k+7} 1_{6k+9} 2_{6k+8}^{-1} 2_{6k+10}^{-1}$. Therefore m_1 is in one of the following polynomials

$$\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}, \tag{4.1}$$
$$\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1}, \tag{4.2}$$
$$\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+5} 1_{6k+7} 1_{6k+9} 2_{6k+8}^{-1} 2_{6k+10}^{-1}. \tag{4.3}$$

Subcase 2.1. Let m_1 be in (4.1). If $m_1 = 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}$, then

$$m = m_1 m_2 = 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+1} 1_{6k+9} 2_{6k+16} \cdots 2_{6k+6\ell+4} \tag{4.4}$$

is dominant. Suppose that

$$m_1 \neq 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}.$$

Then $m_1 = n_1 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}$, where n_1 is a non-highest monomial in $\chi_q(2_6 2_{12} \cdots 2_{6k-6})$. Since n_1 is right negative, 1_{6k+1} or 1_{6k+9} should cancel a factor of n_1 with a negative power. Using the FM algorithm, we see that there exists a factor 1_{6k-1}^2 or 1_{6k+7}^2 in a monomial in $\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}$. By Lemma 3.1, neither 1_{6k-1}^2 nor 1_{6k+7}^2 appear. This is a contradiction.

Subcase 2.2. Let m_1 be in (4.2). If $m_1 = 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1}$, then $m = m_1 m_2$ is not dominant. Suppose that $m_1 \neq 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1}$. Then $m_1 = n_1 1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1}$, where n_1 is a non-highest monomial in $\chi_q(2_6 2_{12} \cdots 2_{6k-6})$. Since n_1 is right negative, 1_{6k+9} or 2_{6k+2} should cancel a factor of n_1 with a negative power. Using the

FM algorithm, we see that there exists either a factor 1^2_{6k+7} or a factor 2^2_{6k-4} in a monomial in $\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+1} 1_{6k+9} 2^{-1}_{6k+10}$. By Lemma 3.1, neither 1^2_{6k+7} nor 2^2_{6k-4} appear. This is a contradiction.

Subcase 2.3. Let m_1 be in (4.3). If $m_1 = 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+5} 1_{6k+7} 1_{6k+9} 2^{-1}_{6k+8} 2^{-1}_{6k+10}$, then $m = m_1 m_2$ is not dominant. Suppose that $m_1 \neq 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+5} 1_{6k+7} 1_{6k+9} 2^{-1}_{6k+8} 2^{-1}_{6k+10}$. Then we have $m_1 = n_1 1_{6k+5} 1_{6k+7} 1_{6k+9} 2^{-1}_{6k+8} 2^{-1}_{6k+10}$, where n_1 is a non-highest monomial in $\chi_q(2_6 2_{12} \cdots 2_{6k-6})$. Since n_1 is right negative, 1_{6k+5} or 1_{6k+7} or 1_{6k+9} should cancel a factor of n_1 with a negative power. Using the FM algorithm, we see that there exists a factor 1_{6k+7} or 1_{6k+5} or 1^2_{6k+3} in a monomial in $\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+1} 1_{6k+9} 2^{-1}_{6k+10}$. By Lemma 3.1, 1_{6k+7}, 1_{6k+5}, and 1^2_{6k+3} do not appear. This is a contradiction.

Therefore the only dominant monomials in $\chi_q(m'_1)\chi_q(m'_2)$ are m_+ and (4.4).

The second tensor product is $L(m''_1) \otimes L(m''_2)$, where

$$m''_1 = 2_6 2_{12} \cdots 2_{6k-6}, \qquad m''_2 = 2_{6k} 2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell+4}.$$

The monomial (4.4) is

$$n = m_+ A^{-1}_{2,6k+3} A^{-1}_{1,6k+6} A^{-1}_{1,6k+4} A^{-1}_{2,6k+7}. \tag{4.5}$$

Since $A_{i,a}$, $i \in I$, $a \in \mathbb{C}^\times$ are algebraically independent, the expression (4.5) of n of the form $m_+ \prod_{i \in I,\, a \in \mathbb{C}^\times} A^{-v_{i,a}}_{i,a}$, where $v_{i,a}$ are some integers, is unique. Suppose that the monomial n is in $\chi_q(m''_1)\chi_q(m''_2)$. Then $n = n_1 n_2$, where $n_i \in \chi_q(m''_i)$, $i = 1, 2$. By the expression (4.5), we have $n_1 = m''_1$ and

$$n_2 = m''_2 A^{-1}_{2,6k+3} A^{-1}_{1,6k+6} A^{-1}_{1,6k+4} A^{-1}_{2,6k+7}.$$

By the FM algorithm, the monomial $m''_2 A^{-1}_{2,6k+3} A^{-1}_{1,6k+6} A^{-1}_{1,6k+4} A^{-1}_{2,6k+7}$ is not in $\chi_q(m''_2)$. This contradicts the fact that $n_2 \in \chi_q(m''_2)$. Therefore n is not in $\chi_q(m''_1)\chi_q(m''_2)$.

4.2 The case of $\mathcal{B}^{(s)}_{k,\ell}$

Let $m_+ = B^{(s)}_{k,\ell}$ with $k, \ell \in \mathbb{Z}_{\geq 1}$. Without loss of generality, we can assume that $s = 6$. Then

$$m_+ = (2_6 2_{12} \cdots 2_{6k})(1_{6k+7} 1_{6k+9} \cdots 1_{6k+2\ell+5}).$$

Let $U = I \times \{aq^s : s \in \mathbb{Z}, s < 6k + 2\ell + 6\}$. Clearly, all monomials in the polynomial $\chi_q(m_+) - \operatorname{trunc}_{m_+\mathcal{Q}^-_U} \chi_q(m_+)$ are right-negative. Therefore it is sufficient to show that the truncated q-character $\operatorname{trunc}_{m_+\mathcal{Q}^-_U} \chi_q(m_+)$ is special.

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+, \qquad m_1 = m_0 A^{-1}_{2,6k+3}, \quad m_2 = m_1 A^{-1}_{2,6k-3}, \quad \ldots, \quad m_k = m_{k-1} A^{-1}_{2,9}.$$

It is clear that $\mathcal{M}$ satisfies the conditions in Theorem 2.2. Therefore

$$\operatorname{trunc}_{m_+\mathcal{Q}^-_U} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and $\operatorname{trunc}_{m_+\mathcal{Q}^-_U} \chi_q(m_+)$ is special.

4.3 The case of $\mathcal{D}_{k,\ell}^{(s)}$

Let $m_+ = D_{k,\ell}^{(s)}$ with k, $\ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we can assume that $s = 0$. Then

$$m_+ = (2_0 2_6 \cdots 2_{6k-6}) 1_{6k+1} (2_{6k+8} 2_{6k+14} \cdots 2_{6k+6\ell+2}).$$

Case 1. $k = 0$. Let $U = I \times \{aq^s : s \in \mathbb{Z}, s < 6\ell + 5\}$. Clearly, all monomials in $\chi_q(m_+) - \operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ are right-negative. Therefore it is sufficient to show that $\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ is special.

Let

$$M = \{m_+, m_+ A_{1,2}^{-1}\}.$$

It is clear that $\mathcal{M}$ satisfies the conditions in Theorem 2.2. Therefore

$$\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and $\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ is special.

Case 2. $k > 0$. Let

$$m_1' = 2_0 2_6 \cdots 2_{6k-6} 1_{6k+1}, \qquad m_2' = 2_{6k+8} 2_{6k+14} \cdots 2_{6k+6\ell+2},$$
$$m_1'' = 2_0 2_6 \cdots 2_{6k-6}, \qquad m_2'' = 1_{6k+1} 2_{6k+8} 2_{6k+14} \cdots 2_{6k+6\ell+2}.$$

Then $\mathscr{M}(L(m_+)) \subset \mathscr{M}(\chi_q(m_1')\chi_q(m_2')) \cap \mathscr{M}(\chi_q(m_1'')\chi_q(m_2''))$.

By using similar arguments as the case of $\mathcal{C}_{k,\ell}^{(s)}$, we show that the only dominant monomials in $\chi_q(m_1')\chi_q(m_2')$ are m_+ and

$$n = 2_0 2_6 \cdots 2_{6k-6} 1_{6k+5} 1_{6k+7} 2_{6k+14} 2_{6k+20} \cdots 2_{6k+6\ell+2} = m_+ A_{1,6k+2}^{-1} A_{2,6k+5}^{-1}.$$

Moreover, n is not in $\chi_q(m_1'')\chi_q(m_2'')$. Therefore the only dominant monomial in $\chi_q(m_+)$ is m_+.

4.4 The case of $\mathcal{E}_{k,\ell}^{(s)}$

Let $m_+ = E_{k,\ell}^{(s)}$ with k, $\ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we can assume that $s = 1$. Suppose that $\ell = 2r + 1$, $r \geq 0$ and $k = 3p$, $p \geq 1$. The cases of $\ell = 2r$, $r \geq 1$, or $k = 0$ or $k = 3p + 1$, $p \geq 0$ or $k = 3p + 2$, $p \geq 0$ are similar.

Then

$$m = (1_1 1_3 \cdots 1_{6p-1})(2_{6p+4} 2_{6p+10} \cdots 2_{6p+6r-2} 2_{6p+6r+4})(2_{6p+6} 2_{6p+12} \cdots 2_{6p+6r}).$$

Let $U = I \times \{aq^s : s \in \mathbb{Z}, s < 6p + 6r + 3\}$. Clearly, all monomials in the polynomial $\chi_q(m_+) - \operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ are right-negative. Therefore it is sufficient to show that the truncated q-character $\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ is special.

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+, \quad m_1 = m_0 A_{1,6p}^{-1}, \quad m_2 = m_1 A_{1,6p-2}^{-1}, \quad \ldots, \quad m_{3p} = m_{3p-1} A_{1,2}^{-1},$$
$$m_{3p+1} = m_{3p} A_{2,6p-4}^{-1}, \quad m_{3p+2} = m_{3p+1} A_{2,6p-10}^{-1}, \quad \ldots, \quad m_{4p} = m_{4p-2} A_{2,6}^{-1}.$$

It is clear that $\mathcal{M}$ satisfies the conditions in Theorem 2.2. Therefore

$$\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and $\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ is special.

4.5 The case of $\mathcal{F}_{k,\ell}^{(s)}$

Let $m_+ = F_{k,\ell}^{(s)}$ with $k,\ \ell \in \mathbb{Z}_{\geq 1}$. Without loss of generality, we can assume that $s = 1$. Then

$$m_+ = (1_1 1_3 \cdots 1_{2k-1})(1_{2k+7} 1_{2k+9} \cdots 1_{2k+2\ell+5}).$$

Case 1. $k = 1$. Let $U = I \times \{aq^s : s \in \mathbb{Z}, s < 2\ell + 8\}$. Clearly, all monomials in $\chi_q(m_+) - \operatorname{trunc}_{m_+\mathcal{Q}_U^-} \chi_q(m_+)$ are right-negative. Therefore it is sufficient to show that $\operatorname{trunc}_{m_+\mathcal{Q}_U^-} \chi_q(m_+)$ is special.

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+, \qquad m_1 = m_0 A_{1,2}^{-1}, \qquad m_2 = m_1 A_{2,5}^{-1}.$$

It is clear that $\mathcal{M}$ satisfies the conditions in Theorem 2.2. Therefore

$$\operatorname{trunc}_{m_+\mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and $\operatorname{trunc}_{m_+\mathcal{Q}_U^-} \chi_q(m_+)$ is special.

