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Multi-Hamiltonian Structures on Spaces of Closed Equicentroaffine Plane Curves Associated to Higher KdV Flows

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Abstract. Higher KdV flows on spaces of closed equicentroaffine plane curves are studied and it is shown that the flows are described as certain multi-Hamiltonian systems on the spaces. Multi-Hamiltonian systems describing higher mKdV flows are also given on spaces of closed Euclidean plane curves via the geometric Miura transformation.

Key words: motions of curves; equicentroaffine curves; KdV hierarchy; multi-Hamiltonian systems

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1 Introduction

A motion of a curve is a smooth one-parameter family of connected curves in a space. It is known that many differential equations related to integrable systems can be linked with special motions of curves [10, 11, 12, 29]. For example, for a special motion of an inextensible curve in the Euclidean plane, the curvature evolves according to the modified Korteweg–de Vries (mKdV) equation [19] (cf. Section 4 below). There are a lot of preceding studies on motions of curves related to Euclidean geometry and the mKdV equation. See [24, 30, 32] and references therein. For special motions of a space curve, it is also known that the nonlinear Schrödinger equation appears [16]. In [13, 14], the authors studied motions of a curve in the complex hyperbola under which the curvature evolves according to the Burgers equation.

In this paper, we shall study motions of an equicentroaffine plane curve. Under a special motion of an equicentroaffine plane curve, the equicentroaffine curvature evolves according to the Korteweg–de Vries (KdV) equation. In order to explain the above motion geometrically, Pinkall [28] introduced the natural presymplectic form on the space of closed equicentroaffine plane curves with fixed enclosing area, and showed that the equicentroaffine curvature evolves according to the KdV equation when the flow is generated by the total equicentroaffine curvature. Furthermore, the result has been generalized to the case of higher KdV flows (cf. [9, 15]).

On the other hand, it is known that a lot of completely integrable systems are described as bi-Hamiltonian systems, from which the existence of many first integrals can be deduced (Magri's theorem [22, 27]). In this context, many of motions of curves as above have been studied from the viewpoint of bi-Hamiltonian systems recently [1, 2, 3, 4, 5, 6, 7, 8, 21, 23, 24, 31]. The purpose of this paper is to construct a multi-Hamiltonian structure associated to the higher KdV flows on each level set of Hamiltonian functions in a geometric way (Theorem 7). Moreover, we shall also introduce multi-Hamiltonian structures associated to the higher mKdV flows on the spaces of closed Euclidean plane curves via the geometric Miura transformation.

2 A bi-Hamiltonian structure on the space of closed equicentroaffine curves

Throughout this paper all maps are assumed to be smooth.

For a regular plane curve γ whose velocity vector is transversal to the position vector at each point, we can choose the parameter s of γ as $\det\begin{pmatrix}\gamma(s)\\ \gamma_s(s)\end{pmatrix}\equiv 1$ holds. A plane curve γ provided with such a parameter s is called an *equicentroaffine* plane curve. For an equicentroaffine plane curve γ, we can define a function κ, called the *equicentroaffine curvature*, by $\gamma_{ss}=-\kappa\gamma$.

We set the space $\mathcal{M}$ of closed equicentroaffine plane curves by

$$\mathcal{M}=\left\{\gamma : S^1\to\mathbb{R}^2\setminus\{0\}\,\middle|\,\det\begin{pmatrix}\gamma\\ \gamma_s\end{pmatrix}=1\right\},$$

where $S^1=\mathbb{R}/2\pi\mathbb{Z}$. Let $\gamma(\,\cdot\,,t)\in\mathcal{M}$ be a one-parameter family of closed equicentroaffine plane curves. As in [28], the motion vector field γ_t is represented as

$$\gamma_t=-\frac{1}{2}\alpha_s\gamma+\alpha\gamma_s,\qquad \alpha:\ S^1\to\mathbb{R},\tag{1}$$

and the equicentroaffine curvature κ evolves as

$$\kappa_t=\Omega\alpha_s=\frac{1}{2}\alpha_{sss}+2\kappa\alpha_s+\kappa_s\alpha,\tag{2}$$

where

$$\Omega=\frac{1}{2}D_s^2+2\kappa+\kappa_sD_s^{-1},\qquad D_s=\frac{\partial}{\partial s},$$

is the *recursion operator of the KdV equation*:

$$\kappa_t=\Omega\kappa_s=\frac{1}{2}\kappa_{sss}+3\kappa\kappa_s.$$

Hence when we choose the one-parameter family $\gamma(\,\cdot\,,t)$ as $\alpha=D_s^{-1}\Omega^{n-1}\kappa_s$, we obtain the nth KdV equation for κ:

$$\kappa_t=\Omega^n\kappa_s.\tag{3}$$

The tangent space of $\mathcal{M}$ at $\gamma\in\mathcal{M}$ is described as

$$T_\gamma\mathcal{M}=\left\{-\frac{1}{2}\alpha_s\gamma+\alpha\gamma_s\,\middle|\,\alpha:S^1\to\mathbb{R}\right\},$$

and we can define a presymplectic form ω_0 on $\mathcal{M}$ by

$$\omega_0(X,Y)=\int_{S^1}\det\begin{pmatrix}X\\ Y\end{pmatrix}ds,\qquad X,Y\in T_\gamma\mathcal{M}.$$

