Integrable Systems Related to Deformed $\mathfrak{so}(5)$

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Abstract. We investigate a family of integrable Hamiltonian systems on Lie–Poisson spaces $\mathcal{L}_+(5)$ dual to Lie algebras $\mathfrak{so}_{\lambda,\alpha}(5)$ being two-parameter deformations of $\mathfrak{so}(5)$. We integrate corresponding Hamiltonian equations on $\mathcal{L}_+(5)$ and $T^*\mathbb{R}^5$ by quadratures as well as discuss their possible physical interpretation.

Key words: integrable Hamiltonian systems; Casimir functions; Lie algebra deformation; symplectic dual pair; momentum map

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1 Introduction

The notion of compatible Poisson structures on a manifold M, firstly introduced by Magri in [12], leads to one of the most productive methods of construction of functions on M being in involution. This method was used by many authors to integrate various Hamiltonian systems, see, e.g., monograph [19] for interesting examples as well as a huge number of references therein.

A pencil of Lie brackets on vector space $\mathfrak{g}$ defines compatible Lie–Poisson structures on the dual $\mathfrak{g}^*$ to $\mathfrak{g}$. For the treatment of this case see [19, Chapter 7, Section 44]. One can find many examples of Hamiltonian systems on Lie–Poisson space $\mathfrak{g}^*$ obtained in this way in [2, 3, 10, 12, 13, 17, 18, 20, 21].

In [15, Section 3, Proposition 6] we investigate compatible Lie–Poisson structures on space $\mathcal{L}_+$ of uppertriangular Hilbert–Schmidt operators. Since this case includes all finite-dimensional cases $\mathcal{L}_+(n)$, $n \in \mathbb{N}$, we will come to finite-dimensional integrable Hamiltonian systems related to various Lie algebras whose Lie brackets depends on a finite number of real parameters. Within this context in the present paper we consider a two-parameter family of Lie algebras $\mathfrak{so}_{\lambda,\alpha}(5)$, $\lambda, \alpha \in \mathbb{R}$, which contains physically important subcases such as Poincaré algebra, Galilean algebra, de Sitter algebra, anti-de Sitter algebra, special orthogonal algebra $\mathfrak{so}(5)$ and Euclidean algebra $\mathfrak{e}(4)$. We arrange all these cases in the table below

1)	$\lambda > 0$	$\wedge$	$\alpha > 0$	$\mathfrak{so}(5)$
2)	$\lambda < 0$	$\wedge$	$\alpha > 0$	$\mathfrak{so}(3,2) \simeq \mathfrak{sp}(2,\mathbb{R})$
3)	$\lambda < 0$	$\wedge$	$\alpha < 0$	$\mathfrak{so}(1,4)$
4)	$\lambda < 0$	$\wedge$	$\alpha = 0$	$\mathfrak{p}(1,3)$ (Poincaré algebra)
5)	$\lambda = 0$	$\wedge$	$\alpha = 0$	Galilean algebra
6)	$\lambda > 0$	$\wedge$	$\alpha = 0$	$\mathfrak{e}(4)$ (Euclidean algebra)
7)	$\lambda = 0$	$\wedge$	$\alpha > 0$	$(\mathfrak{so}(2) \times \mathfrak{so}(3)) \ltimes \mathrm{Mat}_{3\times 2}(\mathbb{R})$
8)	$\lambda = 0$	$\wedge$	$\alpha < 0$	$(\mathfrak{so}(1,1) \times \mathfrak{so}(3)) \ltimes \mathrm{Mat}_{3\times 2}(\mathbb{R})$

The physical importance of the above Lie algebras motived us to investigate related Hamiltonian systems. The Hamiltonian systems connected with Euclidean, Galilean and Poincaré Lie algebras, specified by the condition $\alpha = 0$, were studied (integrated) in [6]. We will study here other cases, characterized by the condition $\alpha\lambda \neq 0$, i.e. the ones corresponding to $\mathfrak{so}(5)$,

$\mathfrak{so}(1,4)$ and $\mathfrak{so}(3,2)$. Let us mention that the construction of integrals of motion in involution on $\mathcal{L}_+(n)$ proposed in [15] for $n = 4$ leads to linear Hamiltonian systems. The Hamiltonian systems obtained for the case $n > 5$ depend on more deformation parameters and thus are more difficult to be handled.

The main results of the paper are the following ones. In Section 2 we construct and integrate by quadratures a Hamiltonian system on Lie–Poisson space $\mathcal{L}_+(5)$ with Poisson bracket $\{\cdot,\cdot\}_{\lambda,\alpha}$ defined by (2.2) and Hamiltonian defined by (2.11).

In Section 3 we find the momentum map $\mathcal{J} : T^*\mathbb{R}^5 \to \mathcal{L}_+(5) \cong \mathfrak{so}(5)$ of the cotangent bundle $T^*\mathbb{R}^5$ into Lie–Poisson space $\mathcal{L}_+(5)$. Then $T^*\mathbb{R}^5 \setminus \mathcal{J}^{-1}(0)$ is shown to be the total space of $\mathrm{GL}(2,\mathbb{R})$-principal bundle over Grassmannian $G(2,5)$. We also define other momentum map $\mathcal{I} : T^*\mathbb{R}^5 \to \mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{sl}(2,\mathbb{R})^*$ and show that $T^*\mathbb{R}^5$ and Lie–Poisson spaces $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(5)$ form symplectic dual pair in the sense of definition presented in [5, Chapter IV, Section 9.3]. Further the spliting of $\mathcal{J}(T^*\mathbb{R}^5 \setminus \mathcal{J}^{-1}(0))$ on co-adjoint $\mathrm{SO}_{\lambda,\alpha}$-orbits is given.

The lifting of the Hamiltonian system (2.12)–(2.15) on the symplectic manifold $T^*\mathbb{R}^5$, see Hamiltonian (4.2) and Hamilton equations (4.3), is integrated in Section 4. We present some examples of the physical interpretation of the system given by (4.2) in Section 4 as well.

2   Compatible Poisson structures related to deformed $\mathfrak{so}_{\lambda,\alpha}(5)$

By definition two Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ on a manifold M are compatible if any linear combination $b_1\{\cdot,\cdot\}_1 + b_2\{\cdot,\cdot\}_2$ is also a Poisson bracket. If $\{\cdot,\cdot\}_2$ is not a scalar multiple of $\{\cdot,\cdot\}_1$ then using well elaborated methods, e.g. see [1, 12, 19], one can construct integrable Hamiltonian systems on M. In this section, basing on the paper [15], we define such systems in the case when M is the vector space of strictly uppertriangular 5×5 matrices $\mathcal{L}_+(5)$.