Case 2. $k > 1$. Let

$$m_1' = 1_1 1_3 \cdots 1_{2k-1}, \qquad m_2' = 1_{2k+7} 1_{2k+9} \cdots 1_{2k+2\ell+5},$$
$$m_1'' = 1_1 1_3 \cdots 1_{2k-3}, \qquad m_2'' = 1_{2k-1} 1_{2k+7} 1_{2k+9} \cdots 1_{2k+2\ell+5}.$$

Then $\mathscr{M}(L(m_+)) \subset \mathscr{M}(\chi_q(m_1')\chi_q(m_2')) \cap \mathscr{M}(\chi_q(m_1'')\chi_q(m_2''))$.

By using similar arguments as the case of $\mathcal{C}_{k,\ell}^{(s)}$, we can show that the only dominant monomials in $\chi_q(m_1')\chi_q(m_2')$ are m_+ and

$$n_1 = 1_1 1_3 \cdots 1_{2k-3} 1_{2k+3} 1_{2k+9} 1_{2k+11} \cdots 1_{2k+2\ell+5} = m_+ A_{1,2k}^{-1} A_{2,2k+3}^{-1} A_{1,2k+6}^{-1},$$
$$n_2 = 1_1 1_3 \cdots 1_{2k-3} 1_{2k+7} 1_{2k+9} 1_{2k+13} 1_{2k+15} \cdots 1_{2k+2\ell+5} = n_1 A_{1,2k+4}^{-1} A_{2,2k+7}^{-1} A_{1,2k+10}^{-1},$$
$$\begin{aligned} n_3 &= 1_1 1_3 \cdots 1_{2k-5} 1_{2k+7} 1_{2k+13} 1_{2k+15} \cdots 1_{2k+2\ell+5} \\ &= n_2 A_{1,2k-2}^{-1} A_{2,2k+1}^{-1} A_{1,2k+4}^{-1} A_{1,2k+2}^{-1} A_{2,2k+5}^{-1} A_{1,2k+8}^{-1}, \end{aligned}$$
$$\begin{aligned} n_4 &= 1_1 1_3 \cdots 1_{2k-7} 1_{2k+13} 1_{2k+15} \cdots 1_{2k+2\ell+5} \\ &= n_3 A_{1,2k-4}^{-1} A_{2,2k-1}^{-1} A_{1,2k+2}^{-1} A_{1,2k}^{-1} A_{2,2k+3}^{-1} A_{1,2k+6}^{-1}. \end{aligned}$$

Moreover, n_1, n_2, n_3, n_4 are not in $\chi_q(m_1')\chi_q(m_2')$. Therefore the only dominant monomial in $\chi_q(m_+)$ is m_+.

5 Proof of Theorem 3.4

We use the FM algorithm to classify dominant monomials in $\chi_q(\mathcal{L})\chi_q(\mathcal{R})$, $\chi_q(\mathcal{T})\chi_q(\mathcal{B})$, and $\chi_q(\mathcal{S})$.

5.1 Classification of dominant monomials in $\chi_q(\mathcal{L})\chi_q(\mathcal{R})$ and $\chi_q(\mathcal{T})\chi_q(\mathcal{B})$

Lemma 5.1. *We have the following cases.*

(1) *Let* $M = B_{k,\ell-1}^{(s)} B_{k-1,\ell}^{(s+6)}$, $k \geq 1$, $\ell \geq 1$. *Then dominant monomials in* $\chi_q\left(\mathcal{B}_{k,\ell-1}^{(s)}\right)\chi_q\left(\mathcal{B}_{k-1,\ell}^{(s+6)}\right)$ *are*

$$M_0 = M, \quad M_1 = M A_{1,s+6k+2\ell-2}^{-1},$$

$$M_2 = M_1 A^{-1}_{1,s+6k+2\ell-4}, \quad \dots, \quad M_{\ell-1} = M_{\ell-2} A^{-1}_{1,s+6k+2},$$
$$M_\ell = M_{\ell-1} A^{-1}_{2,s+6k-3} A^{-1}_{1,s+6k}, \quad M_{\ell+1} = M_\ell A^{-1}_{2,s+6k-9},$$
$$M_{\ell+2} = M_{\ell+1} A^{-1}_{2,s+6k-15}, \quad \dots, \quad M_{k+\ell-1} = M_{k+\ell-2} A^{-1}_{2,s+3}.$$

The dominant monomials in $\chi_q\left(\mathcal{B}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{B}^{(s+6)}_{k-1,\ell-1}\right)$ *are* $M_0, \dots, M_{k+\ell-2}$.

(2) *Let* $M = C^{(s)}_{k,\ell-1} C^{(s+6)}_{k-1,\ell}$, $k \geq 1$, $\ell \geq 1$. *Then dominant monomials in* $\chi_q\left(\mathcal{C}^{(s)}_{k,\ell-1}\right)\chi_q\left(\mathcal{C}^{(s+6)}_{k-1,\ell}\right)$ *are*

$$M_0 = M, \quad M_1 = M A^{-1}_{2,s+6k+6\ell-5}, \quad M_2 = M_1 A^{-1}_{2,s+6k+6\ell-11}, \quad \dots,$$
$$M_{\ell-1} = M_{\ell-2} A^{-1}_{2,s+6k+7}, \quad M_\ell = M_{\ell-1} A^{-1}_{2,s+6k-3} A^{-1}_{1,s+6k} A^{-1}_{1,s+6k-2} A^{-1}_{2,s+6k+1},$$
$$M_{\ell+1} = M_\ell A^{-1}_{2,s+6k-9}, \quad M_{\ell+2} = M_{\ell+1} A^{-1}_{2,s+6k-15}, \quad \dots, \quad M_{k+\ell-1} = M_{k+\ell-2} A^{-1}_{2,s+3}.$$

The dominant monomials in $\chi_q\left(\mathcal{C}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{C}^{(s+6)}_{k-1,\ell-1}\right)$ *are* $M_0, \dots, M_{k+\ell-2}$.

(3) *Let* $M = D^{(s)}_{0,\ell-1} B^{(s+8)}_{\ell,0}$, $\ell \geq 1$. *Then dominant monomials in* $\chi_q\left(\mathcal{D}^{(s)}_{0,\ell-1}\right)\chi_q\left(\mathcal{B}^{(s+8)}_{\ell,0}\right)$ *are*

$$M_0 = M, \quad M_1 = M A^{-1}_{2,s+6\ell-1}, \quad M_2 = M_1 A^{-1}_{2,s+6\ell-7}, \quad \dots,$$
$$M_{\ell-1} = M_{\ell-2} A^{-1}_{2,s+11}, \quad M_\ell = M_{\ell-1} A^{-1}_{1,s+6k+2} A^{-1}_{2,s+6k+5}.$$

The dominant monomials in $\chi_q(\mathcal{D}^{(s)}_{0,\ell})\chi_q(\mathcal{B}^{(s+8)}_{\ell-1,0})$ *are* $M_0, \dots, M_{\ell-1}$.

(4) *Let* $M{=}D^{(s)}_{k,\ell-1} D^{(s+6)}_{k-1,\ell}$, $k \geq 1$, $\ell \geq 1$. *Then dominant monomials in* $\chi_q\left(\mathcal{D}^{(s)}_{k,\ell-1}\right)\chi_q\left(\mathcal{D}^{(s+6)}_{k-1,\ell}\right)$ *are*

$$M_0 = M, \quad M_1 = M A^{-1}_{2,s+6k+6\ell-1},$$
$$M_2 = M_1 A^{-1}_{2,s+6k+6\ell-7}, \quad \dots, \quad M_{\ell-1} = M_{\ell-2} A^{-1}_{2,s+6k+11},$$
$$M_\ell = M_{\ell-1} A^{-1}_{1,s+6k+2} A^{-1}_{2,s+6k+5}, \quad M_{\ell+1} = M_\ell A^{-1}_{2,s+6k-3} A^{-1}_{1,s+6k},$$
$$M_{\ell+2} = M_{\ell+1} A^{-1}_{2,s+6k-9}, \quad \dots, \quad M_{k+\ell} = M_{k+\ell-1} A^{-1}_{2,s+3}.$$

The dominant monomials in $\chi_q\left(\mathcal{D}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{D}^{(s+6)}_{k-1,\ell-1}\right)$ *are* $M_0, \dots, M_{k+\ell-1}$.

(5) *Let* $M = E^{(s)}_{k,\ell-1} E^{(s+2)}_{k-1,\ell}$, $k \geq 1$, $\ell \geq 1$.

If $\ell = 2r+1$, *then dominant monomials in* $\chi_q\left(\mathcal{E}^{(s)}_{k,\ell-1}\right)\chi_q\left(\mathcal{E}^{(s+2)}_{k-1,\ell}\right)$ *are*

$$M_0 = M, \quad M_1 = M A^{-1}_{2,s+2k+3\ell-3},$$
$$M_2 = M_1 A^{-1}_{2,s+2k+3\ell-9}, \quad \dots, \quad M_r = M_{r-1} A^{-1}_{2,s+2k+6},$$
$$M_{r+1} = M_r A^{-1}_{1,s+2k-1} A^{-1}_{1,s+2k-3} A^{-1}_{2,s+2k}, \quad M_{r+2} = M_{r+1} A^{-1}_{1,s+2k-5},$$
$$M_{r+3} = M_{r+2} A^{-1}_{1,s+2k-7}, \quad \dots, \quad M_{k+r-1} = M_{k+r-2} A^{-1}_{1,s+1}.$$

The dominant monomials in $\chi_q\left(\mathcal{E}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{E}^{(s+2)}_{k-1,\ell-1}\right)$ *are* $M_0, \dots, M_{k+r-2}$.

If $\ell = 2r$, *then dominant monomials in* $\chi_q\left(\mathcal{E}^{(s)}_{k,\ell-1}\right)\chi_q\left(\mathcal{E}^{(s+2)}_{k-1,\ell}\right)$ *are*

$$M_0 = M, \quad M_1 = M A^{-1}_{2,s+2k+3\ell-4},$$
$$M_2 = M_1 A^{-1}_{2,s+2k+3\ell-10}, \quad \dots, \quad M_{r-1} = M_{r-2} A^{-1}_{2,s+2k+8},$$

$$M_r = M_{r-1}A^{-1}_{1,s+2k-1}A^{-1}_{2,s+2k+2}, \quad M_{r+1} = M_r A^{-1}_{1,s+2k-3},$$
$$M_{r+2} = M_{r+1}A^{-1}_{1,s+2k-5}, \quad \dots, \quad M_{k+r-1} = M_{k+r-2}A^{-1}_{1,s+1}.$$

The dominant monomials in $\chi_q\left(\mathcal{E}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{E}^{(s+2)}_{k-1,\ell-1}\right)$ *are* $M_0,\dots,M_{k+r-2}$.

(6) *Let* $M = F^{(s)}_{k,\ell-1}F^{(s+2)}_{k-1,\ell}$, $k\ge 1$, $\ell \ge 1$. *Then dominant monomials in* $\chi_q\left(\mathcal{F}^{(s)}_{k,\ell-1}\right)\chi_q\left(\mathcal{F}^{(s+2)}_{k-1,\ell}\right)$ *are*

$$M_0 = M, \quad M_1 = MA^{-1}_{1,s+2k+2\ell+3},$$
$$M_2 = M_1A^{-1}_{1,s+2k+2\ell+1}, \quad \dots, \quad M_{\ell-1} = M_{\ell-2}A^{-1}_{1,s+2k+7},$$
$$M_\ell = M_{\ell-1}A^{-1}_{1,s+2k-1}A^{-1}_{2,s+2k+2}A^{-1}_{1,s+2k+5}, \quad M_{\ell+1} = M_\ell A^{-1}_{1,s+2k-3},$$
$$M_{\ell+2} = M_{\ell+1}A^{-1}_{1,s+2k-5}, \quad \dots, \quad M_{k+\ell-1} = M_{k+\ell-2}A^{-1}_{1,s+1}.$$

The dominant monomials in $\chi_q\left(\mathcal{F}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{F}^{(s+2)}_{k-1,\ell-1}\right)$ *are* $M_0,\dots,M_{k+\ell-2}$.