When X and Y are given by

$$X=-\frac{1}{2}\alpha_s\gamma+\alpha\gamma_s,\qquad Y=-\frac{1}{2}\beta_s\gamma+\beta\gamma_s,\qquad \alpha,\beta:\ S^1\to\mathbb{R},\tag{4}$$

a direct calculation shows that

$$\omega_0(X,Y)=\int_{S^1}\alpha\beta_s ds,$$

from which we see that the kernel of ω_0 at γ is $\mathbb{R}\cdot\gamma_s$.

It is known that the higher KdV equation (3) as well as (2) has an infinite series of conserved quantities $\{H_m\}_{m\in\mathbb{N}}$ given in the form of

$$H_m = \int_{S^1} h_m(\kappa, \kappa_s, \kappa_{ss}, \dots)ds,$$

where h_m is a polynomial in κ and its derivatives up to order m, for example,

$$h_1 = \kappa, \qquad h_2 = \frac{1}{2}\kappa^2, \qquad h_3 = \frac{1}{2}\kappa^3 - \frac{1}{4}\kappa_s^2$$

(see [17, 20, 25, 26]). Moreover, by using the conserved quantity, nth KdV equation (3) can be expressed as

$$\kappa_t = D_s \frac{\delta H_{n+2}}{\delta \kappa}, \tag{5}$$

where $\delta H_{n+2}/\delta\kappa$ is the *variational derivative* of H_{n+2}:

$$\frac{\delta H_{n+2}}{\delta \kappa} = \frac{\partial h_{n+2}}{\partial \kappa} - D_s \frac{\partial h_{n+2}}{\partial \kappa_s} + D_s^2 \frac{\partial h_{n+2}}{\partial \kappa_{ss}} - \cdots .$$

The expression (5) played an important role in computation in [15], where we studied the higher KdV flows on the space of closed equicentroaffine curves as Hamiltonian systems; using the above presymplectic structure ω_0, we gave the Hamiltonian flows associated with the higher KdV equations. The paper [15] deals also with the geometric Miura transformation as is mentioned in Section 5 below.

For each $n \in \mathbb{N}$, we define a vector field X_n on $\mathcal{M}$ by

$$(X_n)_\gamma = -\frac{1}{2}\left(\Omega^{n-1}\kappa_s\right)\gamma + \left(D_s^{-1}\Omega^{n-1}\kappa_s\right)\gamma_s, \qquad \gamma \in \mathcal{M}.$$

Regarding $\{H_m\}_{m\in\mathbb{N}}$ as functions on $\mathcal{M}$ by substituting the equicentroaffine curvature of γ for κ, we have the following proposition, which is essentially due to Pinkall [28] in the case $n = 1$.

Proposition 1 ([15])**.** *For each $n \in \mathbb{N}$, X_n is a Hamiltonian vector field for H_n with respect to ω_0, i.e., $dH_n = \omega_0(X_n, \cdot)$ holds. Hence H_n is a Hamiltonian function for the nth KdV flow $\gamma_t = X_n$.*

Now, we define another form ω_1 on $\mathcal{M}$ by

$$\omega_1(X, Y) = \int_{S^1} \det \begin{pmatrix} X \\ (D_s^2 + \kappa)Y \end{pmatrix} ds, \qquad X, Y \in T_\gamma\mathcal{M},$$

which is represented as

$$\omega_1(X, Y) = \int_{S^1} \alpha\Omega\beta_s ds \tag{6}$$

for X, Y given by (4). The following shows that ω_0 and ω_1 with $\{H_m\}_{m\in N}$ define a bi-Hamiltonian structure on $\mathcal{M}$ (cf. [22, 27]).