Let us consider the vector space $\mathfrak{so}_{\lambda,\alpha}(5)$ of matrices

$$\boldsymbol{X} := \begin{pmatrix} 0 & \alpha b & \alpha\lambda\vec{u}^\top \\ -b & 0 & \lambda\vec{w}^\top \\ -\vec{u} & -\vec{w} & \delta \end{pmatrix} \in \mathrm{Mat}_{5\times 5}(\mathbb{R})$$

with fixed parameters $\alpha, \lambda \in \mathbb{R}$ and $b \in \mathbb{R}$, $\vec{u}, \vec{w} \in \mathbb{R}^3$, $\delta \in \mathfrak{so}(3)$. One easily verifies that $\mathfrak{so}_{\lambda,\alpha}(5)$ is a Lie algebra with respect to the standard matrix commutator.

Using the pairing

$$\langle \boldsymbol{X}, \kappa \rangle := \mathrm{Tr}(\kappa \boldsymbol{X}), \tag{2.1}$$

between $\boldsymbol{X} \in \mathfrak{so}_{\lambda,\alpha}(5)$ and

$$\kappa = \left(\begin{array}{cc|c} 0 & a & \vec{x}^\top \\ 0 & 0 & \vec{y}^\top \\ \hline \vec{0} & \vec{0} & \boldsymbol{\mu} \end{array}\right) \in \mathcal{L}_+(5),$$

where $a \in \mathbb{R}$, $\vec{x}, \vec{y} \in \mathbb{R}^3$ and

$$\boldsymbol{\mu} = \begin{pmatrix} 0 & \mu_3 & -\mu_2 \\ 0 & 0 & \mu_1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{R}),$$

we will identify $\mathcal{L}_+(5)$ with the dual $\mathfrak{so}_{\lambda,\alpha}(5)^*$ of $\mathfrak{so}_{\lambda,\alpha}(5)$.

In the coordinates $(a, \vec{x}, \vec{y}, \boldsymbol{\mu})$ Lie–Poisson bracket for $f, g \in C^\infty(\mathcal{L}_+(5))$ is given by the formula

$$\{f,g\}_{\lambda,\alpha} = \mathrm{Tr}\left(\kappa\left[\frac{\partial f}{\partial \kappa}, \frac{\partial g}{\partial \kappa}\right]\right) = \lambda a\left(\frac{\partial f}{\partial \vec{x}} \cdot \frac{\partial g}{\partial \vec{y}} - \frac{\partial f}{\partial \vec{y}} \cdot \frac{\partial g}{\partial \vec{x}}\right)$$

$$+ \vec{\mu} \cdot \left(\alpha\lambda \left(\frac{\partial f}{\partial \vec{x}} \times \frac{\partial g}{\partial \vec{x}} \right) + \lambda \left(\frac{\partial f}{\partial \vec{y}} \times \frac{\partial g}{\partial \vec{y}} \right) + \left(\frac{\partial f}{\partial \vec{\mu}} \times \frac{\partial g}{\partial \vec{\mu}} \right) \right)$$

$$+ \frac{\partial g}{\partial a} \vec{x} \cdot \frac{\partial f}{\partial \vec{y}} - \frac{\partial f}{\partial a} \vec{x} \cdot \frac{\partial g}{\partial \vec{y}} - \alpha \frac{\partial g}{\partial a} \vec{y} \cdot \frac{\partial f}{\partial \vec{x}} + \alpha \frac{\partial f}{\partial a} \vec{y} \cdot \frac{\partial g}{\partial \vec{x}}$$

$$+ \vec{x} \cdot \left(\frac{\partial f}{\partial \vec{x}} \times \frac{\partial g}{\partial \vec{\mu}} + \frac{\partial f}{\partial \vec{\mu}} \times \frac{\partial g}{\partial \vec{x}} \right) + \vec{y} \cdot \left(\frac{\partial f}{\partial \vec{y}} \times \frac{\partial g}{\partial \vec{\mu}} + \frac{\partial f}{\partial \vec{\mu}} \times \frac{\partial g}{\partial \vec{y}} \right), \tag{2.2}$$

where $\vec{\mu} = (\mu_1, \mu_2, \mu_3)^\top$. Let us note that this bracket belongs to the family of Lie–Poisson brackets investigated in [15].

According to Proposition 3 from [15] the global Casimirs for the bracket $\{\cdot, \cdot\}_{\lambda, \alpha}$ are as follows

$$c_1 = \vec{x}^{\,2} + \alpha \vec{y}^{\,2} + \alpha\lambda \vec{\mu}^{\,2} + \lambda a^2, \tag{2.3}$$

$$c_2 = \alpha\lambda (\vec{\mu} \cdot \vec{y})^2 + \lambda (\vec{\mu} \cdot \vec{x})^2 + (\lambda a \vec{\mu} - \vec{x} \times \vec{y})^2. \tag{2.4}$$

Choosing a Hamiltonian $H \in C^\infty(\mathcal{L}_+(5))$ we obtain Hamilton equations

$$\frac{da}{dt} = \alpha \vec{y} \cdot \frac{\partial H}{\partial \vec{x}} - \vec{x} \cdot \frac{\partial H}{\partial \vec{y}}, \tag{2.5}$$

$$\frac{d\vec{x}}{dt} = -\alpha \frac{\partial H}{\partial a} \vec{y} + \alpha\lambda \frac{\partial H}{\partial \vec{x}} \times \vec{\mu} + \lambda a \frac{\partial H}{\partial \vec{y}} + \frac{\partial H}{\partial \vec{\mu}} \times \vec{x}, \tag{2.6}$$

$$\frac{d\vec{y}}{dt} = \frac{\partial H}{\partial a} \vec{x} + \lambda \frac{\partial H}{\partial \vec{y}} \times \vec{\mu} - \lambda a \frac{\partial H}{\partial \vec{x}} + \frac{\partial H}{\partial \vec{\mu}} \times \vec{y}, \tag{2.7}$$

$$\frac{d\vec{\mu}}{dt} = -\vec{x} \times \frac{\partial H}{\partial \vec{x}} - \vec{y} \times \frac{\partial H}{\partial \vec{y}} - \vec{\mu} \times \frac{\partial H}{\partial \vec{\mu}} \tag{2.8}$$

on Lie–Poisson space $\mathcal{L}_+(5)$. We will construct a family of Hamiltonians depending on two real parameters, which are completely integrable.

To this end we observe that Poisson brackets $\{\cdot, \cdot\}_{\lambda, \alpha}$ and $\{\cdot, \cdot\}_{\epsilon, \beta}$ are compatible if $\alpha = \beta$ or $\lambda = \epsilon$, see [15, Proposition 4] what means that the linear combination of these brackets is a Lie–Poisson bracket. In this paper we consider the case when $\lambda \neq \epsilon$ and $\alpha = \beta$. Since the case $\alpha = 0$ was considered in [6] we will not discuss it here. The bi-Hamiltonian systems given by the Lie–Poisson bracket $\{\cdot, \cdot\}_{1, \alpha}$ and the constant Lie–Poisson bracket were studied in [7].