In each case, for each i, *the multiplicity of* M_i *in the corresponding product of* q*-characters is* 1.

Proof. We prove the case of $\chi_q\left(\mathcal{C}^{(s)}_{k,\ell-1}\right)\chi_q\left(\mathcal{C}^{(s+6)}_{k-1,\ell}\right)$. The other cases are similar. Let $m_1' = C^{(s)}_{k,\ell-1}$, $m_2' = C^{(s+6)}_{k-1,\ell}$. Without loss of generality, we assume that $s=6$. Then

$$m_1' = (2_6 2_{12}\cdots 2_{6k})(2_{6k+10}2_{6k+16}\cdots 2_{6k+6\ell-2}),$$
$$m_2' = (2_{12}\cdots 2_{6k})(2_{6k+10}2_{6k+16}\cdots 2_{6k+6\ell-2}2_{6k+6\ell+4}).$$

Let $m = m_1m_2$ be a dominant monomial, where $m_i \in \chi_q(m_i')$, $i=1,2$. Denote by $m_3 = 2_{6k+10}2_{6k+16}\cdots 2_{6k+6\ell+4}$. If $m_2 \in \chi_q(2_{12}\cdots 2_{6k})(\chi_q(m_3)-m_3)$, then $m=m_1m_2$ is right negative and hence m is not dominant. Therefore $m_2 \in \chi_q(2_{12}\cdots 2_{6k})m_3$.

Suppose that $m_2 \in \mathscr{M}(L(m_2'))\cap\mathscr{M}(\chi_q(2_{12}\cdots 2_{6k-6})(\chi_q(2_{6k})-2_{6k})m_3)$. By the FM algorithm for $L(m_2')$ and Lemma 3.1, m_2 must have a factor 2^{-1}_{6k+6} or 1^{-1}_{6k+7} or 2^{-1}_{6k+8}. By Lemma 3.1, m_1 does not have the factors 2_{6k+6} and 2_{6k+8}. Therefore m_2 cannot have factors 2^{-1}_{6k+6} and 2^{-1}_{6k+8} since $m=m_1m_2$ is dominant. Hence 1^{-1}_{6k+7} is a factor of m_2. Since $m=m_1m_2$ is dominant, we need to cancel 1^{-1}_{6k+7} using a factor in m_1. By Lemma 3.1, the only possible way to cancel 1^{-1}_{6k+7} by m_1 is to use the factor $1_{6k+5}1_{6k+7}1_{6k+9}2^{-1}_{6k+8}2^{-1}_{6k+10}$ or $1_{6k+5}1_{6k+7}1^{-1}_{6k+11}2^{-1}_{6k+8}$ of m_1 coming from $\chi_q(2_{6k})$. Since 2^{-1}_{6k+8} cannot be canceled by any monomials in $\chi_q(2_62_{12}\cdots 2_{6k-6})$, we have the factor 2^{-1}_{6k+8} in $m=m_1m_2$ and hence m is not dominant. Therefore $m_2 \in \mathscr{M}(L(2_{12}\cdots 2_{6k-6}))2_{6k}m_3$. By the FM algorithm, $m_2 = m_2'$.

If

$$m_1 \in \chi_q(2_6\cdots 2_{6k}2_{6k+10}2_{6k+16}\cdots 2_{6k+6\ell-8})(\chi_q(2_{6k+6\ell-2}) - 2_{6k+6\ell-2} - 2^{-1}_{6k+6\ell+4}1_{6k+6\ell-1}1_{6k+6\ell+1}1_{6k+6\ell+3}),$$

then $m=m_1m_2$ is right-negative and hence not dominant. Therefore m_1 is in one of the following polynomials

$$\chi_q(2_6\cdots 2_{6k}2_{6k+10}2_{6k+16}\cdots 2_{6k+6\ell-8})2_{6k+6\ell-2}, \tag{5.1}$$
$$\chi_q(2_6\cdots 2_{6k}2_{6k+10}2_{6k+16}\cdots 2_{6k+6\ell-8})2^{-1}_{6k+6\ell+4}1_{6k+6\ell-1}1_{6k+6\ell+1}1_{6k+6\ell+3}. \tag{5.2}$$

If m_1 is in (5.1), then $m_1 = m_1'$. The dominant monomial we obtain is $M_0 = m_1'm_2'$. If m_1 is the highest monomial in (5.2), then we obtain the dominant monomial $M_1 = m_1m_2'$. Suppose that m_1 is in

$$\mathscr{M}(L(m_1'))\cap\mathscr{M}(\chi_q(2_6\cdots 2_{6k}2_{6k+10}2_{6k+16}\cdots 2_{6k+6\ell-14})(\chi_q(2_{6k+6\ell-8})-2_{6k+6\ell-8}) \times 2^{-1}_{6k+6\ell+4}1_{6k+6\ell-1}1_{6k+6\ell+1}1_{6k+6\ell+3}).$$

By the FM algorithm for $L(m_1')$,

$$m_1 \in \chi_q(2_6 \cdots 2_{6k} 2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell-14}) \\ \times \big(2^{-1}_{6k+6\ell-2} 1_{6k+6\ell-7} 1_{6k+6\ell-5} 1_{6k+6\ell-3}\big)\big(2^{-1}_{6k+6\ell+4} 1_{6k+6\ell-1} 1_{6k+6\ell+1} 1_{6k+6\ell+3}\big).$$

We obtain the dominant monomial $M_2 = m_1 m_2'$. Continue this procedure, we obtain dominant monomials $M_3, \dots, M_{\ell-1}$ and the remaining dominant monomials are of the form $m_1 m_2'$, where m_1 is a non-highest monomial in

$$\mathscr{M}(L(m_1')) \cap \mathscr{M}(L(2_6 \cdots 2_{6k})) 2^{-1}_{6k+16} 2^{-1}_{6k+22} \cdots 2^{-1}_{6k+6\ell+4} 1_{6k+11} 1_{6k+13} \cdots 1_{6k+6\ell+3}.$$

Suppose that m_1 is a non-highest monomial in the above set. Since the non-highest monomials in $\chi_q(2_6 \cdots 2_{6k})$ are right-negative, we need cancellations of factors with negative powers of some monomial in $\chi_q(2_6 \cdots 2_{6k})$ with $2_{6k+10} 1_{6k+11} 1_{6k+13} \cdots 1_{6k+6\ell+3}$. The only cancellation can happen is to cancel 2_{6k+10} or 1_{6k+11}. Since 1^2_{6k+9} does not appear in $\chi_q(2_6 \cdots 2_{6k})$, 1_{6k+11} cannot be canceled. Therefore we need a cancellation with 2_{6k+10}. The only monomials in $\chi_q(2_6 \cdots 2_{6k})$ which can cancel 2_{6k+10} is in one of the following polynomials

$$\chi_q(2_6 \cdots 2_{6k-6}) 1_{6k+1} 1_{6k+9} 2^{-1}_{6k+10}, \\ \chi_q(2_6 \cdots 2_{6k-6}) 1^{-1}_{6k+3} 1_{6k+9} 2_{6k+2} 2^{-1}_{6k+10}, \\ \chi_q(2_6 \cdots 2_{6k-6}) 1_{6k+5} 1_{6k+7} 1_{6k+9} 2^{-1}_{6k+8} 2^{-1}_{6k+10}.$$

Therefore m_1 is in one of the following sets

$$\mathscr{M}(L(m_1')) \cap \mathscr{M}(L(2_6 \cdots 2_{6k-6})) 1_{6k+1} 1_{6k+9} 2^{-1}_{6k+10} \cdots 2^{-1}_{6k+6\ell+4} 1_{6k+11} \cdots 1_{6k+6\ell+3}, \tag{5.3}$$

$$\mathscr{M}(L(m_1')) \cap \mathscr{M}(L(2_6 \cdots 2_{6k-6})) \\ \times 1^{-1}_{6k+3} 1_{6k+9} 2_{6k+2} 2^{-1}_{6k+10} \cdots 2^{-1}_{6k+6\ell+4} 1_{6k+11} \cdots 1_{6k+6\ell+3}, \tag{5.4}$$

$$\mathscr{M}(L(m_1')) \cap \mathscr{M}(L(2_6 \cdots 2_{6k-6})) \\ \times 1_{6k+5} 1_{6k+7} 1_{6k+9} 2^{-1}_{6k+8} 2^{-1}_{6k+10} \cdots 2^{-1}_{6k+6\ell+4} 1_{6k+11} \cdots 1_{6k+6\ell+3}. \tag{5.5}$$

If m_1 is in (5.4), then we need to cancel 1^{-1}_{6k+3}. We have

$$\mathscr{M}(L(2_6 \cdots 2_{6k-6})) \subset \mathscr{M}(\chi_q(2_6 \cdots 2_{6k-12}) \chi_q(2_{6k-6})).$$

By Lemma 3.1, only the monomials

$$1_{6k-5} 1_{6k+3} 2^{-1}_{6k+4}, \qquad 1^{-1}_{6k-3} 1_{6k+3} 2_{6k-4} 2^{-1}_{6k+4}, \qquad 1_{6k-1} 1_{6k+1} 1_{6k+3} 2^{-1}_{6k+2} 2^{-1}_{6k+4}$$

in $\chi_q(2_{6k-6})$ can cancel 1^{-1}_{6k+3}. But these monomials have the factor 2^{-1}_{6k+4} which cannot be canceled by any monomials in $\chi_q(2_6 \cdots 2_{6k-12})$ or by m_2'. Hence m_1 is not in (5.4).

If m_1 is in (5.5), then we need to cancel 2^{-1}_{6k+8}. But 2^{-1}_{6k+8} cannot be canceled by any monomials in $\chi_q(2_6 \cdots 2_{6k-6})$ or by m_2'. Therefore m_1 is not in (5.5). Hence m_1 is in (5.3).

If m_1 is the highest monomial in (5.3) with respect to $\leq$ defined in (2.7), then $m_1 m_2' = M_\ell$. Suppose that m_1 a non-highest monomial in (5.3). By the FM algorithm, m_1 must be in

$$\chi_q(2_6 \cdots 2_{6k-12}) 2^{-1}_{6k} 1_{6k-5} 1_{6k-3} 1_{6k-1} 1_{6k+1} 1_{6k+9} 2^{-1}_{6k+10} \cdots 2^{-1}_{6k+6\ell+4} 1_{6k+11} \cdots 1_{6k+6\ell+3}.$$

If m_1 is the highest monomial in the above set, then $m_1 m_2' = M_{\ell+1}$. Continue this procedure, we can show that the only remaining dominant monomials are $M_{\ell+2}, \dots, M_{k+\ell-1}$.

It is clear that the multiplicity of M_i, $i = 1, \dots, k+\ell-1$, in $\chi_q(m_1)\chi_q(m_2)$ is 1. ■

5.2 Products of sources are special

Lemma 5.2. *Let $[\mathcal{S}]$ be the last summand in one of the relations (3.4)–(3.11). Then $\mathcal{S}$ is special.*

Proof. We give a proof for $\mathcal{S}$ in the last line of (3.7) and in the last line of (3.11). The other cases are similar.