Theorem 2. *The form ω_1 is a presymplectic form on $\mathcal{M}$. For each $n \in \mathbb{N}$, X_n is a Hamiltonian vector field for H_{n+1} with respect to ω_1.*

Proof. For two functions F and G on $\mathcal{M}$ of the form

$$F = \int_{S^1} f(\kappa, \kappa_s, \kappa_{ss}, \dots) ds, \qquad G = \int_{S^1} g(\kappa, \kappa_s, \kappa_{ss}, \dots) ds, \tag{7}$$

we set

$$\{F, G\}_1 = \int_{S^1} \frac{\delta F}{\delta \kappa} \Omega D_s \frac{\delta G}{\delta \kappa} ds.$$

Then from [18, 22], we see that $\{\,\cdot\,,\,\cdot\,\}_1$ provides a Poisson bracket with

$$X_n = -\{H_{n+1},\,\cdot\,\}_1.$$

We put $\widetilde{\alpha}_F = \delta F/\delta\kappa$ and $(\widetilde{X}_F)_\gamma = -(1/2)(\widetilde{\alpha}_F)_s\gamma + \widetilde{\alpha}_F\gamma_s$. Since the differentiation of F along a motion $\gamma_t = X_\gamma = -(1/2)\alpha_s\gamma + \alpha\gamma_s$ is given as

$$XF = \frac{dF}{dt} = \int_{S^1} \frac{\delta F}{\delta\kappa}\kappa_t ds = \int_{S^1} \frac{\delta F}{\delta\kappa}\Omega\alpha_s ds,$$

we have

$$\omega_1(\widetilde{X}_F, X) = \int_{S^1} \frac{\delta F}{\delta\kappa}\Omega\alpha_s ds = XF = dF(X)$$

and

$$\omega_1(\widetilde{X}_F, \widetilde{X}_G) = \int_{S^1} \frac{\delta F}{\delta\kappa}\Omega D_s \frac{\delta G}{\delta\kappa} = \{F, G\}_1.$$

Hence ω_1 is skew-symmetric and its closedness follows from the Jacobi identity for $\{\,\cdot\,,\,\cdot\,\}_1$ since for functions F, G and $H = \int_{S^1} h(\kappa, \kappa_s, \kappa_{ss}, \dots) ds$ on $\mathcal{M}$ we have

$$d\omega\big(\widetilde{X}_F, \widetilde{X}_G, \widetilde{X}_H\big) = 2\big(\{\{F, G\}_1, H\}_1 + \{\{G, H\}_1, F\}_1 + \{\{H, F\}_1, G\}_1\big) = 0.$$

Moreover, since

$$\widetilde{X}_F G = \int_{S^1} \frac{\delta G}{\delta\kappa}\Omega D_s \frac{\delta F}{\delta\kappa} ds = \{G, F\}_1 = -\{F, G\}_1,$$

we obtain $X_n = \widetilde{X}_{H_{n+1}}$ and hence

$$\omega_1(X_n,\,\cdot\,) = \omega_1\big(\widetilde{X}_{H_{n+1}},\,\cdot\,\big) = dH_{n+1}.$$

Therefore X_n is a Hamiltonian vector field for H_{n+1} with respect to ω_1. ■

The special linear group of degree two $\mathrm{SL}(2;\mathbb{R})$ acts on $\mathcal{M}$ as $\mathcal{M} \ni \gamma \mapsto A\gamma \in \mathcal{M}$ ($A \in \mathrm{SL}(2;\mathbb{R})$). Two elements of $\mathcal{M}$ belong to the same orbit if and only if their equicentroaffine curvatures coincide. Hence ω_1 is invariant under the action of $\mathrm{SL}(2;\mathbb{R})$. Moreover, the kernel of ω_1 at γ is the tangent space of the orbit $\mathrm{SL}(2;\mathbb{R}) \cdot \gamma$; indeed for a one-parameter family $\gamma(\cdot, t) \in \mathcal{M}$, it follows from (2) and (6) that the tangent vector (1) belongs to the kernel of ω_1 if and only if $\kappa_t = 0$, that is, κ is independent of t and hence $\gamma(\cdot, t)$ is contained in an $\mathrm{SL}(2;\mathbb{R})$-orbit. As a consequence, ω_1 defines a symplectic form on the quotient space $\mathcal{M}/\mathrm{SL}(2;\mathbb{R})$.

We consider another action on $\mathcal{M}$ given by

$$\mathcal{M} \ni \gamma \mapsto \gamma(\,\cdot + \sigma) \in \mathcal{M}, \qquad \sigma \in S^1. \tag{8}$$

It is obvious that this S^1-action is presymplectic, that is, it leaves ω_1 invariant. Moreover, the action is Hamiltonian as we see in the proof of the following theorem.