By Magri method [12] it can be shown that Casimir functions of the Poisson bracket $\{\cdot, \cdot\}_{\epsilon, \alpha}$:

$$h_1 = \vec{x}^{\,2} + \alpha \vec{y}^{\,2} + \alpha\epsilon \vec{\mu}^{\,2} + \epsilon a^2, \tag{2.9}$$

$$h_2 = \alpha\epsilon (\vec{\mu} \cdot \vec{y})^2 + \epsilon (\vec{\mu} \cdot \vec{x})^2 + (\epsilon a \vec{\mu} - \vec{x} \times \vec{y})^2. \tag{2.10}$$

are in involution with respect to the Poisson bracket $\{\cdot, \cdot\}_{\lambda, \alpha}$.

For Hamiltonian

$$H = \gamma h_1 + \nu h_2, \tag{2.11}$$

where $\gamma, \nu \in \mathbb{R}$, equations (2.5)–(2.8) take the form

$$\frac{da}{dt} = 0, \tag{2.12}$$

$$\frac{d\vec{\mu}}{dt} = 0, \tag{2.13}$$

$$\frac{d\vec{x}}{dt} = 2(\lambda - \epsilon) \big(\gamma\alpha (a\vec{y} + \vec{x} \times \vec{\mu}) + \nu (\alpha\vec{\mu} \times ((\vec{x} \times \vec{y}) \times \vec{y})$$

$$+ \alpha\epsilon a \vec{\mu}^{\,2} \vec{y} + \epsilon a^2 \vec{x} \times \vec{\mu} + a(\vec{x} \times \vec{y}) \times \vec{x}) \big), \tag{2.14}$$

$$\frac{d\vec{y}}{dt} = 2(\lambda - \epsilon)\big(\gamma(-a\vec{x} + \alpha\vec{y} \times \vec{\mu}) + \nu(\vec{\mu} \times ((\vec{y} \times \vec{x}) \times \vec{x})$$
$$- \epsilon a\vec{\mu}^2\vec{x} + \epsilon a^2\vec{y} \times \vec{\mu} + a(\vec{x} \times \vec{y}) \times \vec{y})\big). \tag{2.15}$$

One can verify functions $h_1, h_2, \vec{\mu}^2, \mu_3, a, h_1 - c_1, h_2 - c_2 \in C^\infty(\mathcal{L}_+(5))$ to be integrals of motion here which are in involution. Recall that c_1 and c_2 are Casimir functions defined in (2.3), (2.4). Since generic symplectic leaves of $\mathcal{L}_+(5)$ have dimension eight then for the integrability of the above Hamiltonian system it is enough to possess four functionally independent integrals of motion being in involution with respect to the Poisson bracket $\{\cdot, \cdot\}_{\lambda, \alpha}$. For example one of the possible choices of four integrals of motion is

$$I_1 := a, \qquad I_2 := \mu_3, \qquad I_3 := h_1 - c_1 = (\epsilon - \lambda)\big(\alpha\vec{\mu}^2 + a^2\big),$$
$$I_4 := h_2 - c_2 = \alpha(\epsilon - \lambda)(\vec{\mu} \cdot \vec{y})^2 + (\epsilon - \lambda)(\vec{\mu} \cdot \vec{x})^2$$
$$+ \big(\epsilon^2 - \lambda^2\big)a^2\vec{\mu}^2 - 2(\epsilon - \lambda)a\vec{\mu} \cdot (\vec{x} \times \vec{y}). \tag{2.16}$$

One easily verifies that the following proposition is valid.

Proposition 2.1. *The Jacobi matrix $DI(a, \vec{\mu}, \vec{x}, \vec{y})$ of the map $I : \mathcal{L}_+(5) \to \mathbb{R}^4$ defined by (2.16) has rank smaller than four if and only if*

$$\mu_i\left((\vec{\mu} \cdot \vec{x})\vec{\mu} + a\vec{\mu} \times \vec{y}\right) = 0 \quad \wedge \quad \mu_i\left(\alpha(\vec{\mu} \cdot \vec{y})\vec{\mu} - a\vec{\mu} \times \vec{x}\right) = 0, \qquad i = 1, 2. \tag{2.17}$$

From (2.17) we conclude that a, μ_3, $h_1 - c_1$, $h_2 - c_2$ are integrals of motion functionally independent almost everywhere. There are the other choices of four integrals of motion for example a, $\vec{\mu}^2$, h_1, h_2, which are also functionally independent almost everywhere. However, the proof of this property is technically more difficult than in the case (2.16).

Now we integrate the Hamiltonian equations (2.12)–(2.15) by quadratures. For this reason we mention that $\vec{\mu}$ and a are integrals of motion. Hamiltonian (2.11) is invariant with respect to the action of the rotation group SO(3) defined by

$$(a, \vec{x}, \vec{y}, \vec{\mu}) \to (a, O\vec{x}, O\vec{y}, O\vec{\mu}),$$

where $O \in \mathrm{SO}(3)$. The above motivates us to use the following SO(3)-invariant coordinates

$$x := \vec{\mu} \cdot \vec{x}, \qquad y := \vec{\mu} \cdot \vec{y}, \qquad f := 2\vec{x} \cdot \vec{y}, \tag{2.18}$$

in order to solve (2.12)–(2.15). In these coordinates equations (2.14), (2.15) (for the case $\alpha \neq 0$, $a \neq 0$) reduce to the following three equations

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = (\lambda - \epsilon)\nu a \begin{pmatrix} -f & \alpha K \pm \sqrt{\alpha}\sqrt{C - f^2} \\ -K \pm \frac{1}{\sqrt{\alpha}}\sqrt{C - f^2} & f \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{2.19}$$

$$\frac{df}{dt} = \pm 2(\lambda - \epsilon)\alpha\nu a\sqrt{C - f^2}\left(\frac{1}{a^2}\big(x^2 + \alpha y^2\big) + D\right), \tag{2.20}$$

where the constants C, D and K are expressed is terms of Casimirs (2.3), (2.4) and integrals of motion a, h_1, h_2 and $\vec{\mu}^2$ in the following way

$$C = \alpha^{-1}\big(c_1 - \alpha\lambda\vec{\mu}^2 - \lambda a^2\big)^2 - 4\left(\frac{\lambda h_2 - \epsilon c_2}{\lambda - \epsilon} + \lambda\epsilon a^2\vec{\mu}^2\right),$$

$$D = \frac{h_2 - c_2}{a^2(\lambda - \epsilon)} - 2\epsilon\vec{\mu}^2 - \frac{2\gamma}{\nu}, \qquad K = \alpha^{-1}\big(c_1 - \alpha\lambda\vec{\mu}^2 - \lambda a^2\big) + 2\epsilon\vec{\mu}^2 + \frac{2\gamma}{\nu}.$$