Let $S_1 = \chi_q(\mathcal{C}^{(s+1)}_{r+1,p})\chi_q(\mathcal{B}^{(s+3)}_{r+1,3p-1})\chi_q(\mathcal{B}^{(s+5)}_{r+p+1,0})$. Let

$$n_1 = 2_{s+1}2_{s+7}\cdots 2_{s+6r-5}2_{s+6r+1}, \qquad n_1' = 2_{s+6r+11}2_{s+6r+17}\cdots 2_{s+6r+6p+5},$$
$$n_2 = 2_{s+3}2_{s+9}\cdots 2_{s+6r-3}2_{s+6r+3}, \qquad n_2' = 1_{s+6r+10}1_{s+6r+12}\cdots 1_{s+6r+6p+6},$$
$$n_3 = 2_{s+5}2_{s+11}\cdots 2_{s+6r+6p+5}.$$

Then $C^{(s+1)}_{r+1,p} = n_1n_1'$, $B^{(s+3)}_{r+1,3p-1} = n_2n_2'$, $B^{(s+5)}_{r+p+1,0} = n_3$. Let $m' = m_1m_2m_3$ be a dominant monomial, where

$$m_1 \in \mathscr{M}\Big(\mathcal{C}^{(s+1)}_{r+1,p}\Big), \qquad m_2 \in \mathscr{M}\Big(\mathcal{B}^{(s+3)}_{r+1,3p-1}\Big), \qquad m_3 \in \mathscr{M}\Big(\mathcal{B}^{(s+5)}_{r+p+1,0}\Big).$$

If $m_3 \neq B^{(s+5)}_{r+p+1,0}$ or $m_1 \in \chi_q(n_1)(\chi_q(n_1') - n_1')$ or $m_2 \in \chi_q(n_2)(\chi_q(n_2') - n_2')$, then m' is right-negative which contradicts the fact that m' is dominant. Therefore $m_3 = B^{(s+5)}_{r+p+1,0}$, $m_1 \in \chi_q(n_1)n_1'$, and $m_2 \in \chi_q(n_2)n_2'$.

If m_2 is in

$$\mathscr{M}(L(n_2n_2')) \cap \mathscr{M}(\chi_q(2_{s+3}2_{s+9}\cdots 2_{s+6r-3})(\chi_q(2_{s+6r+3}) - 2_{s+6r+3})n_2'), \tag{5.6}$$

then

$$m_2 \in \chi_q(2_{s+3}2_{s+9}\cdots 2_{s+6r-3})2^{-1}_{s+6r+9}1_{s+6k+4}1_{s+6k+6}1_{s+6k+8}n_2'.$$

By Lemma 3.1, the factor 2^{-1}_{s+6r+9} cannot be canceled by any monomial in either $\chi_q(n_1)$ or $\chi_q(2_{s+3}2_{s+9}\cdots 2_{s+6r-3})$. It is clear that 2^{-1}_{s+6r+9} cannot be canceled by n_1', n_2', n_3. Therefore 2^{-1}_{s+6r+9} cannot be canceled. Hence m_2 is not in (5.6). Thus m_2 must be in

$$\mathscr{M}(L(n_2n_2')) \cap \mathscr{M}(L(2_{s+3}2_{s+9}\cdots 2_{s+6r-3}))2_{s+6r+3}n_2'.$$

Therefore $m_2 = B^{(s+3)}_{r+1,3p-1}$.

Suppose that $m_1 \neq C^{(s+1)}_{r+1,p}$. Then $m_1 = m_1'n_1'$, where m_1' is a non-highest monomial in $\chi_q(n_1)$. Since the non-highest monomials in $\chi_q(n_1)$ are right-negative, we need a cancellation with $n_1'n_2'm_3$. The only cancellation can happen is to cancel $2_{s+6r+11}$ in n_1', or cancel one of 2_{s+6r+3}, $1_{s+6r+10}$ in n_2n_2', or cancel one of 2_{s+6r+5}, $2_{s+6r+11}$ in m_3. By the FM algorithm, $2_{s+6r+11}$ cannot be canceled. By Lemma 3.1, $1_{s+6r+10}$, 2_{s+6r+3} and 2_{s+6r+5} cannot be canceled. This is a contradiction. Therefore $m_1 = C^{(s+1)}_{r+1,p}$.

Therefore the only dominant monomial in S_1 is $C^{(s+1)}_{r+1,p}B^{(s+3)}_{r+1,3p-1}B^{(s+5)}_{r+p+1,0}$.

Let $S_2 = \chi_q\Big(\mathcal{D}^{(s+1)}_{r+1,\lfloor\frac{\ell}{3}\rfloor}\Big)\chi_q\Big(\mathcal{C}^{(s+3)}_{r+1,\lfloor\frac{\ell+1}{3}\rfloor}\Big)\chi_q\Big(\mathcal{B}^{(s+5)}_{r,0}\Big)\chi_q\Big(\mathcal{B}^{(s+6r+17)}_{\lfloor\frac{\ell-1}{3}\rfloor,0}\Big)$, $r \geq 0$, and $\ell = 3p$, $p \geq 1$. The cases of $\ell = 3p+1$, $p \geq 0$ and $\ell = 3p+2$, $p \geq 0$ are similar. Let

$$n_1 = 2_{s+1}2_{s+7}\cdots 2_{s+6r+1}1_{s+6r+8}, \qquad n_1' = 2_{s+6r+15}2_{s+6r+21}\cdots 2_{s+6r+6p+9},$$
$$n_2 = 2_{s+3}2_{s+9}\cdots 2_{s+6r-3}2_{s+6r+3}, \qquad n_2' = 2_{s+6r+13}2_{s+6r+20}\cdots 2_{s+6r+6p+7},$$
$$n_3 = 2_{s+5}2_{s+11}\cdots 2_{s+6r-1}, \qquad n_4 = 2_{s+6r+17}2_{s+6r+23}\cdots 2_{s+6r+6p+5}.$$

Then $D^{(s+1)}_{r+1,\lfloor\frac{\ell}{3}\rfloor} = n_1n_1'$, $C^{(s+3)}_{r+1,\lfloor\frac{\ell+1}{3}\rfloor} = n_2n_2'$, $B^{(s+5)}_{r,0} = n_3$, $B^{(s+6r+17)}_{\lfloor\frac{\ell-1}{3}\rfloor,0} = n_4$.

Let $m' = m_1m_2m_3m_4$ be a dominant monomial, where

$$m_1 \in \mathscr{M}\Big(\mathcal{D}^{(s+1)}_{r+1,\lfloor\frac{\ell}{3}\rfloor}\Big), \qquad m_2 \in \mathscr{M}\Big(\mathcal{C}^{(s+3)}_{r+1,\lfloor\frac{\ell+1}{3}\rfloor}\Big),$$
$$m_3 \in \mathscr{M}\Big(\mathcal{B}^{(s+5)}_{r,0}\Big), \qquad m_4 \in \mathscr{M}\Big(\mathcal{B}^{(s+6r+17)}_{\lfloor\frac{\ell-1}{3}\rfloor,0}\Big).$$

If $m_4 \neq n_4$ or $m_1 \in \chi_q(n_1)(\chi_q(n'_1) - n'_1)$ or $m_2 \in \chi_q(n_2)(\chi_q(n'_2) - n'_2)$, then m' is right-negative which contradicts the fact that m' is dominant. Therefore $m_4 = n_4$, $m_1 \in \chi_q(n_1)n'_1$, and $m_2 \in \chi_q(n_2)n'_2$.

If

$$m_1 \in \mathscr{M}(L(n_1n'_1)) \cap \mathscr{M}(\chi_q(2_{s+1}2_{s+7}\cdots 2_{s+6r+1})(\chi_q(1_{s+6r+8}) - 1_{s+6r+8})n'_1), \tag{5.7}$$

then by the FM algorithm for $L(n_1n'_1)$,

$$m_1 \in \chi_q(2_{s+1}2_{s+7}\cdots 2_{s+6r+1})1^{-1}_{s+6r+10}2_{s+6r+9}n'_1.$$

It is clear that $1^{-1}_{s+6r+10}$ is not canceled by n'_1, n'_2, n_4, and any monomial in $\chi_q(n_3)$. By the FM algorithm for $\chi_q(n_2n'_2)$, $1^{-1}_{s+6r+10}$ cannot be canceled by any monomial in $\chi_q(n_2n'_2)$. Therefore, by Lemma 3.1, $1^{-1}_{s+6r+10}$ can only be canceled by one of the factors

$$1_{s+6r+2}1_{s+6r+10}2^{-1}_{s+6r+11},$$
$$1^{-1}_{s+6r+4}1_{s+6r+10}2_{s+6r+3}2^{-1}_{s+6r+11}, \qquad 1_{s+6r+6}1_{s+6r+8}1_{s+6r+10}2^{-1}_{s+6r+9}2^{-1}_{s+6r+11}$$

coming from $\chi_q(2_{s+6r+1})$, where 2_{s+6r+1} is in n_1. But then $2^{-1}_{s+6r+11}$ cannot be canceled. This contradicts the fact that m' is dominant. Hence m_1 is not in (5.6). Thus m_1 must be in

$$\mathscr{M}(L(n_1n'_1)) \cap \mathscr{M}(L(n_1n'_1)) \cap \mathscr{M}(L(2_{s+1}2_{s+7}\cdots 2_{s+6r+1}))1_{s+6r+8}n'_1.$$

If m_1 is in

$$\mathscr{M}(L(n_1n'_1)) \cap \mathscr{M}(L(n_1n'_1)) \cap \mathscr{M}(\chi_q(2_{s+1} \times 2_{s+7}\cdots 2_{s+6r-5})(\chi_q(2_{s+6r+1}) - 2_{s+6r+1})1_{s+6r+8}n'_1).$$

Then

$$m_1 \in \chi_q(2_{s+1}2_{s+7}\cdots 2_{s+6r-5})2^{-1}_{s+6r+7}1_{s+6r+2}1_{s+6r+4}1_{s+6r+6}1_{s+6r+8}n'_1.$$

The only possible way to cancel 2^{-1}_{s+6r+7} is to use one of the terms

$$1_{s+6r+4}1^{-1}_{s+6r+8}1^{-1}_{s+6r+10}2_{s+6r+7},$$
$$1^{-1}_{s+6r+6}1^{-1}_{s+6r+8}1^{-1}_{s+6r+10}2_{s+6r+5}2_{s+6r+7}, \qquad 2_{s+6r+7}2^{-1}_{s+6r+11}$$

in $\chi_q(2_{s+6r+3})$, where 2_{s+6r+3} is in n_2. But then we have to cancel $1^{-1}_{s+6r+10}$ or $2^{-1}_{s+6r+11}$. But $1^{-1}_{s+6r+10}$ and $2^{-1}_{s+6r+11}$ cannot be canceled. This is a contradiction. Therefore m_1 must be in

$$\mathscr{M}(L(n_1n'_1)) \cap \mathscr{M}(L(2_{s+1}2_{s+7}\cdots 2_{s+6r-5}))2_{s+6r+1}1_{s+6r+8}n'_1.$$

Hence $m_1 = n_1n'_1$.