Theorem 3. *The moment map μ_1 for the S^1-action (8) with respect to ω_1 is given by*

$$\mu_1(\gamma)\left(\frac{\partial}{\partial\sigma}\right) = H_1(\gamma), \qquad \gamma \in \mathcal{M}. \tag{9}$$

Proof. The fundamental vector field $\underline{A}$ on $\mathcal{M}$ corresponding to $\partial/\partial\sigma \in \mathrm{Lie}(S^1)$ is given by $\underline{A}_\gamma = \gamma_s$ ($\gamma \in \mathcal{M}$). For any tangent vector $\gamma_t = -(1/2)\alpha_s\gamma + \alpha\gamma_s$, we have

$$\omega_1(\underline{A}, \gamma_t) = \omega_1(\gamma_s, \gamma_t) = \int_{S^1} \Omega\alpha_s ds = \int_{S^1} \kappa_t ds = \frac{d}{dt} H_1(\gamma) = dH_1(\gamma_t),$$

which implies (9) by the definition of the moment map. ∎

Remark 4. Let Φ_n^τ be the flow generated by X_n, that is, $\Phi_n^{\cdot}$ is a one-parameter transformation group of $\mathcal{M}$ such that

$$\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \Phi_n^\tau(\gamma) = (X_n)_\gamma, \qquad \gamma \in \mathcal{M}.$$

As an $\mathbb{R}$-action on $\mathcal{M}$, $\Phi_n^{\cdot}$ is Hamiltonian with respect to ω_0 and the corresponding moment map is given by H_n.

3 Multi-Hamiltonian structures on the level sets of Hamiltonians

For a given sequence of real numbers $C = \{c_k\}_{k\in\mathbb{N}}$, we define subsets $\mathcal{M}(C_m)$ ($m = 1, 2, \dots$) of $\mathcal{M}$ by

$$\mathcal{M}(C_m) = H_1^{-1}(c_1) \cap \cdots \cap H_m^{-1}(c_m).$$

In the following, we assume that each $\mathcal{M}(C_m)$ is not an empty set.

Lemma 5. *For functions α, β on S^1, if $D_s^{-1}\Omega D_s\alpha$ is determined as a function on S^1, then we have*

$$\int_{S^1} \left(D_s^{-1}\Omega D_s\alpha\right) \cdot \beta_s ds = \int_{S^1} \alpha\Omega\beta_s ds. \tag{10}$$

Proof. Noting $\Omega D_s = (1/2)D_s^3 + \kappa D_s + D_s\kappa$, we can easily verify (10) by integration by parts. ∎

Proposition 6. *For $\gamma \in \mathcal{M}(C_m)$ and $X = -(1/2)\alpha_s\gamma + \alpha\gamma_s \in T_\gamma\mathcal{M}(C_m)$, $\Omega\alpha_s, \Omega^2\alpha_s, \dots,$ $\Omega^{m+1}\alpha_s$ are defined as functions on S^1 and $\int_{S^1} \Omega^k\alpha_s ds = 0$ for any $k = 1, 2, \dots, m$.*

Proof. We shall show the proposition by induction on m. In the case $m = 1$, $\Omega\alpha_s = (1/2)\alpha_{sss} + 2\kappa\alpha_s + \kappa_s\alpha$ is a function on S^1 and we have

$$\int_{S^1} \Omega\alpha_s ds = \int_{S^1} \kappa\alpha_s ds = \omega_0(X_1, X) = dH_1(X),$$

which vanishes since $X \in T_\gamma\mathcal{M}(C_1) = \mathrm{Ker}(dH_1)_\gamma$; moreover, this implies that $D_s^{-1}\Omega\alpha_s$, and consequently $\Omega^2\alpha_s = ((1/2)D_s^2 + 2\kappa + \kappa_s D_s^{-1})\Omega\alpha_s$ are defined on S^1.

We assume that the proposition holds for $m = l$ for some $l \geq 1$. Then, for $X \in T_\gamma\mathcal{M}(C_{l+1}) = T_\gamma\mathcal{M}(C_l) \cap \mathrm{Ker}(dH_{l+1})_\gamma$, by using (10) we get

$$\int_{S^1} \Omega^{l+1}\alpha_s ds = \int_{S^1} \kappa\Omega^l\alpha_s ds = \int_{S^1} \left(\left(D_s^{-1}\Omega D_s\right)^l\kappa\right)\cdot\alpha_s ds = \int_{S^1} \left(D_s^{-1}\Omega^l\kappa_s\right)\cdot\alpha_s ds$$
$$= \omega_0(X_{l+1}, X) = dH_{l+1}(X) = 0,$$

which implies that $\Omega^{l+2}\alpha_s$ is determined as a function on S^1 in the same way as in the case $m = 1$. ■