Introducing new variables φ, ψ and r by

$$f := \sqrt{C}\cos\varphi, \qquad x := e^r\sqrt{\alpha}\cos\left(\frac{1}{2}(\psi \mp \varphi)\right), \qquad y := e^r\sin\left(\frac{1}{2}(\psi \mp \varphi)\right)$$

and substituting them into (2.19) and (2.20) we obtain

$$\frac{dr}{dt} = -\nu a\sqrt{C}(\lambda - \epsilon)\cos\psi, \tag{2.21}$$

$$\frac{d\psi}{dt} = 2\nu a\sqrt{C}(\lambda - \epsilon)\left(\sin\psi - K\sqrt{\frac{\alpha}{C}} - \left(\frac{\alpha^2}{a^2\sqrt{C}}e^{2r} + \frac{D\alpha}{\sqrt{C}}\right)\right), \tag{2.22}$$

$$\frac{d\varphi}{dt} = \mp 2\alpha\nu a(\lambda - \epsilon)\left(\frac{\alpha}{a^2}e^{2r} + D\right). \tag{2.23}$$

Now from (2.21) and (2.22) we have

$$\frac{1}{2}\frac{g'(t)}{1 - g^2(t)} + Eg(t) - 4\nu^2 a^2 C(\lambda - \epsilon)^2 g^2(t) =: R = \text{const}, \tag{2.24}$$

where

$$g(t) := \sin\psi(t), \qquad E = 8\nu^2 a^2\sqrt{C}(\lambda - \epsilon)^2(K\sqrt{\alpha} \pm D\alpha).$$

Separating variables in (2.24) we find

$$t = \int \frac{dg}{\sqrt{(g^2 - 1)(Eg - 4\nu^2 a^2 C(\lambda - \epsilon)^2 g^2 - R)}},$$

where constant R is defined by (2.24). Functions $x(t)$, $y(t)$, $f(t)$ are expressed by means of elliptic function $g(t)$ as follows

$$f(t) = \sqrt{C}\cos\left(\mp 2(\lambda - \epsilon)\alpha\nu a\left(D(t - t_0) + \frac{\alpha}{a^2}\int_{t_0}^t e^{-2\nu a\sqrt{C}(\lambda - \epsilon)\int_{s_0}^s \sqrt{1 - g^2(z)}dz}ds\right)\right),$$

$$x(t) = \sqrt{\alpha}e^{-\nu a\sqrt{C}(\lambda - \epsilon)\int_{t_0}^t \sqrt{1 - g^2(s)}ds}\cos\left((\lambda - \epsilon)\nu a\sqrt{C}\left(\int_{t_0}^t g(s)ds - K\sqrt{\frac{\alpha}{C}}(t - t_0)\right)\right),$$

$$y(t) = e^{-\nu a\sqrt{C}(\lambda - \epsilon)\int_{t_0}^t \sqrt{1 - g^2(s)}ds}\sin\left((\lambda - \epsilon)\nu a\sqrt{C}\left(\int_{t_0}^t g(s)ds - K\sqrt{\frac{\alpha}{C}}(t - t_0)\right)\right).$$

Now, without loss of generality, we can assume $\vec{\mu} = (0, 0, \mu)$. Then we obtain that

$$x_3(t) = \frac{1}{\mu}x(t), \qquad y_3(t) = \frac{1}{\mu}y(t). \tag{2.25}$$

One obtains the other coordinate functions $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$ from algebraic equations

$$\frac{1}{2}f(t) - \frac{1}{\mu^2}x(t)y(t) = x_1(t)y_1(t) + x_2(t)y_2(t),$$

$$h_1 - \alpha\epsilon\mu^2 - \epsilon a^2 - \frac{1}{\mu^2}(x^2(t) + \alpha y^2(t)) = x_1^2(t) + x_2^2(t) + \alpha y_1^2(t) + \alpha y_2^2(t),$$

$$\left(x_1^2(t) + x_2^2(t) + \frac{1}{\mu^2}x^2(t)\right)\left(y_1^2(t) + y_2^2(t) + \frac{1}{\mu^2}y^2(t)\right) = \frac{\epsilon c_2 - \lambda h_2}{\epsilon - \lambda} + \epsilon\lambda a^2\mu^2 + \frac{1}{4}f^2(t),$$

$$\frac{c_2 - h_2}{\lambda - \epsilon} - \alpha y^2(t) - x^2(t) - (\epsilon + \lambda)a^2\mu^2 = 2a\mu\left(x_2(t)y_1(t) - x_1(t)y_2(t)\right) \tag{2.26}$$

which follow from (2.4) (2.9), (2.10), (2.18).

In order to integrate equations (2.12)–(2.15) in the case when $a = 0$ we note that functions

$$g_1 := \vec{\mu} \cdot \vec{x}(t), \qquad g_2 := \vec{\mu} \cdot \vec{y}(t), \qquad g_3 := (\vec{x}(t) \times \vec{y}(t))^2 = c_2 - \lambda g_1 - \alpha \lambda g_2 \tag{2.27}$$

are independent of the parameter $t \in \mathbb{R}$. We note also that functions

$$f_1 := \vec{\mu} \cdot (\vec{x} \times \vec{y}), \qquad f_2 := \vec{x} \cdot \vec{y}, \qquad f_3 := \vec{x}^2 - \alpha \vec{y}^2, \tag{2.28}$$

satisfy system of equations

$$\begin{aligned}
\frac{d}{dt} f_1 &= 2(\lambda - \epsilon)\nu\big(\alpha g_2^2 - g_1^2\big)f_2 + 2(\lambda - \epsilon)\nu g_1 g_2 f_3, \\
\frac{d}{dt} f_2 &= 2(\lambda - \epsilon)\nu f_1 f_3, \\
\frac{d}{dt} f_3 &= -8(\lambda - \epsilon)\alpha\nu f_1 f_2.
\end{aligned} \tag{2.29}$$

From (2.29) we find

$$4\alpha f_2^2 + f_3^2 = M = \text{const}, \qquad f_1^2 + \frac{1}{4\alpha}\big(\alpha g_2^2 - g_1^2\big)f_3 - 2g_1 g_2 f_2 = N = \text{const},$$

and thus the equation

$$\frac{d}{dt} f_3 = \pm 2(\lambda - \epsilon)\alpha\nu\sqrt{\alpha^{-1}\big(M - f_3^2\big)\left(4N - \alpha^{-1}\big(\alpha g_2^2 - g_1^2\big)f_3 \pm 4g_1 g_2 \sqrt{\alpha^{-1}\big(M - f_3^2\big)}\right)}$$

holds. This equation is solved by quadratures. Finally we find $\vec{x}(t)$ and $\vec{y}(t)$ solving the algebraic system of equation given by (2.27), (2.28).