By the FM algorithm, when we compute the q-character for $\chi_q(n_2n'_2)$, we can only choose one of the following terms

$$2_{s+6r+3}, \qquad 1_{s+6r+4}1_{s+6r+6}1_{s+6r+8}2^{-1}_{s+6r+9}, \qquad 1_{s+6r+4}1_{s+6r+6}1^{-1}_{s+6r+10},$$

$$1_{s+6r+4}1^{-1}_{s+6r+8}1^{-1}_{s+6r+10}2_{s+6r+7},$$
$$1^{-1}_{s+6r+6}1^{-1}_{s+6r+8}1^{-1}_{s+6r+10}2_{s+6r+5}2_{s+6r+7}, \qquad 2^{-1}_{s+6r+11}2_{s+6r+7}$$

in $\chi_q(2_{s+6r+3})$. Since 2^{-1}_{s+6r+9}, $1^{-1}_{s+6r+10}$, and $2^{-1}_{s+6r+11}$ cannot be canceled, we can only choose 2_{s+6r+3}. Therefore m_2 is in

$$\mathscr{M}(L(n_2n_2')) \cap \mathscr{M}(L(2_{s+3}2_{s+9}\cdots 2_{s+6r-3}))2_{s+6r+3}n_2'.$$

Therefore $m_2 = n_2n_2'$.

If m_3 is in

$$\mathscr{M}(L(n_3)) \cap \mathscr{M}(\chi_q(2_{s+5}2_{s+11}\cdots 2_{s+6r-7})(\chi_q(2_{s+6r-1}) - 2_{s+6r-1})),$$

then, by Lemma 3.1, $m = m_1m_2m_3m_4$ is non-dominant since $m_1 = n_1n_1', m_2 = n_2n_2', m_4 = n_4$. This contradicts the fact that m is dominant. Therefore m_3 is in

$$\mathscr{M}(L(n_3)) \cap \mathscr{M}(L(2_{s+5}2_{s+11}\cdots 2_{s+6r-7}))2_{s+6r-1}.$$

Hence $m_3 = n_3$.

Therefore the only dominant monomial in S_2 is $D^{(s+1)}_{r+1,\lfloor \frac{\ell}{3}\rfloor} C^{(s+3)}_{r+1,\lfloor \frac{\ell+1}{3}\rfloor} B^{(s+5)}_{r,0} B^{(s+6r+17)}_{\lfloor \frac{\ell-1}{3}\rfloor,0}$. ■

5.3 Proof of Theorem 3.4

By Lemmas 5.1 and 5.2, the dominant monomials in the q-characters of the left hand side and of the right hand side of every relation in Theorem 3.4 are the same. The theorem follows.

6 Proof of Theorem 3.5

By Lemma 5.2, $\mathcal{S}$ is special and hence irreducible. Therefore we only have to show that $\mathcal{T}\otimes\mathcal{B}$ is irreducible. It suffices to prove that for each non-highest dominant monomial M in $\mathcal{T}\otimes\mathcal{B}$, we have $\mathscr{M}(L(M)) \not\subset \mathscr{M}(\mathcal{T}\otimes\mathcal{B})$. The idea is similar as in [10, 18, 21]. Recall that the dominant monomials in $\mathcal{T}\otimes\mathcal{B}$ are described by Lemma 5.1.

Lemma 6.1. *We consider the same cases as in Lemma 5.1. In each case M_i are the dominant monomials described by that lemma.*

(1) *For $k \geq 1$, $\ell \geq 1$, let*

$$n_1 = M_1A^{-1}_{1,s+6k+2\ell-2}, \quad n_2 = M_2A^{-1}_{1,s+6k+2\ell-4}, \quad \ldots,$$
$$n_{\ell-1} = M_{\ell-1}A^{-1}_{1,s+6k+2}, \quad n_\ell = M_\ell A^{-1}_{2,s+6k-3}A^{-1}_{1,s+6k},$$
$$n_{\ell+1} = M_{\ell+1}A^{-1}_{2,s+6k-9}, \quad \ldots, \quad n_{k+\ell-2} = M_{k+\ell-2}A^{-1}_{2,s+9}.$$

Then for $i = 1,\ldots,k+\ell-2$, $n_i \in \chi_q(M_i)$ and $n_i \notin \chi_q\left(\mathcal{B}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{B}^{(s+6)}_{k-1,\ell-1}\right)$.

(2) *For $k \geq 1$, $\ell \geq 1$, let*

$$n_1 = M_1A^{-1}_{2,s+6k+6\ell-5}, \quad n_2 = M_2A^{-1}_{2,s+6k+6\ell-11}, \quad \ldots,$$
$$n_{\ell-1} = M_{\ell-1}A^{-1}_{2,s+6k+7}, \quad n_\ell = M_\ell A^{-1}_{2,s+6k-3}A^{-1}_{1,s+6k}A^{-1}_{1,s+6k-2}A^{-1}_{2,s+6k+1},$$
$$n_{\ell+1} = M_{\ell+1}A^{-1}_{2,s+6k-9}, \quad \ldots, \quad n_{k+\ell-2} = M_{k+\ell-2}A^{-1}_{2,s+9}.$$

Then for $i = 1,\ldots,k+\ell-2$, $n_i \in \chi_q(M_i)$ and $n_i \notin \chi_q\left(\mathcal{C}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{C}^{(s+6)}_{k-1,\ell-1}\right)$.

(3) *For $\ell \geq 1$, let*

$$n_1 = M_1 A^{-1}_{2,s+6\ell-1}, \quad n_2 = M_2 A^{-1}_{2,s+6\ell-7}, \quad \ldots, \quad n_{\ell-1} = M_{\ell-1} A^{-1}_{2,s+11}.$$

Then for $i = 1, \ldots, \ell$, $n_i \in \chi_q(M_i)$ and $n_i \notin \chi_q\left(\mathcal{D}^{(s)}_{0,\ell}\right)\chi_q\left(\mathcal{B}^{(s+8)}_{\ell-1,0}\right)$.

(4) *For $k \geq 1$, $\ell \geq 1$, let*

$$\begin{aligned} &n_1 = M_1 A^{-1}_{2,s+6k+6\ell-1}, \quad n_2 = M_2 A^{-1}_{2,s+6k+6\ell-7}, \quad \ldots, \quad n_{\ell-1} = M_{\ell-1} A^{-1}_{2,s+6k+11}, \\ &n_\ell = M_\ell A^{-1}_{1,s+6k+2} A^{-1}_{2,s+6k+5}, \quad n_{\ell+1} = M_{\ell+1} A^{-1}_{2,s+6k-3} A^{-1}_{1,s+6k}, \\ &n_{\ell+2} = M_{\ell+2} A^{-1}_{2,s+6k-9}, \quad \ldots, \quad n_{k+\ell-1} = M_{k+\ell-1} A^{-1}_{2,s+9}. \end{aligned}$$

Then for $i = 1, \ldots, k+\ell-1$, $n_i \in \chi_q(M_i)$ and $n_i \notin \chi_q\left(\mathcal{D}^{(s+6)}_{k-1,\ell-1}\right)\chi_q\left(\mathcal{D}^{(s)}_{k,\ell}\right)$.

(5) *For $k \geq 0, \ell = 2r+1, r \geq 0$, let*

$$\begin{aligned} &n_1 = M_1 A^{-1}_{2,s+2k+3\ell-3}, \quad n_2 = M_2 A^{-1}_{2,s+2k+3\ell-9}, \quad \ldots, \\ &n_r = M_r A^{-1}_{2,s+2k+3}, \quad n_{r+1} = M_{r+1} A^{-1}_{1,s+2k-1} A^{-1}_{1,s+2k-3} A^{-1}_{2,s+2k}, \\ &n_{r+2} = M_{r+2} A^{-1}_{1,s+2k-5}, \quad \ldots, \quad n_{k+r-2} = M_{k+r-2} A^{-1}_{1,s+3}. \end{aligned}$$

Then for $i = 1, \ldots, r+k-2$, $n_i \in \chi_q(M_i)$ and $n_i \notin \chi_q\left(\mathcal{E}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{E}^{(s+2)}_{k-1,\ell-1}\right)$.

For $k \geq 0, \ell = 2r, r \geq 1$, let

$$\begin{aligned} &n_1 = M_1 A^{-1}_{2,s+2k+3\ell-4}, \quad n_2 = M_2 A^{-1}_{2,s+2k+3\ell-10}, \quad \ldots, \\ &n_{r-1} = M_{r-1} A^{-1}_{2,s+2k+8}, \quad n_r = M_r A^{-1}_{1,s+2k-1} A^{-1}_{2,s+2k+2}, \\ &n_{r+1} = M_{r+1} A^{-1}_{1,s+2k-3}, \quad \ldots, \quad n_{k+r-2} = M_{k+r-2} A^{-1}_{1,s+3}. \end{aligned}$$

Then for $i = 1, \ldots, r+k-2$, $n_i \in \chi_q(M_i)$ and $n_i \notin \chi_q\left(\mathcal{E}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{E}^{(s+2)}_{k-1,\ell-1}\right)$.

(6) *For $k \geq 1$, $\ell \geq 1$, let*

$$\begin{aligned} &n_1 = M_1 A^{-1}_{1,s+2k+2\ell+3}, \quad n_2 = M_2 A^{-1}_{1,s+2k+2\ell+1}, \quad \ldots, \\ &n_{\ell-1} = M_{\ell-1} A^{-1}_{1,s+2k+7}, \quad n_\ell = M_\ell A^{-1}_{1,s+2k-1} A^{-1}_{2,s+2k+2} A^{-1}_{1,s+2k+5}, \\ &n_{\ell+1} = M_{\ell+1} A^{-1}_{1,s+2k-3}, \quad \ldots, \quad n_{k+\ell-2} = M_{k+\ell-2} A^{-1}_{1,s+3}. \end{aligned}$$

Then $i = 1, \ldots, k+\ell-2$, $n_i \in \chi_q(M_i)$ and $n_i \notin \chi_q\left(\mathcal{F}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{F}^{(s+2)}_{k-1,\ell-1}\right)$.

Proof. We give a proof in the case of $\chi_q\left(\mathcal{C}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{C}^{(s+6)}_{k-1,\ell-1}\right)$. The other cases are similar. By definition, we have

$$\begin{aligned} C^{(s)}_{k,\ell} &= (2_s 2_{s+6} \cdots 2_{s+6k-6})(2_{s+6k+4} 2_{s+6k+10} \cdots 2_{s+6k+6\ell-8} 2_{s+6k+6\ell-2}), \\ C^{(s+6)}_{k-1,\ell-1} &= (2_{s+6} 2_{s+12} \cdots 2_{s+6k-6})(2_{s+6k+4} 2_{s+6k+10} \cdots 2_{s+6k+6\ell-8}), \\ M_1 &= C^{(s)}_{k,\ell} C^{(s+6)}_{k-1,\ell-1} A^{-1}_{2,s+6k+6\ell-5} \\ &= C^{(s)}_{k,\ell} C^{(s+6)}_{k-1,\ell-1} 2^{-1}_{s+6k+6\ell-8} 2^{-1}_{s+6k+6\ell-2} 1_{s+6k+6\ell-7} 1_{s+6k+6\ell-5} 1_{s+6k+6\ell-3}. \end{aligned}$$

By $U_{q^2}(\hat{\mathfrak{sl}}_2)$ argument, it is clear that $n_1 = M_1 A^{-1}_{2,s+6k+6\ell-5}$ is in $\chi_q(M_1)$.

If n_1 is in $\chi_q\left(\mathcal{C}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{C}^{(s+6)}_{k-1,\ell-1}\right)$, then $C^{(s)}_{k,\ell}A^{-1}_{2,s+6k+6\ell-5}$ is in $\chi_q\left(\mathcal{C}^{(s)}_{k,\ell}\right)$ which is impossible by the FM algorithm for $\mathcal{C}^{(s)}_{k,\ell}$. Similarly, $n_i \in \chi_q(M_i)$, $i = 2, \dots, \ell-1$, but $n_2, \dots, n_{\ell-1}$ are not in $\chi_q\left(\mathcal{C}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{C}^{(s+6)}_{k-1,\ell-1}\right)$.