From Proposition 6, we can define a tensor field ω_{m+1} of type $(0,2)$ on $\mathcal{M}(C_m)$ by

$$\omega_{m+1}(X, Y) = \int_{S^1} \alpha\Omega^{m+1}\beta_s ds,$$

which is shown to be skew-symmetric by using (10). Furthermore, in a similar way to the proof of Theorem 2, we see that ω_{m+1} is a presymplectic form and X_n is a Hamiltonian vector field for the Hamiltonian function H_{n+m+1} with respect to ω_{m+1}; indeed, for functions F, G given by (7) and for an integer k, putting

$$\{F, G\}_k = \int_{S^1} \frac{\delta F}{\delta\kappa}\Omega^k D_s \frac{\delta G}{\delta\kappa} ds,$$

we have a family of Poisson brackets $\{\,\cdot\,,\,\cdot\,\}_k$ with

$$X_n = -\{H_{n+2-k},\,\cdot\,\}_k.$$

Setting $\widetilde{\alpha}_F = D_s^{-1}\Omega^{-m}D_s(\delta F/\delta\kappa)$ and $(\widetilde{X}_F)_\gamma = -(1/2)(\widetilde{\alpha}_F)_s\gamma + \widetilde{\alpha}_F\gamma_s$, we have

$$\omega_{m+1}\big(\widetilde{X}_F, X\big) = dF(X) \qquad \text{and} \qquad \omega_{m+1}\big(\widetilde{X}_F, \widetilde{X}_G\big) = \{F, G\}_{1-m},$$

which implies that ω_{m+1} is presymplectic. Moreover, since

$$\widetilde{X}_F G = -\{F, G\}_{1-m}$$

holds, we have $X_n = \widetilde{X}_{H_{n+m+1}}$ and

$$\omega_{m+1}(X_n,\,\cdot\,) = \omega_{m+1}\big(\widetilde{X}_{H_{n+m+1}},\,\cdot\,\big) = dH_{n+m+1}.$$

Hence X_n is a Hamiltonian vector field of H_{n+m+1} with respect to ω_{m+1}.

Besides ω_{m+1}, we have $m+1$ more presymplectic forms on $\mathcal{M}(C_m)$ by restricting ω_0, ω_1 on $\mathcal{M}$ and ω_{k+1}'s on $\mathcal{M}(C_k)$'s for $k = 1, 2, \ldots, m-1$ to $\mathcal{M}(C_m)$; we denote them by the same symbols. By the discussion so far, we obtain the following theorem.

Theorem 7. *On $\mathcal{M}(C_m)$, for each $n \in \mathbb{N}$ and $k = 0, 1, \ldots, m+1$, X_n is a Hamiltonian vector field for H_{n+k} with respect to ω_k, that is, the set $\big(\{H_n\}_{n\in\mathbb{N}}, \{\omega_k\}_{k=0}^{m+1}\big)$ is a multi-Hamiltonian system on $\mathcal{M}(C_m)$ describing the higher KdV flows.*

As on $\mathcal{M}$, we have the following theorem for a Hamiltonian S^1-action on $\mathcal{M}(C_m)$:

$$\mathcal{M}(C_m) \ni \gamma \mapsto \gamma(\,\cdot\,+\sigma) \in \mathcal{M}(C_m), \qquad \sigma \in S^1.$$

Theorem 8. *The moment map μ_{m+1} for the S^1-action on $\mathcal{M}(C_m)$ with respect to ω_{m+1} is given by*

$$\mu_{m+1}(\gamma)\left(\frac{\partial}{\partial\sigma}\right) = H_{m+1}(\gamma), \qquad \gamma \in \mathcal{M}(C_m).$$

Remark 9. We can define ω_{m+1} in a manner similar to the definitions of ω_0 and ω_1. We put a map ϕ from $T_\gamma\mathcal{M}$ to the space of all vector fields along γ as

$$\phi X = -\alpha_s\gamma, \qquad X = -\frac{1}{2}\alpha_s\gamma + \alpha\gamma_s.$$

For any tangent vector X of $\mathcal{M}$, $(D_s^2+\kappa)X$ has no γ_s-component and it belongs to the image of ϕ if X is tangent to $\mathcal{M}(C_1)$. Then for $X \in T_\gamma\mathcal{M}(C_1)$ we have

$$\phi^{-1}\big(D_s^2+\kappa\big)X = -\frac{1}{2}(\Omega\alpha_s)\gamma + \big(D_s^{-1}\Omega\alpha_s\big)\gamma_s$$

and

$$\big(D_s^2+\kappa\big)\phi^{-1}\big(D_s^2+\kappa\big)X = -\big(\Omega^2\alpha_s\big)\gamma.$$