3 Symplectic dual pair

In this section we will consider the case $\alpha\lambda \neq 0$. Using Plücker embedding we will define momentum map $\mathcal{J} : T^*\mathbb{R}^5 \to \mathcal{L}_+(5) \cong \mathfrak{so}(5)$ for the canonical action of $\mathrm{SO}_{\lambda,\alpha}(5)$, defined on the cotangent bundle $T^*\mathbb{R}^5$ by (3.1). We will discuss various geometric structures of $T^*\mathbb{R}^5 \setminus \mathcal{J}^{-1}(0)$ crucial for the integration of Hamiltonian system defined by the Hamiltonian $h := H \circ \mathcal{J} : T^*\mathbb{R}^5 \setminus \mathcal{J}^{-1}(0) \to \mathbb{R}$ presented in (4.2).

Due to the assumption $\alpha\lambda \neq 0$ we consider the matrix Lie group

$$\mathrm{SO}_{\lambda,\alpha}(5) = \big\{g \in \mathrm{Mat}_{5\times 5}(\mathbb{R}) : g^\top \eta_{\lambda,\alpha} g = \eta_{\lambda,\alpha}\big\},$$

where

$$\eta_{\lambda,\alpha} = \left(\begin{array}{cc|c} \alpha\lambda & 0 & \vec{0}^\top \\ 0 & \lambda & \vec{0}^\top \\ \hline \vec{0} & \vec{0} & \mathbf{1} \end{array}\right).$$

We introduce the canonical Hamiltonian action of $\mathrm{SO}_{\lambda,\alpha}(5)$ on the cotangent vector bundle $T^*\mathbb{R}^5$ with canonical symplectic form $\mathrm{d}\gamma$, defined for $g \in \mathrm{SO}_{\lambda,\alpha}(5)$ and $(q, p) \in T^*\mathbb{R}^5 \cong \mathbb{R}^5 \times \mathbb{R}^{5*}$ by

$$\Phi_g \begin{pmatrix} q \\ p \end{pmatrix} := \left(\begin{array}{c|c} g\mathbf{1}_5 & 0 \\ \hline 0 & (g^{-1})^\top \mathbf{1}_5 \end{array}\right) \begin{pmatrix} q \\ p \end{pmatrix}, \tag{3.1}$$

where $\mathbf{1}_5$ is unit 5×5 matrix and

$$\gamma = p_{-1}dq_{-1} + p_0 dq_0 + \vec{p} \cdot d\vec{q}. \tag{3.2}$$

Let us note that (3.1) is the lift of the action of $SO_{\lambda,\alpha}(5)$ from the base space $\mathbb{R}^5$ to the cotangent bundle $T^*\mathbb{R}^5$. So, (3.1) is a Hamiltonian action, see, e.g., [11, Chapter IV, Proposition 1.19]. In (3.2) we used the following notation $q^\top = (q_{-1}, q_0, \vec{q}^\top)$ and $p^\top = (p_{-1}, p_0, \vec{p}^\top)$. Since the case $\lambda\alpha \neq 0$ is considered, instead of the pairing (2.1) we will use a non-degenerate pairing

$$\mathfrak{so}(5) \times \widetilde{\mathfrak{so}}_{\lambda,\alpha}(5) \ni (\varrho, \mathbf{Y}) \longrightarrow \frac{1}{2}\operatorname{Tr}(\eta_{\lambda,\alpha}\mathbf{Y}\varrho) \in \mathbb{R}, \tag{3.3}$$

where $\mathfrak{so}(5) = \{\varrho \in \operatorname{Mat}_{5\times 5}(\mathbb{R}) : \varrho^\top + \varrho = 0\}$ and $\widetilde{\mathfrak{so}}_{\lambda,\alpha}(5) = \{\mathbf{Y} \in \operatorname{Mat}_{5\times 5}(\mathbb{R}) : (\eta_{\lambda,\alpha}\mathbf{Y})^\top + \eta_{\lambda,\alpha}\mathbf{Y} = 0\}$ is the Lie algebra of $SO_{\lambda,\alpha}(5)$. Using (3.3) we will identify $\mathfrak{so}_{\lambda,\alpha}(5)^*$ with $\mathfrak{so}(5)$. Note here that one has isomorphisms $\widetilde{\iota} : \mathfrak{so}_{\lambda,\alpha}(5) \longrightarrow \widetilde{\mathfrak{so}}_{\lambda,\alpha}(5)$ and $\iota : \mathcal{L}_+(5) \longrightarrow \mathfrak{so}(5)$ given by

$$\mathbf{Y} = \eta_{\lambda,\alpha}^{-1}\mathbf{X}\eta_{\lambda,\alpha}, \qquad \varrho = \eta_{\lambda,\alpha}^{-1}\kappa - \left(\eta_{\lambda,\alpha}^{-1}\kappa\right)^\top \tag{3.4}$$

respectively, which intertwine the pairings (2.1) and (3.3).

From the identity

$$\operatorname{Tr}\left(\eta_{\lambda,\alpha}g\mathbf{Y}g^{-1}\varrho\right) = \operatorname{Tr}\left(\eta_{\lambda,\alpha}\mathbf{Y}g^{-1}\varrho\left(g^{-1}\right)^\top\right),$$

we find that

$$\operatorname{Ad}_{g^{-1}}^*\varrho = g\varrho g^\top$$

for $g \in SO_{\lambda,\alpha}(5)$. Now we define momentum map $\mathcal{J} : T^*\mathbb{R}^5 \to \mathfrak{so}(5)$ as the Plücker map

$$\mathcal{J}(q,p) := q\left(\eta_{\lambda,\alpha}^{-1}p\right)^\top - \left(\eta_{\lambda,\alpha}^{-1}p\right)q^\top, \tag{3.5}$$

where $q, \eta_{\lambda,\alpha}^{-1}p \in \mathbb{R}^5$ and

$$\mathcal{J} \circ \Phi_g = \operatorname{Ad}_{g^{-1}}^* \circ \mathcal{J}$$

for $g \in SO_{\lambda,\alpha}(5)$. We find from (3.4) and (3.5) that

$$a = \alpha q_{-1}p_0 - q_0 p_{-1}, \qquad \vec{x} = \alpha\lambda q_{-1}\vec{p} - p_{-1}\vec{q}, \qquad \vec{y} = \lambda q_0\vec{p} - p_0\vec{q}, \qquad \vec{\mu} = \vec{q} \times \vec{p}. \tag{3.6}$$