By definition,

$$M_\ell = (2_s 2_{s+6} \cdots 2_{s+6k-6})(2_{s+6}2_{s+12}\cdots 2_{s+6k-12})(1_{s+6k-5}1_{s+6k+3}1_{s+6k+5}\cdots 1_{s+6k+6\ell-3}).$$

Let $U = \left\{(1, aq^{s+6k}), (1, aq^{s+6k-3}), (2, aq^{s+6k-2}), (2, aq^{s+6k+1})\right\} \subset I \times \mathbb{C}^\times$. Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = M_\ell, \qquad m_1 = m_0 A^{-1}_{2,s+6k-3}, \qquad m_2 = m_1 A^{-1}_{1,s+6k},$$
$$m_3 = m_2 A^{-1}_{1,s+6k-2}, \qquad m_4 = m_3 A^{-1}_{2,s+6k+1}.$$

It is clear that $\mathcal{M}$ satisfies the conditions in Theorem 2.2. Therefore

$$\mathrm{trunc}_{m+\mathcal{Q}^-_U}(\chi_q(M_\ell)) = \sum_{m\in\mathcal{M}} m$$

and hence $n_\ell = M_\ell A^{-1}_{2,s+6k-3}A^{-1}_{1,s+6k}A^{-1}_{1,s+6k-2}A^{-1}_{2,s+6k+1}$ is in $\chi_q(M_\ell)$.

If n_ℓ is in $\chi_q\left(\mathcal{C}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{C}^{(s+6)}_{k-1,\ell-1}\right)$, then $C^{(s)}_{k,\ell}A^{-1}_{2,s+6k-3}A^{-1}_{1,s+6k}A^{-1}_{1,s+6k-2}A^{-1}_{2,s+6k+1}$ is in $\chi_q\left(\mathcal{C}^{(s)}_{k,\ell}\right)$ which is impossible by the FM algorithm for $\mathcal{C}^{(s)}_{k,\ell}$. Similarly, we show that for $i = \ell+1, \dots, k+\ell-2$, $n_i \in \chi_q(M_i)$ and $n_i \notin \chi_q\left(\mathcal{C}^{(s)}_{k,\ell}\right)\chi_q\left(\mathcal{C}^{(s+6)}_{k-1,\ell-1}\right)$. ■

7 The second part of the extended T-system

Let $\tilde{B}^{(s)}_{k,\ell}$, $\tilde{C}^{(s)}_{k,\ell}$, $\tilde{D}^{(s)}_{k,\ell}$, $\tilde{E}^{(s)}_{k,\ell}$, $\tilde{F}^{(s)}_{k,\ell}$ be the monomials obtained from $B^{(s)}_{k,\ell}$, $C^{(s)}_{k,\ell}$, $D^{(s)}_{k,\ell}$, $E^{(s)}_{k,\ell}$, $F^{(s)}_{k,\ell}$ by replacing i_a with i_{-a}, $i = 1, 2$. Namely,

$$\tilde{B}^{(s)}_{k,\ell} = \left(\prod_{i=0}^{\ell-1} 1_{-s-6k-2i-1}\right)\left(\prod_{i=0}^{k-1} 2_{-s-6i}\right), \qquad \tilde{C}^{(s)}_{k,\ell} = \left(\prod_{i=0}^{\ell-1} 2_{-s-6k-6i-4}\right)\left(\prod_{i=0}^{k-1} 2_{-s-6i}\right),$$
$$\tilde{D}^{(s)}_{k,\ell} = \left(\prod_{i=0}^{\ell-1} 2_{-s-6k-6i-8}\right) 1_{-s-6k-1}\left(\prod_{i=0}^{k-1} 2_{-s-6i}\right),$$
$$\tilde{F}^{(s)}_{k,\ell} = \left(\prod_{i=0}^{\ell-1} 1_{-s-2k-2i-6}\right)\left(\prod_{i=0}^{k-1} 1_{-s-2i}\right),$$
$$\tilde{E}^{(s)}_{k,\ell} = \left(\prod_{i=0}^{\lfloor\frac{\ell-2}{2}\rfloor} 2_{-s-2k-6i-5}\right)\left(\prod_{i=0}^{\lfloor\frac{\ell-1}{2}\rfloor} 2_{-s-2k-6i-3}\right)\left(\prod_{i=0}^{k-1} 1_{-s-2i}\right).$$

Note that, in particular, for $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$, we have the following trivial relations

$$\tilde{\mathcal{B}}^{(s)}_{k,0} = \tilde{\mathcal{C}}^{(s)}_{k,0} = \tilde{\mathcal{C}}^{(s-4)}_{0,k}, \qquad \tilde{\mathcal{D}}^{(s)}_{k,0} = \tilde{\mathcal{B}}^{(s)}_{k,1}, \qquad \tilde{\mathcal{E}}^{(s)}_{k,0} = \tilde{\mathcal{B}}^{(s-1)}_{0,k} = \tilde{\mathcal{F}}^{(s-6)}_{0,k} = \tilde{\mathcal{F}}^{(s)}_{k,0}. \tag{7.1}$$

We also have $\mathcal{D}^{(s)}_{0,k} = \tilde{\mathcal{B}}^{(-s-6k-2)}_{k,1}$, $\tilde{\mathcal{D}}^{(s)}_{0,k} = \mathcal{B}^{(-s-6k-2)}_{k,1}$, $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$.

Note that $\tilde{\mathcal{B}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{D}}^{(s)}_{0,\ell}$, $\tilde{\mathcal{D}}^{(s)}_{k,0}$ are minimal affinizations. In general, the modules $\tilde{\mathcal{B}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{C}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{D}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{E}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{F}}^{(s)}_{k,\ell}$ are not special. For example, we have the following proposition.

Proposition 7.1. *The module $\tilde{\mathcal{B}}^{(0)}_{3,1} = L(1_0 1_2 1_4 2_{11})$ is not special.*

Proof. Suppose that $L(1_0 1_2 1_4 2_{11})$ is special. Then the FM algorithm applies to $L(1_0 1_2 1_4 2_{11})$. Therefore, by the FM algorithm, the monomials

$$1_0 1_2 1_4 2_{11},\quad 1_0 1_2 1_6^{-1} 2_5 2_{11},\quad 1_0 1_4^{-1} 1_6^{-1} 2_3 2_5 2_{11},\quad 1_2^{-1} 1_4^{-1} 1_6^{-1} 2_1 2_3 2_5 2_{11},\quad 2_7^{-1} 2_3 2_5 2_{11},$$
$$2_7^{-1} 2_9^{-1} 1_4 1_6 1_8 2_5 2_{11},\quad 2_7^{-1} 1_4 1_6 1_{10}^{-1} 2_5 2_{11},\quad 1_4 1_8^{-1} 1_{10}^{-1} 2_5 2_{11},\quad 1_6^{-1} 1_8^{-1} 1_{10}^{-1} 2_5^2 2_{11},\quad 2_5$$

are in $\mathscr{M}(L(1_0 1_2 1_4 2_{11}))$. Hence $\mathscr{M}(L(1_0 1_2 1_4 2_{11}))$ has at least two dominant monomials $1_0 1_2 1_4 2_{11}$ and 2_5. This contradicts the assumption that $L(1_0 1_2 1_4 2_{11})$ is special. ■

Theorem 7.2. *The modules $\tilde{\mathcal{B}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{C}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{D}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{E}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{F}}^{(s)}_{k,\ell}$, $s \in \mathbb{Z}$, $k, l, \in \mathbb{Z}_{\geq 0}$ are anti-special.*

Proof. This theorem can be proved using the dual arguments of the proof of Theorem 3.3. ■

Lemma 7.3. *Let $\iota : \mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{P}$ be a homomorphism of rings such that $Y_{1,aq^s} \mapsto Y^{-1}_{1,aq^{12-s}}$, $Y_{2,aq^s} \mapsto Y^{-1}_{2,aq^{12-s}}$ for all $a \in \mathbb{C}^\times$, $s \in \mathbb{Z}$. Then*

$$\chi_q\Big(\tilde{\mathcal{B}}^{(s)}_{k,\ell}\Big) = \iota\Big(\chi_q\Big(\mathcal{B}^{(s)}_{k,\ell}\Big)\Big), \qquad \chi_q\Big(\tilde{\mathcal{C}}^{(s)}_{k,\ell}\Big) = \iota\Big(\chi_q\Big(\mathcal{C}^{(s)}_{k,\ell}\Big)\Big),$$
$$\chi_q\Big(\tilde{\mathcal{D}}^{(s)}_{k,\ell}\Big) = \iota\Big(\chi_q\Big(\mathcal{D}^{(s)}_{k,\ell}\Big)\Big), \qquad \chi_q\Big(\tilde{\mathcal{E}}^{(s)}_{k,\ell}\Big) = \iota\Big(\chi_q\Big(\mathcal{E}^{(s)}_{k,\ell}\Big)\Big), \qquad \chi_q\Big(\tilde{\mathcal{F}}^{(s)}_{k,\ell}\Big) = \iota\Big(\chi_q\Big(\mathcal{F}^{(s)}_{k,\ell}\Big)\Big).$$

Proof. Let m_+ be one of $B^{(s)}_{k,\ell}, C^{(s)}_{k,\ell}, D^{(s)}_{k,\ell}, E^{(s)}_{k,\ell}, F^{(s)}_{k,\ell}$. Then $\chi_q(\tilde{m_+})$ can be computed by the FM algorithm starting from the lowest weight using $A_{i,a}$ with $i \in I$, $a \in \mathbb{C}^\times$. The procedure is dual to the computation of $\chi_q(m_+)$ which starts from m_+ using $A^{-1}_{i,a}$ with $i \in I$, $a \in \mathbb{C}^\times$. The highest (resp. lowest) l-weight in $\chi_q(m_+)$ is sent to the lowest (resp. highest) l-weight in $\chi_q(\tilde{m_+})$ by ι. ■

Note that Lemma 7.3 can also proved using the Cartan involution in [1].

The modules $\tilde{\mathcal{B}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{C}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{D}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{E}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{F}}^{(s)}_{k,\ell}$ satisfy the same relations as in Theorem 3.4 but the roles of left and right modules are exchanged. More precisely, we have the following theorem.