Hence

$$\int_{S^1} \det\begin{pmatrix} X \\ (D_s^2+\kappa)\phi^{-1}(D_s^2+\kappa)Y \end{pmatrix} ds = \omega_2(X,Y)$$

holds. More generally, $[\phi^{-1}(D_s^2+\kappa)]^m X$ can be defined for any tangent vector X of $\mathcal{M}(C_m)$ and we obtain

$$\int_{S^1} \det\begin{pmatrix} X \\ (D_s^2+\kappa)\big[\phi^{-1}(D_s^2+\kappa)\big]^m Y \end{pmatrix} ds = \omega_{m+1}(X,Y)$$

on $\mathcal{M}(C_m)$. We note that this formula is valid in the case ω_1 ($m=0$) and even in the case ω_0 ($m=-1$) since

$$\int_{S^1} \det\begin{pmatrix} X \\ \phi Y \end{pmatrix} ds = \omega_0(X,Y).$$

4 A bi-Hamiltonian structure on the space of closed curves in the Euclidean plane

We denote by $\mathbb{E}^2$ the Euclidean plane equipped with the standard inner product $\langle\,\cdot\,,\,\cdot\,\rangle$, and we set the space $\hat{\mathcal{M}}$ of closed curves in the Euclidean plane $\mathbb{E}^2$ by

$$\hat{\mathcal{M}} = \big\{\hat\gamma : S^1 \to \mathbb{E}^2 \,\big|\, \langle\hat\gamma_s(s), \hat\gamma_s(s)\rangle \equiv 1\big\}.$$

For $\hat\gamma \in \hat{\mathcal{M}}$, the curvature $\hat\kappa$ is defined by $T_s = \hat\kappa N$, where $T = \hat\gamma_s$ is the velocity vector field and N is the left-oriented unit normal vector field along $\hat\gamma$.

Let $\hat\gamma(\,\cdot\,,t) \in \hat{\mathcal{M}}$ be a one-parameter family of closed curves in $\mathbb{E}^2$. Then $\hat\gamma_t$ is represented as

$$\hat\gamma_t = \lambda T + \mu N, \qquad \lambda, \mu :\ S^1 \to \mathbb{R}, \quad \lambda_s = \hat\kappa\mu,$$

and the curvature $\hat\kappa$ evolves as

$$\hat\kappa_t = \mu_{ss} + \hat\kappa\lambda_s + \hat\kappa_s\lambda = \hat\Omega(2\mu),$$

where

$$\hat\Omega = \frac{1}{2}\big(D_s^2 + \hat\kappa^2 + \hat\kappa_s D_s^{-1}\hat\kappa\big)$$

is the *recursion operator of the mKdV equation*:

$$\hat{\kappa}_t = \hat{\Omega}\hat{\kappa}_s = \frac{1}{2}\hat{\kappa}_{sss} + \frac{3}{4}\hat{\kappa}^2\hat{\kappa}_s.$$

Hence when we choose $\mu = (1/2)\hat{\Omega}^{n-1}\hat{\kappa}_s$, we have the nth mKdV equation for $\hat{\kappa}$:

$$\hat{\kappa}_t = \hat{\Omega}^n\hat{\kappa}_s. \tag{11}$$

The tangent space of $\hat{\mathcal{M}}$ at $\hat{\gamma} \in \hat{\mathcal{M}}$ is described as

$$T_{\hat{\gamma}}\hat{\mathcal{M}} = \{\lambda T + \mu N \mid \lambda, \mu : S^1 \to \mathbb{R},\ \lambda_s = \hat{\kappa}\mu\},$$

and we can define a presymplectic form $\hat{\omega}_0$ on $\hat{\mathcal{M}}$ by

$$\hat{\omega}_0(X, Y) = \int_{S^1} \langle D_s X, Y\rangle ds, \qquad X, Y \in T_{\hat{\gamma}}\hat{\mathcal{M}}.$$

When X and Y are given by

$$X = \lambda T + \mu N, \qquad Y = \tilde{\lambda} T + \tilde{\mu} N, \qquad \lambda, \mu, \tilde{\lambda}, \tilde{\mu} :\ S^1 \to \mathbb{R}, \tag{12}$$

we have

$$\hat{\omega}_0(X, Y) = \int_{S^1} (\hat{\kappa}\lambda + \mu_s)\tilde{\mu} ds,$$

and we see that the kernel of $\hat{\omega}_0$ at $\hat{\gamma}$ is $\mathbb{R} \cdot \hat{\gamma}_s$.

As in the case of the higher KdV equation (3), the nth mKdV equation (11) can be written as

$$\hat{\kappa}_t = D_s \frac{\delta \hat{H}_{n+2}}{\delta \hat{\kappa}}$$

for an infinite series of conserved quantities $\{\hat{H}_m\}_{m\in\mathbb{N}}$ expressed in the form of

$$\hat{H}_m = \int_{S^1} \hat{h}_m(\hat{\kappa}, \hat{\kappa}_s, \hat{\kappa}_{ss}, \dots) ds,$$

where $\hat{h}_m$ is a polynomial in $\hat{\kappa}$ and its derivatives up to order m, for example,

$$\hat{h}_1 = \frac{1}{4}\hat{\kappa}^2, \qquad \hat{h}_2 = \frac{1}{32}\hat{\kappa}^4 - \frac{1}{8}\hat{\kappa}_s^2, \qquad \hat{h}_3 = \frac{1}{128}\hat{\kappa}^6 - \frac{5}{32}\hat{\kappa}^2\hat{\kappa}_s^2 + \frac{1}{16}\hat{\kappa}_{ss}^2.$$