A Hamiltonian action of the group $SL(2,\mathbb{R})$ on $(T^*\mathbb{R}^5, d\gamma)$ is defined by

$$\Psi_A\begin{pmatrix} q \\ p \end{pmatrix} := \left(\begin{array}{c|c} a\mathbf{1}_5 & b\eta_{\lambda,\alpha}^{-1} \\ \hline c\eta_{\lambda,\alpha} & d\mathbf{1}_5 \end{array}\right)\begin{pmatrix} q \\ p \end{pmatrix},$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. The map $\mathcal{I} : T^*\mathbb{R}^5 \to \mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{sl}(2,\mathbb{R})^*$ given by

$$\mathcal{I}(q,p) = \begin{pmatrix} d_3 & -d_1 \\ d_2 & -d_3 \end{pmatrix} := \begin{pmatrix} q_{-1}p_{-1} + q_0 p_0 + \vec{q} \cdot \vec{p} & -\left(\alpha\lambda q_{-1}^2 + \lambda q_0^2 + \vec{q}^2\right) \\ \frac{1}{\alpha\lambda}\left(p_{-1}^2 + \alpha p_0^2 + \alpha\lambda\vec{p}^2\right) & -\left(q_{-1}p_{-1} + q_0 p_0 + \vec{q} \cdot \vec{p}\right) \end{pmatrix} \tag{3.7}$$

is an equivariant map for this action, i.e.

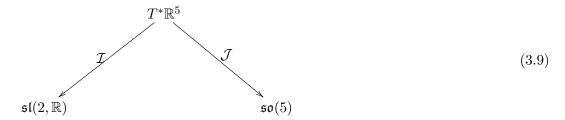$$\mathcal{I} \circ \Psi_A = \operatorname{Ad}_{A^{-1}}^* \circ \mathcal{I},$$

where

$$\mathrm{Ad}^*_{A^{-1}}(\vec{d}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d_3 & -d_1 \\ d_2 & -d_3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

So, $\mathcal{I} : T^*\mathbb{R}^5 \to \mathfrak{sl}(2,\mathbb{R})$ is a momentum map. As usual the vector space isomorphism of Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with its dual $\mathfrak{sl}(2,\mathbb{R})^*$ is defined by the trace. Let us recall that Lie–Poisson bracket for $\mathfrak{sl}(2,\mathbb{R})$ is given by the formula

$$\{f,g\}_{\mathfrak{sl}(2,\mathbb{R})} = 2d_3 \left(\frac{\partial f}{\partial d_1} \frac{\partial g}{\partial d_2} - \frac{\partial f}{\partial d_2} \frac{\partial g}{\partial d_1} \right) + d_1 \left(\frac{\partial f}{\partial d_1} \frac{\partial g}{\partial d_3} - \frac{\partial f}{\partial d_3} \frac{\partial g}{\partial d_1} \right) +$$
$$+ d_2 \left(\frac{\partial f}{\partial d_3} \frac{\partial g}{\partial d_2} - \frac{\partial f}{\partial d_2} \frac{\partial g}{\partial d_3} \right). \tag{3.8}$$

Proposition 3.1. *For both momentum maps mentioned above the following holds:*

(i) *They prove to be Poisson maps, i.e. arrows in the diagram*

$$\tag{3.9}$$

are morphisms of Poisson manifolds.

(ii) *The momentum maps' fibers $\mathcal{I}^{-1}(\vec{d})$ and $\mathcal{J}^{-1}(\rho)$ over $\vec{d} = (d_1, d_2, d_3)^\top \in \mathfrak{sl}(2,\mathbb{R})$ and $\rho \in \mathfrak{so}(5)$ are symplectically orthogonal, i.e.*

$$\{\mathcal{I}^*(C^\infty(\mathfrak{sl}(2,\mathbb{R}))), \mathcal{J}^*(C^\infty(\mathfrak{so}(5)))\} = 0, \tag{3.10}$$

where $\{\cdot,\cdot\}$ is the canonical Poisson bracket on $T^*\mathbb{R}^5$.

Proof. The property (3.10) follow from Leibniz rule and relations

$$\{d_k, a\} = \{d_k, \vec{\mu}\} = \{d_k, \vec{x}\} = \{d_k, \vec{y}\} = 0,$$

where d_k and a, $\vec{\mu}$, $\vec{x}$, $\vec{y}$ are given by (3.7) and (3.6), respectively. ∎

From the above properties of $\mathcal{I}$ and $\mathcal{J}$ we conclude that diagram (3.9) realizes symplectic dual pair. For the definition of symplectic dual pair see [5, Chapter IV, Section 9.3].

We will consider $T^*\mathbb{R}^5$ as union of two complementary subsets

$$T^*\mathbb{R}^5 = T^*_{\mathrm{sing}}\mathbb{R}^5 \cup T^*_{\mathrm{reg}}\mathbb{R}^5,$$

where the subset $T^*_{\mathrm{sing}}\mathbb{R}^5$ consists of the pairs $(q,p) \in T^*_{\mathrm{sing}}\mathbb{R}^5$ such that $q \in \mathbb{R}^5$ and $\eta_{\lambda,\alpha}^{-1} p \in \mathbb{R}^5$ are linearly dependent while $(q,p) \in T^*_{\mathrm{reg}}\mathbb{R}^5$ iff q and $\eta_{\lambda,\alpha} p$ are linearly independent. Note that $T^*_{\mathrm{sing}}\mathbb{R}^5 = \mathcal{J}^{-1}(0)$ and so, it is closed in $T^*\mathbb{R}^5$.

The function

$$c := \det \mathcal{I}(q,p) = d_1 d_2 - d_3^2 \tag{3.11}$$

is a Casimir of the Poisson bracket (3.8) and the equality

$$\delta_{\lambda,\alpha} := c \circ \mathcal{I} = \frac{1}{\alpha\lambda} \left(c_1 \circ \iota^{-1} \circ \mathcal{J} \right), \tag{3.12}$$

is valid, where c_1 is Casimir function defined in (2.3). See (3.4) for definition of $\iota : \mathcal{L}_+(5) \longrightarrow \mathfrak{so}(5)$. The function $\delta_{\lambda,\alpha}$ as well as the subsets $T^*_{\mathrm{sing}}\mathbb{R}^5$ and $T^*_{\mathrm{reg}}\mathbb{R}^5$ are invariant with respect to the action of the groups $\Phi(\mathrm{SO}_{\lambda,\alpha}(5))$ and $\Psi(\mathrm{SL}(2,\mathbb{R}))$. Let us also mention that

$$\Psi_A \circ \Phi_g = \Phi_g \circ \Psi_A$$

for $A \in \mathrm{SL}(2,\mathbb{R})$ and $g \in \mathrm{SO}_{\lambda,\alpha}(5)$.

We will present other important facts in the following

Proposition 3.2.