Theorem 7.4. *For $s \in \mathbb{Z}$, $k, \ell \in \mathbb{Z}_{\geq 1}$, $t \in \mathbb{Z}_{\geq 2}$, we have the following relations in $\operatorname{Rep}(U_q\hat{\mathfrak{g}})$.*

$$\Big[\tilde{\mathcal{B}}^{(s+6)}_{k-1,\ell}\Big]\Big[\tilde{\mathcal{B}}^{(s)}_{k,\ell-1}\Big] = \Big[\tilde{\mathcal{B}}^{(s)}_{k,\ell}\Big]\Big[\tilde{\mathcal{B}}^{(s+6)}_{k-1,\ell-1}\Big] + \Big[\tilde{\mathcal{E}}^{(s+1)}_{3k-1,\lceil\frac{2\ell-2}{3}\rceil}\Big]\Big[\tilde{\mathcal{B}}^{(s+6k+6)}_{\lfloor\frac{\ell-1}{3}\rfloor,0}\Big],$$
$$\Big[\tilde{\mathcal{E}}^{(s)}_{0,\ell}\Big] = \Big[\tilde{\mathcal{B}}^{(s+3)}_{\lfloor\frac{\ell+1}{2}\rfloor,0}\Big]\Big[\tilde{\mathcal{B}}^{(s+5)}_{\lfloor\frac{\ell}{2}\rfloor,0}\Big], \qquad \Big[\tilde{\mathcal{E}}^{(s)}_{1,\ell}\Big] = \Big[\tilde{\mathcal{D}}^{(s-1)}_{0,\lfloor\frac{\ell}{2}\rfloor}\Big]\Big[\tilde{\mathcal{B}}^{(s+5)}_{\lfloor\frac{\ell+1}{2}\rfloor,0}\Big],$$
$$\Big[\tilde{\mathcal{E}}^{(s+2)}_{t-1,\ell}\Big]\Big[\tilde{\mathcal{E}}^{(s)}_{t,\ell-1}\Big] = \Big[\tilde{\mathcal{E}}^{(s)}_{t,\ell}\Big]\Big[\tilde{\mathcal{E}}^{(s+2)}_{t-1,\ell-1}\Big]$$
$$+\begin{cases} \Big[\tilde{\mathcal{D}}^{(s+1)}_{r,p-1}\Big]\Big[\tilde{\mathcal{B}}^{(s+3)}_{r+p,0}\Big]\Big[\tilde{\mathcal{B}}^{(s+5)}_{r,3p-2}\Big], & \text{if} \quad t=3r+2,\ \ell=2p-1,\\ \Big[\tilde{\mathcal{B}}^{(s+1)}_{r+p+1,0}\Big]\Big[\tilde{\mathcal{C}}^{(s+3)}_{r,p}\Big]\Big[\tilde{\mathcal{B}}^{(s+5)}_{r,3p-1}\Big], & \text{if} \quad t=3r+2,\ \ell=2p,\\ \Big[\tilde{\mathcal{B}}^{(s+1)}_{r+1,3p-2}\Big]\Big[\tilde{\mathcal{D}}^{(s+3)}_{r,p-1}\Big]\Big[\tilde{\mathcal{B}}^{(s+5)}_{r+p,0}\Big], & \text{if} \quad t=3r+3,\ \ell=2p-1,\\ \Big[\tilde{\mathcal{B}}^{(s+1)}_{r+1,3p-1}\Big]\Big[\tilde{\mathcal{B}}^{(s+3)}_{r+p+1,0}\Big]\Big[\tilde{\mathcal{C}}^{(s+5)}_{r,p}\Big], & \text{if} \quad t=3r+3,\ \ell=2p,\\ \Big[\tilde{\mathcal{B}}^{(s+1)}_{r+p+1,0}\Big]\Big[\tilde{\mathcal{B}}^{(s+3)}_{r+1,3p-2}\Big]\Big[\tilde{\mathcal{D}}^{(s+5)}_{r,p-1}\Big], & \text{if} \quad t=3r+4,\ \ell=2p-1,\\ \Big[\tilde{\mathcal{C}}^{(s+1)}_{r+1,p}\Big]\Big[\tilde{\mathcal{B}}^{(s+3)}_{r+1,3p-1}\Big]\Big[\tilde{\mathcal{B}}^{(s+5)}_{r+p+1,0}\Big], & \text{if} \quad t=3r+4,\ \ell=2p, \end{cases}$$
$$\Big[\tilde{\mathcal{C}}^{(s+6)}_{k-1,\ell}\Big]\Big[\tilde{\mathcal{C}}^{(s)}_{k,\ell-1}\Big] = \Big[\tilde{\mathcal{C}}^{(s)}_{k,\ell}\Big]\Big[\tilde{\mathcal{C}}^{(s+6)}_{k-1,\ell-1}\Big] + \Big[\tilde{\mathcal{F}}^{(s+1)}_{3k-2,3\ell-2}\Big],$$

$$\begin{aligned}
&\left[\tilde{\mathcal{B}}^{(s+8)}_{\ell,0}\right]\left[\tilde{\mathcal{D}}^{(s)}_{0,\ell-1}\right] = \left[\tilde{\mathcal{D}}^{(s)}_{0,\ell}\right]\left[\tilde{\mathcal{B}}^{(s+8)}_{\ell-1,0}\right] + \left[\tilde{\mathcal{B}}^{(s+4)}_{0,3\ell-1}\right],\\
&\left[\tilde{\mathcal{D}}^{(s+6)}_{k-1,\ell}\right]\left[\tilde{\mathcal{D}}^{(s)}_{k,\ell-1}\right] = \left[\tilde{\mathcal{D}}^{(s)}_{k,\ell}\right]\left[\tilde{\mathcal{D}}^{(s+6)}_{k-1,\ell-1}\right] + \left[\tilde{\mathcal{F}}^{(s+1)}_{3k-1,3\ell-1}\right],\\
&\left[\tilde{\mathcal{F}}^{(s+2)}_{k-1,\ell}\right]\left[\tilde{\mathcal{F}}^{(s)}_{k,\ell-1}\right] = \left[\tilde{\mathcal{F}}^{(s)}_{k,\ell}\right]\left[\tilde{\mathcal{F}}^{(s+2)}_{k-1,\ell-1}\right]\\
&\qquad + \begin{cases}
\left[\tilde{\mathcal{B}}^{(s+1)}_{r,0}\right]\left[\tilde{\mathcal{D}}^{(s+3)}_{r,\lfloor\frac{\ell}{3}\rfloor}\right]\left[\tilde{\mathcal{C}}^{(s+5)}_{r,\lfloor\frac{\ell+1}{3}\rfloor}\right]\left[\tilde{\mathcal{B}}^{(s+2k+11)}_{\lfloor\frac{\ell-1}{3}\rfloor,0}\right], & \textit{if} \quad k=3r+1,\\
\left[\tilde{\mathcal{C}}^{(s+1)}_{r+1,\lfloor\frac{\ell+1}{3}\rfloor}\right]\left[\tilde{\mathcal{B}}^{(s+3)}_{r,0}\right]\left[\tilde{\mathcal{D}}^{(s+5)}_{r,\lfloor\frac{\ell}{3}\rfloor}\right]\left[\tilde{\mathcal{B}}^{(s+2k+11)}_{\lfloor\frac{\ell-1}{3}\rfloor,0}\right], & \textit{if} \quad k=3r+2,\\
\left[\tilde{\mathcal{D}}^{(s+1)}_{r+1,\lfloor\frac{\ell}{3}\rfloor}\right]\left[\tilde{\mathcal{C}}^{(s+3)}_{r+1,\lfloor\frac{\ell+1}{3}\rfloor}\right]\left[\tilde{\mathcal{B}}^{(s+5)}_{r,0}\right]\left[\tilde{\mathcal{B}}^{(s+2k+11)}_{\lfloor\frac{\ell-1}{3}\rfloor,0}\right], & \textit{if} \quad k=3r+3.
\end{cases}
\end{aligned}$$

Moreover, the modules corresponding to each summand on the right hand side of the above relations are all irreducible.

Proof. The theorem follows from the relations in Theorem 3.4, Theorem 3.5, and Lemma 7.3. ■

The following proposition is similar to Proposition 3.6.

Proposition 7.5. *Given $\chi_q(1_s)$, $\chi_q(2_s)$, one can obtain the q-characters of $\tilde{\mathcal{B}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{C}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{D}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{E}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{F}}^{(s)}_{k,\ell}$, $s\in\mathbb{Z}$, $k,\ell\in\mathbb{Z}_{\geq 0}$, recursively, by using (7.1), and computing the q-character of the top module through the q-characters of other modules in relations in Theorem 7.4.*

8 Dimensions

In this section, we give dimension formulas for the modules $\mathcal{B}^{(s)}_{k,\ell}$, $\mathcal{C}^{(s)}_{k,\ell}$, $\mathcal{D}^{(s)}_{k,\ell}$, $\mathcal{E}^{(s)}_{k,\ell}$, $\mathcal{F}^{(s)}_{k,\ell}$, $\tilde{\mathcal{B}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{C}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{D}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{E}}^{(s)}_{k,\ell}$, $\tilde{\mathcal{F}}^{(s)}_{k,\ell}$.

Note that dimensions do not depend on the upper index s. Note also that $\dim M = \dim \tilde{M}$ for each $M = \mathcal{B}^{(s)}_{k,\ell}$, $\mathcal{C}^{(s)}_{k,\ell}$, $\mathcal{D}^{(s)}_{k,\ell}$, $\mathcal{E}^{(s)}_{k,\ell}$, $\mathcal{F}^{(s)}_{k,\ell}$.

Theorem 8.1. *Let $s\in\mathbb{Z}$, $k,\ell\in\mathbb{Z}_{\geq 0}$. Then*

$$\begin{aligned}
\dim \mathcal{B}^{(s)}_{k,3\ell} &= (\ell+2)(\ell+1)(1+k)(k+3+\ell)(k+2+\ell)\\
&\quad\times \big(54\ell^3k^3 + 243\ell^2k^3 + 363\ell k^3 + 180k^3 + 2784\ell^2k^2 + 1080k^2 + 162\ell^4k^2\\
&\quad + 2880\ell k^2 + 1134\ell^3k^2 + 162\ell^5k + 1539\ell^4k + 5490\ell^3k + 9132\ell^2k + 7057\ell k\\
&\quad + 2040k + 54\ell^6 + 648\ell^5 + 3069\ell^4 + 7272\ell^3 + 8977\ell^2 + 5380\ell + 1200\big)/14400,\\
\dim \mathcal{B}^{(s)}_{k,3\ell+1} &= (\ell+3)(\ell+2)(\ell+1)(1+k)(k+2+\ell)(k+4+\ell)(k+3+\ell)\\
&\quad\times \big(171\ell k^2 + 120k^2 + 54\ell^2k^2 + 600k + 621\ell^2k + 108\ell^3k\\
&\quad + 1116\ell k + 54\ell^4 + 450\ell^3 + 1341\ell^2 + 1665\ell + 700\big)/14400,\\
\dim \mathcal{B}^{(s)}_{k,3\ell+2} &= (\ell+3)(\ell+2)(\ell+1)(1+k)(k+4+\ell)(k+3+\ell)(2+k+\ell)\\
&\quad\times \big(300k^2 + 261\ell k^2 + 54\ell^2k^2 + 891\ell^2k + 2376\ell k + 2040k\\
&\quad + 108\ell^3k + 54\ell^4 + 630\ell^3 + 2691\ell^2 + 4995\ell + 3400\big)/14400,\\
\dim \mathcal{C}^{(s)}_{k,\ell} &= (\ell+2)(\ell+1)(k+2)(k+1)(k+3+\ell)(k+2+\ell)\\
&\quad\times \big(3k^2 + 3\ell k^2 + 12k + 15\ell k + 3\ell^2k + 3\ell^2 + 12\ell + 10\big)/240,\\
\dim \mathcal{D}^{(s)}_{k,\ell} &= (\ell+2)(\ell+1)(k+2)(k+1)(k+3+\ell)(k+4+\ell)\\
&\quad\times \big(3\ell k^2 + 6k^2 + 3\ell^2k + 30k + 21\ell k + 6\ell^2 + 30\ell + 35\big)/240,
\end{aligned}$$