For each $n \in \mathbb{N}$, we define a vector field $\hat{X}_n$ on $\hat{\mathcal{M}}$ by

$$\big(\hat{X}_n\big)_{\hat{\gamma}} = \frac{1}{2}\big(D_s^{-1}\big(\hat{\kappa}\hat{\Omega}^{n-1}\hat{\kappa}_s\big)\big)T + \frac{1}{2}\big(\hat{\Omega}^{n-1}\hat{\kappa}_s\big)N, \qquad \hat{\gamma} \in \hat{\mathcal{M}},$$

then we have the following.

Proposition 10 ([15])**.** *For each $n \in \mathbb{N}$, $\hat{X}_n$ is a Hamiltonian vector field for $\hat{H}_n$ with respect to $\hat{\omega}_0$. Hence $\hat{H}_n$ is a Hamiltonian function for the nth mKdV flow $\hat{\gamma}_t = \hat{X}_n$.*

In addition, we define another form $\hat{\omega}_1$ on $\hat{\mathcal{M}}$ by

$$\hat{\omega}_1(X, Y) = \int_{S^1} \big\langle D_s X, D_s^2 Y\big\rangle ds, \qquad X, Y \in T_{\hat{\gamma}}\hat{\mathcal{M}},$$

which is represented as

$$\hat{\omega}_1(X, Y) = \int_{S^1} (\hat{\kappa}\lambda + \mu_s)\hat{\Omega}\tilde{\mu} ds$$

for X, Y given by (12). The following theorem is proved in a similar way to the proof of Theorem 2.

Theorem 11. *The form $\hat{\omega}_1$ is a presymplectic form on $\hat{\mathcal{M}}$. For each $n \in \mathbb{N}$, $\hat{X}_n$ is a Hamiltonian vector field for $\hat{H}_{n+1}$ with respect to $\hat{\omega}_1$.*

Note that the Euclidean motion group $E(2) = O(2) \ltimes \mathbb{R}^2$ of $\mathbb{E}^2$ acts on $\hat{\mathcal{M}}$. It is easily verified that $\hat{\omega}_1$ is invariant under the $E(2)$-action and the kernel of $\hat{\omega}_1$ at $T_{\hat{\gamma}}\hat{\mathcal{M}}$ contains the tangent space of the orbit. Hence ω_1 determines a presymplectic form on $\hat{\mathcal{M}}/E(2)$.

As well as on $(\mathcal{M}, \omega_1)$, S^1 acts on $\hat{\mathcal{M}}$ leaving $\hat{\omega}_1$ invariant and the following theorem holds.

Theorem 12. *The moment map $\hat{\mu}_1$ for the S^1-action on $\hat{\mathcal{M}}$ with respect to $\hat{\omega}_1$ is given by*

$$\hat{\mu}_1(\hat{\gamma})\left(\frac{\partial}{\partial\sigma}\right) = \hat{H}_1(\hat{\gamma}), \qquad \hat{\gamma} \in \hat{\mathcal{M}}.$$

5 The geometric Miura transformation and multi-Hamiltonian structures on spaces of closed curves in the Euclidean plane

First, we briefly review the geometric Miura transformation which relates the Hamiltonian structures on $\mathcal{M}$ and on $\hat{\mathcal{M}}$ (see [15] for more details). We consider the complexification of $\mathcal{M}$:

$$\mathcal{M}^{\mathbb{C}} = \left\{\gamma : S^1 \to \mathbb{C}^2 \setminus \{0\} \,\middle|\, \det\begin{pmatrix}\gamma \\ \gamma_s\end{pmatrix} = 1\right\}.$$

We determine the curvature of $\gamma \in \mathcal{M}^{\mathbb{C}}$, (complex) presymplectic forms on $\mathcal{M}^{\mathbb{C}}$, etc. by the same formulas as in the case of $\mathcal{M}$, hence we use the same symbols $\kappa, \omega_0, \omega_1, \dots$ to denote them.

By identifying the range $\mathbb{E}^2$ of $\hat{\gamma} \in \hat{\mathcal{M}}$ with a complex plane $\mathbb{C}$, we define the *geometric Miura transformation* $\Phi : \hat{\mathcal{M}} \to \mathcal{M}^{\mathbb{C}}$ by

$$\Phi(\hat{\gamma}) = (-\hat{\gamma}_s)^{-\frac{1}{2}}(\hat{\gamma}, 1), \qquad \hat{\gamma} \in \hat{\mathcal{M}}.$$

The curvature κ of $\Phi(\hat{\gamma})$ is related with the curvature $\hat{\kappa}$ of $\hat{\gamma}$ by the Miura transformation:

$$\kappa = \frac{\sqrt{-1}}{2}\hat{\kappa}_s + \frac{1}{4}\hat{\kappa}^2. \tag{13}$$

Moreover, we have the following.