(i) *For $A \in \mathrm{GL}(2,\mathbb{R})$ one has*

$$(\mathcal{I} \circ \Psi_A)(q,p)\boldsymbol{\epsilon} = A\mathcal{I}(q,p)\boldsymbol{\epsilon} A^\top, \tag{3.13}$$

where $(q,p) \in \mathbb{R} \times \mathbb{R}^{5*}$ and $\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(ii) *The fibres $\Gamma_s := \delta_{\lambda,\alpha}^{-1}(s)$, $s \in \mathbb{R}$, of $\delta_{\lambda,\alpha} : T^*\mathbb{R}^5 \to \mathbb{R}$ are 9-dimensional submanifolds of $T^*_{\mathrm{reg}}\mathbb{R}^5$ invariant with respect to the subgroup $\mathrm{SL}_\pm(2,\mathbb{R}) \subset \mathrm{GL}(2,\mathbb{R})$, consisting of such $A \in \mathrm{GL}(2,\mathbb{R})$ that $\det A = \pm 1$; they are also invariant with respect to the group $\mathrm{SO}_{\lambda,\alpha}(5)$.*

(iii) *The fibres $\mathcal{I}^{-1}(\vec{d})$ of $\mathcal{I} : T^*\mathbb{R}^5 \to \mathfrak{sl}(2,\mathbb{R})$, $\vec{d} \in \mathbb{R}^3 \cong \mathfrak{sl}(2,\mathbb{R})$ defined by equations*

$$d_1 = \alpha\lambda q_{-1}^2 + \lambda q_0^2 + \vec{q}^{\,2}, \tag{3.14}$$

$$d_2 = \frac{1}{\alpha\lambda}p_{-1}^2 + \frac{1}{\lambda}p_0^2 + \vec{p}^{\,2}, \tag{3.15}$$

$$d_3 = q_{-1}p_{-1} + q_0 p_0 + (\vec{q} \cdot \vec{p}) \tag{3.16}$$

are 7-dimensional submanifolds of $T^*_{\mathrm{reg}}\mathbb{R}^5$. They are also invariant with respect to the action of $\mathrm{SO}_{\lambda,\alpha}(5)$ and the action of stabilizer subgroup $\mathrm{SL}(2,\mathbb{R})_{\vec{d}}$.

Proof. Equivariance property (3.13) and the facts that fibres $\Gamma_s = \delta_{\lambda,\alpha}^{-1}(s)$ and $\mathcal{I}^{-1}(\vec{d})$, for $\vec{d} \neq \vec{0}$, are submanifolds of $T^*_{\mathrm{reg}}\mathbb{R}^5$ can be easily verified by the direct calculations. From (3.11) and (3.13) one obtains

$$(\delta_{\lambda,\alpha} \circ \Psi_A)(q,p) = (\det A)^2 \delta_{\lambda,\alpha}(q,p).$$

So, submanifold $\Gamma_s \subset T^*_{\mathrm{reg}}\mathbb{R}^5$ is invariant with respect to $\Psi(\mathrm{SL}_\pm(2,\mathbb{R}))$. ∎

For $A \in \mathrm{GL}(2,\mathbb{R})$ one has

$$\mathcal{J}(\Psi_A(q,p)) = \det A\, \mathcal{J}(q,p).$$

Thus according to the theory of Grassmannians, see, e.g., [8, Chapter I, Section 5], we note that the momentum map (3.5) defines the Plücker embeding $\mathcal{P} : G(2,5) \to \mathbb{P}(\bigwedge^2 \mathbb{R}^5) \cong \mathbb{P}(\mathfrak{so}(5))$ of the Grassmannian $G(2,5)$ of the 2-dimensional vector subspaces of $\mathbb{R}^5$, spanned by vectors $q, \eta_{\lambda,\alpha}^{-1}p \in \mathbb{R}^5$. Thus the image $\mathcal{J}(T^*_{\mathrm{reg}}\mathbb{R}^5)$ of $T^*_{\mathrm{reg}}\mathbb{R}^5$ in $\mathfrak{so}(5)$ is described by the Plücker relations

$$\lambda a\vec{\mu} - \vec{x} \times \vec{y} = 0, \qquad \vec{\mu} \cdot \vec{x} = 0, \qquad \vec{\mu} \cdot \vec{y} = 0, \tag{3.17}$$

which one obtains directly from (3.6). We also observe that $T^*_{\mathrm{reg}}\mathbb{R}^5$ has structure of the $\mathrm{GL}(2,\mathbb{R})$-principal bundle, i.e. it is the total space of Stiefel principal bundle

$$
\begin{array}{l}
\mathrm{GL}(2,\mathbb{R}) \longrightarrow T^*_{\mathrm{reg}}\mathbb{R}^5 \\
\qquad\qquad\qquad\Big\downarrow{\scriptstyle \pi} \\
\qquad\qquad G(2,5)
\end{array}
\tag{3.18}
$$

over $G(2,5)$, for definition of Stiefel bundle see [14]. Equations (3.17) define 7-dimensional submanifold $\mathcal{J}\left(T^*_{\mathrm{reg}}\mathbb{R}^5\right)$ in $\mathfrak{so}(5)$ which is invariant with respect to the multiplication $\mathfrak{so}(5) \ni \varrho \to r\varrho \in \mathfrak{so}(5)$ of ϱ by $r \in \mathbb{R} \setminus \{0\}$. So, one has the $(\mathbb{R} \setminus \{0\})$-principal bundle
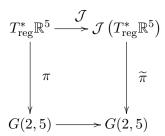
$$
\begin{array}{ccc}
\mathbb{R} \setminus \{0\} & \longrightarrow & \mathcal{J}\left(T^*_{\mathrm{reg}}\mathbb{R}^5\right) \\
& & \Big\downarrow \\
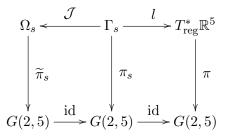& & \mathcal{P}\left(G(2,5)\right) \cong G(2,5)
\end{array}
\tag{3.19}
$$

over the Grassmannian $G(2,5)$. Let us note here that (3.19) is the determinant bundle of the bundle (3.18). Thus one has the surjective morphism

$$
\begin{array}{ccc}
T^*_{\mathrm{reg}}\mathbb{R}^5 & \xrightarrow{\ \mathcal{J}\ } & \mathcal{J}\left(T^*_{\mathrm{reg}}\mathbb{R}^5\right) \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \widetilde{\pi}} \\
G(2,5) & \longrightarrow & G(2,5)
\end{array}
$$

of the principal bundles defined by the momentum map $\mathcal{J} : T^*_{\mathrm{reg}}\mathbb{R}^5 \to \mathfrak{so}(5)$ and the determinant map $\det : \mathrm{GL}(2,\mathbb{R}) \to \mathbb{R} \setminus \{0\}$.