$$\begin{aligned}
\dim \mathcal{E}^{(s)}_{3k,2\ell} = {} & (\ell+2)(\ell+1)(k+1)(k+\ell+1)(k+\ell+2)^2(k+\ell+3)^2 \\
& \times \big(27k^4\ell^2 + 81k^4\ell + 54k^4 + 81k^3\ell^3 + 468k^3\ell^2 + 825k^3\ell \\
& + 432k^3 + 81k^2\ell^4 + 711k^2\ell^3 + 2184k^2\ell^2 + 2754k^2\ell + 1179k^2 + 27k\ell^5 \\
& + 342k\ell^4 + 1593k\ell^3 + 3438k\ell^2 + 3435k\ell + 1260k + 18\ell^5 \\
& + 180\ell^4 + 696\ell^3 + 1296\ell^2 + 1160\ell + 400\big)/28800, \\
\dim \mathcal{E}^{(s)}_{3k,2\ell+1} = {} & (\ell+3)(\ell+2)(\ell+1)(k+1)(k+\ell+4)(k+\ell+2)^2(k+\ell+3)^2 \\
& \times \big(27k^4\ell + 54k^4 + 81k^3\ell^2 + 414k^3\ell + 510k^3 + 81k^2\ell^3 \\
& + 684k^2\ell^2 + 1842k^2\ell + 1611k^2 + 27k\ell^4 + 342k\ell^3 \\
& + 1512k\ell^2 + 2808k\ell + 1875k + 18\ell^4 + 180\ell^3 + 642\ell^2 + 960\ell + 500\big)/28800, \\
\dim \mathcal{E}^{(s)}_{3k+1,2\ell} = {} & (\ell+2)(\ell+1)(k+1)(k+\ell+4)(k+\ell+2)^2(k+\ell+3)^2 \\
& \times \big(27k^4\ell^2 + 81k^4\ell + 54k^4 + 81k^3\ell^3 + 477k^3\ell^2 + 852k^3\ell \\
& + 450k^3 + 81k^2\ell^4 + 747k^2\ell^3 + 2373k^2\ell^2 + 3069k^2\ell + 1341k^2 \\
& + 27k\ell^5 + 387k\ell^4 + 1935k\ell^3 + 4353k\ell^2 + 4461k\ell + 1665k \\
& + 36\ell^5 + 360\ell^4 + 1374\ell^3 + 2490\ell^2 + 2140\ell + 700\big)/28800, \\
\dim \mathcal{E}^{(s)}_{3k+1,2\ell+1} = {} & (\ell+3)(\ell+2)(\ell+1)(k+1)(k+\ell+2)(k+\ell+3)^2(k+\ell+4)^2 \\
& \times \big(27k^4\ell + 54k^4 + 81k^3\ell^2 + 450k^3\ell + 582k^3 + 81k^2\ell^3 + 774k^2\ell^2 \\
& + 2310k^2\ell + 2193k^2 + 27k\ell^4 + 414k\ell^3 + 2124k\ell^2 + 4488k\ell \\
& + 3375k + 36\ell^4 + 396\ell^3 + 1590\ell^2 + 2760\ell + 1750\big)/28800, \\
\dim \mathcal{E}^{(s)}_{3k+2,2\ell} = {} & (\ell+2)(\ell+1)(k+2)(k+1)(k+\ell+4)(k+\ell+2)(k+\ell+3)^2 \\
& \times \big(27k^4\ell^2 + 81k^4\ell + 54k^4 + 108k^3\ell^3 + 648k^3\ell^2 + 1176k^3\ell \\
& + 630k^3 + 162k^2\ell^4 + 1458k^2\ell^3 + 4629k^2\ell^2 + 6057k^2\ell \\
& + 2691k^2 + 108k\ell^5 + 1296k\ell^4 + 5946k\ell^3 + 12942k\ell^2 + 13230k\ell + 4995k \\
& + 27\ell^6 + 405\ell^5 + 2439\ell^4 + 7515\ell^3 + 12429\ell^2 + 10395\ell + 3400\big)/28800, \\
\dim \mathcal{E}^{(s)}_{3k+2,2\ell+1} = {} & (\ell+3)(\ell+2)(\ell+1)(k+2)(k+1)(k+\ell+5)(k+\ell+2) \\
& \times \big(k+\ell+3)^2(k+\ell+4)^2(9k^2\ell + 18k^2 \\
& + 18k\ell^2 + 99k\ell + 128k + 9\ell^3 + 81\ell^2 + 237\ell + 225\big)/9600, \\
\dim \mathcal{F}^{(s)}_{3k,3\ell} = {} & (\ell+2)^2(\ell+1)^2(k+2)^2(k+1)^2(k+\ell+3)^2 \\
& \times \big(27k^4\ell^2 + 81k^4\ell + 54k^4 + 54k^3\ell^3 + 405k^3\ell^2 + 801k^3\ell + 432k^3 + 27k^2\ell^4 \\
& + 405k^2\ell^3 + 1746k^2\ell^2 + 2646k^2\ell + 1179k^2 + 81k\ell^4 + 801k\ell^3 \\
& + 2646k\ell^2 + 3342k\ell + 1260k + 54\ell^4 + 432\ell^3 + 1179\ell^2 + 1260\ell + 400\big)/57600, \\
\dim \mathcal{F}^{(s)}_{3k+1,3\ell} = {} & (\ell+2)^2(\ell+1)^2(k+3)(k+1)(k+2)^2(k+\ell+4)(k+\ell+3) \\
& \times \big(27k^4\ell^2 + 81k^4\ell + 54k^4 + 54k^3\ell^3 + 414k^3\ell^2 + 828k^3\ell + 450k^3 + 27k^2\ell^4 \\
& + 414k^2\ell^3 + 1854k^2\ell^2 + 2907k^2\ell + 1341k^2 + 81k\ell^4 + 864k\ell^3 + 3063k\ell^2 \\
& + 4116k\ell + 1665k + 54\ell^4 + 498\ell^3 + 1563\ell^2 + 1905\ell + 700\big)/57600, \\
\dim \mathcal{F}^{(s)}_{3k+2,3\ell} = {} & (\ell+2)^2(\ell+1)^2(k+3)(k+1)(k+2)^2(k+\ell+4)(k+\ell+3) \\
& \times \big(27k^4\ell^2 + 81k^4\ell + 54k^4 + 54k^3\ell^3 + 504k^3\ell^2 + 1098k^3\ell + 630k^3 + 27k^2\ell^4 \\
& + 558k^2\ell^3 + 3042k^2\ell^2 + 5355k^2\ell + 2691k^2 + 135k\ell^4 + 1764k\ell^3 + 7395k\ell^2
\end{aligned}$$

$$
\begin{aligned}
&+ 11190k\ell + 4995k + 162\ell^4 + 1734\ell^3 + 6249\ell^2 + 8475\ell + 3400\big)/57600,\\
\dim \mathcal{F}^{(s)}_{3k,3\ell+1} &= (\ell+3)(\ell+1)(\ell+2)^2(k+2)^2(k+1)^2(k+\ell+4)(k+\ell+3)\\
&\times \big(27k^4\ell^2 + 81k^4\ell + 54k^4 + 54k^3\ell^3 + 414k^3\ell^2 + 864k^3\ell + 498k^3 + 27k^2\ell^4\\
&+ 414k^2\ell^3 + 1854k^2\ell^2 + 3063k^2\ell + 1563k^2 + 81k\ell^4 + 828k\ell^3 + 2907k\ell^2\\
&+ 4116k\ell + 1905k + 54\ell^4 + 450\ell^3 + 1341\ell^2 + 1665\ell + 700\big)/57600,\\
\dim \mathcal{F}^{(s)}_{3k+1,3\ell+1} &= (\ell+3)(\ell+1)(\ell+2)^2(k+3)(k+1)(k+2)^2(k+\ell+3)(k+\ell+4)\\
&\times \big(27k^4\ell^2 + 81k^4\ell + 54k^4 + 54k^3\ell^3 + 450k^3\ell^2 + 972k^3\ell + 570k^3 + 27k^2\ell^4\\
&+ 450k^2\ell^3 + 2214k^2\ell^2 + 3891k^2\ell + 2061k^2 + 81k\ell^4 + 972k\ell^3 + 3891k\ell^2\\
&+ 6060k\ell + 2985k + 54\ell^4 + 570\ell^3 + 2061\ell^2 + 2985\ell + 1400\big)/57600,\\
\dim \mathcal{F}^{(s)}_{3k+2,3\ell+1} &= (\ell+3)(\ell+1)(\ell+2)^2(k+3)(k+1)(k+2)^2(k+\ell+3)(k+\ell+5)\\
&\times \big(k+\ell+4)^2(9k^2\ell^2 + 27k^2\ell + 18k^2 + 45k\ell^2 + 135k\ell\\
&+ 88k + 54\ell^2 + 164\ell + 105\big)/19200,\\
\dim \mathcal{F}^{(s)}_{3k,3\ell+2} &= (\ell+3)(\ell+1)(\ell+2)^2(k+2)^2(k+1)^2(k+\ell+4)(k+\ell+3)\big(27k^4\ell^2\\
&+ 135k^4\ell + 162k^4 + 54k^3\ell^3 + 558k^3\ell^2 + 1764k^3\ell + 1734k^3 + 27k^2\ell^4\\
&+ 504k^2\ell^3 + 3042k^2\ell^2 + 7395k^2\ell + 6249k^2 + 81k\ell^4 + 1098k\ell^3 + 5355k\ell^2\\
&+ 11190k\ell + 8475k + 54\ell^4 + 630\ell^3 + 2691\ell^2 + 4995\ell + 3400\big)/57600,\\
\dim \mathcal{F}^{(s)}_{3k+1,3\ell+2} &= (\ell+3)(\ell+1)(\ell+2)^2(k+3)(k+1)(k+2)^2(k+\ell+3)(k+\ell+5)\\
&\times \big(k+\ell+4)^2(9k^2\ell^2 + 45k^2\ell + 54k^2 + 27k\ell^2 + 135k\ell\\
&+ 164k + 18\ell^2 + 88\ell + 105\big)/19200,\\
\dim \mathcal{F}^{(s)}_{3k+2,3\ell+2} &= (\ell+3)(\ell+1)(\ell+2)^2(k+3)(k+1)(k+2)^2(k+\ell+4)(k+\ell+5)\\
&\times \big(27k^4\ell^2 + 135k^4\ell + 162k^4 + 54k^3\ell^3 + 630k^3\ell^2 + 2124k^3\ell + 2166k^3\\
&+ 27k^2\ell^4 + 630k^2\ell^3 + 4374k^2\ell^2 + 11661k^2\ell + 10473k^2 + 135k\ell^4\\
&+ 2124k\ell^3 + 11661k\ell^2 + 26748k\ell + 21759k + 162\ell^4 + 2166\ell^3\\
&+ 10473\ell^2 + 21759\ell + 16400\big)/57600.
\end{aligned}
$$

Proof. We check the initial conditions, namely dimensions of $\mathcal{B}^{(s)}_{0,1}$, $\mathcal{B}^{(s)}_{1,0}$. We check the dimensions are compatible with relations (3.1), (3.2), (3.3). We directly check that the formulas satisfy the relations in Theorems 3.4. For the checks we employed the computer algebra system Maple.

The theorem follows since the solution of the extended T-system is unique, see Proposition 3.6. ∎

Acknowledgements

We would like to thank D. Hernandez, B. Leclerc, T. Nakanishi, C.A.S. Young for helpful discussions. JL would like to thank IUPUI Department of Mathematical Sciences for hospitality during his visit when this work was carried out. JL is partially supported by a CSC scholarship, the Natural Science Foundation of Gansu Province (No. 1107RJZA218), and the Fundamental Research Funds for the Central Universities (No. lzujbky-2012-12) from China. The research of EM is supported by the NSF, grant number DMS-0900984.

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