Proposition 13 ([15])**.** *For each $n \in \mathbb{N}$, $\Phi_*\hat{X}_n = X_n$ holds and the Hamiltonian system $(\hat{\omega}_0, \hat{H}_n)$ on $\hat{\mathcal{M}}$ coincides with the pullback of (ω_0, H_n) on $\mathcal{M}^{\mathbb{C}}$ by Φ:*

$$\hat{\omega}_0 = \Phi^*\omega_0, \qquad \hat{H}_n = \Phi^* H_n. \tag{14}$$

For a sequence of real numbers $C = \{c_k\}_{k\in\mathbb{N}}$, the second equation of (14) implies that

$$\hat{\mathcal{M}}(C_m) = \hat{H}_1^{-1}(c_1) \cap \cdots \cap \hat{H}_m^{-1}(c_m) = \Phi^{-1}\big(\mathcal{M}^{\mathbb{C}}(C_m)\big).$$

Therefore, Φ gives a map from $\hat{\mathcal{M}}(C_m)$ to $\mathcal{M}^{\mathbb{C}}(C_m)$ and we have a presymplectic form $\hat{\omega}_{m+1} = \Phi^*\omega_{m+1}$ on $\hat{\mathcal{M}}(C_m)$. Under these settings the following theorems are directly deduced from Theorems 7 and 8.

Theorem 14. *On $\hat{\mathcal{M}}(C_m)$, for each $n \in \mathbb{N}$ and $k = 0, 1, \dots, m+1$, $\hat{X}_n$ is a Hamiltonian vector field for $\hat{H}_{n+k}$ with respect to $\hat{\omega}_k$, that is, the set $\big(\{\hat{H}_n\}_{n\in\mathbb{N}}, \{\hat{\omega}_k\}_{k=0}^{m+1}\big)$ is a multi-Hamiltonian system on $\hat{\mathcal{M}}(C_m)$ describing the higher modified KdV flows.*

Theorem 15. *The moment map $\hat{\mu}_{m+1}$ for the S^1-action on $\hat{\mathcal{M}}(C_m)$ with respect to $\hat{\omega}_{m+1}$ is given by*

$$\hat{\mu}_{m+1}(\hat{\gamma})\left(\frac{\partial}{\partial\sigma}\right) = \hat{H}_{m+1}(\hat{\gamma}), \qquad \hat{\gamma} \in \hat{\mathcal{M}}(C_m).$$

Remark 16. The symplectic form ω_{m+1} can be represented as

$$\hat{\omega}_{m+1}(X,Y) = \int_{S^1} (\hat{\kappa}\lambda + \mu_s)\hat{\Omega}^{m+1}\tilde{\mu} ds, \tag{15}$$

where X and Y are tangent vectors on $\hat{\mathcal{M}}(C_m)$ given by (12). In fact, when κ and $\hat{\kappa}$ are related by (13), a direct calculation shows an identity

$$\left(\sqrt{-1}D_s + \hat{\kappa}\right)\hat{\Omega} = \Omega\left(\sqrt{-1}D_s + \hat{\kappa}\right);$$

thus we have

$$\begin{aligned}
\hat{\omega}_{m+1}(X,Y) = \omega_{m+1}\big(\Phi_* X, \Phi_* Y\big) &= \int_{S^1} \big(\lambda + \sqrt{-1}\mu\big)\Omega^{m+1}\big(\tilde{\lambda} + \sqrt{-1}\tilde{\mu}\big)_s ds \\
&= \int_{S^1} \big(\lambda + \sqrt{-1}\mu\big)\Omega^{m+1}\big(\sqrt{-1}D_s + \hat{\kappa}\big)\tilde{\mu} ds \\
&= \int_{S^1} \big(\lambda + \sqrt{-1}\mu\big)\big(\sqrt{-1}D_s + \hat{\kappa}\big)\hat{\Omega}^{m+1}\tilde{\mu} ds \\
&= \int_{S^1} \big[\big(-\sqrt{-1}D_s + \hat{\kappa}\big)\big(\lambda + \sqrt{-1}\mu\big)\big] \cdot \hat{\Omega}^{m+1}\tilde{\mu} ds \\
&= \int_{S^1} (\hat{\kappa}\lambda + \mu_s)\hat{\Omega}^{m+1}\tilde{\mu} ds.
\end{aligned}$$

We note that (15) implies $\hat{\omega}_{m+1}$ is a real form, though ω_{m+1} on $\mathcal{M}^{\mathbb{C}}(C_m)$ is complex.

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