Since, submanifold $\Gamma_s \subset T^*_{\mathrm{reg}}\mathbb{R}^5$ is invariant with respect to the action of $\mathrm{SL}_\pm(2,\mathbb{R})$ it is a total space of the $\mathrm{SL}_\pm(2,\mathbb{R})$-principal subbundle of the $\mathrm{GL}(2,\mathbb{R})$-principal bundle (3.18). The structural groups morphism in this case is given by the inclusion $\mathrm{SL}_\pm(2,\mathbb{R}) \hookrightarrow \mathrm{GL}(2,\mathbb{R})$.

On the other hand submanifold $\Omega_s := \mathcal{J}(T^*_{\mathrm{reg}}\mathbb{R}^5) \cap c_1^{-1}(\alpha\lambda s) \subset \mathfrak{so}(5)$ is total space of a $\mathbb{Z}_2$-principal bundle over $G(2,5)$. In the subsequent diagram we present the morphisms of the principal bundles mentioned above

$$
\begin{array}{ccccc}
\Omega_s & \xleftarrow{\ \mathcal{J}\ } & \Gamma_s & \xrightarrow{\ l\ } & T^*_{\mathrm{reg}}\mathbb{R}^5 \\
\Big\downarrow{\scriptstyle \widetilde{\pi}_s} & & \Big\downarrow{\scriptstyle \pi_s} & & \Big\downarrow{\scriptstyle \pi} \\
G(2,5) & \xrightarrow{\ \mathrm{id}\ } & G(2,5) & \xrightarrow{\ \mathrm{id}\ } & G(2,5)
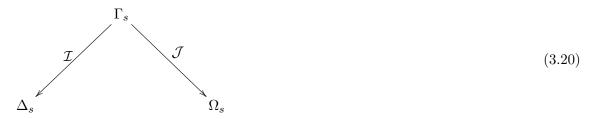\end{array}
$$

The corresponding structural group epimorphism for $\mathcal{J} : \Gamma_s \to \Omega_s$ is $\det : \mathrm{SL}_\pm(2,\mathbb{R}) \to \mathbb{Z}_2 = \{-1,1\}$. The bundle map $\mathcal{J} : \Gamma_s \to \Omega_s$ is a surjective submersion and bundle projection $\widetilde{\pi}_s : \Omega_s \to G(2,5)$ defines a two-fold covering of the Grassmannian $G(2,5)$.

One has the decompositions $\mathrm{GL}(2,\mathbb{R}) = \mathrm{GL}_2(2,\mathbb{R}) \cdot \mathrm{GL}_+(2,\mathbb{R})$ and $\mathrm{SL}_\pm(2,\mathbb{R}) = \mathrm{GL}_2(2,\mathbb{R}) \cdot \mathrm{SL}(2,\mathbb{R})$, where $\mathrm{GL}_2(2,\mathbb{R}) := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathbb{Z}_2$ and $\mathrm{GL}_+(2,\mathbb{R}) := \{A \in \mathrm{GL}(2,\mathbb{R}) : \det A > 0\}$. The map $\psi_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} : T^*_{\mathrm{reg}}\mathbb{R}^5 \to T^*_{\mathrm{reg}}\mathbb{R}^5$ changes the orientation of the frame defined by the pair of vectors $(q, \eta_{\lambda,\alpha}^{-1} p)$ and $\left(\mathcal{J} \circ \psi_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \right)(q,p) = -\mathcal{J}(q,p)$. Hence we can consider $\Omega_s \cong G_+(2,5)$ as the Grassmannian of 2-dimensional subspaces in $\mathbb{R}^5$ with fixed orientation.

One has the double principal bundle structure on Γ_s with structural groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}_{\lambda,\alpha}(5)$ and momentum maps $\mathcal{I}$ and $\mathcal{J}$ being bundle projections

$$
\begin{array}{ccc}
 & \Gamma_s & \\
 {}^{\mathcal{I}}\swarrow & & \searrow{}^{\mathcal{J}} \\
\Delta_s & & \Omega_s
\end{array}
\qquad (3.20)
$$

where $\Delta_s := c^{-1}(s)$ and $s \in \mathbb{R}$. For $s = 0$ we assume by definition that $\vec{0} \notin \Delta_0$. The restriction $\mathrm{d}\gamma|_{\Gamma_s}$ of symplectic form $\mathrm{d}\gamma$ to Γ_s is invariant with respect to $\Psi(\mathrm{SL}_\pm(2,\mathbb{R}))$ and $\Phi(\mathrm{SO}_{\lambda,\alpha}(5))$. So, applying reduction procedure to both these actions one obtains the reduced symplectic manifolds $\Gamma_s/\mathrm{SO}_{\lambda,\alpha}(5) \cong \Delta_s$ and $\Gamma_s/\mathrm{SL}(2,\mathbb{R}) \cong \Omega_s$.

The Hamiltonian flow $\{\sigma_t^{\lambda,\alpha}\}_{t\in\mathbb{R}}$ on $T^*_{\mathrm{reg}}\mathbb{R}^5$ defined by the Hamiltonian $\delta_{\lambda,\alpha} = \frac{1}{\alpha\lambda}(c_1 \circ \iota^{-1} \circ \mathcal{J})$ is described explicitly by expressions (4.12), (4.13) established in Section 4. It preserves fibres $\mathcal{J}^{-1}(\varrho)$ and $\mathcal{I}^{-1}(\vec{d})$ of both momentum maps and on $\mathcal{I}^{-1}(\vec{d})$ it is identical to the action of the stabilizer subgroup $\mathrm{SL}(2,\mathbb{R})_{\vec{d}} \subset \mathrm{SL}(2,\mathbb{R})$. From (3.12) one sees that $\delta_{\lambda,\alpha}$ is the pull-back of the Casimirs c and $\frac{1}{\alpha\lambda}(c_1 \circ \iota^{-1})$. Thus the groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}_{\lambda,\alpha}(5)$ act also on the reduced symplectic manifold $\widetilde{\Gamma}_s := \Gamma_s/\{\sigma_t^{\lambda,\alpha}\}$ by symplectomorphisms. Summarizing the above facts we can formulate

Proposition 3.3. *For any $s \in \mathbb{R}$ one has the symplectic double fibration*

$$
\begin{array}{ccc}
 & \widetilde{\Gamma}_s & \\
 {}^{\widetilde{\mathcal{I}}}\swarrow & & \searrow{}^{\widetilde{\mathcal{J}}} \\
\Delta_s & & \Omega_s
\end{array}
\qquad (3.21)
$$

i.e. all manifolds in (3.21) are symplectic and the maps $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{J}}$ are surjective Poisson submersions. Moreover $\widetilde{\mathcal{I}}$-fibres are symplectically orthogonal to the $\widetilde{\mathcal{J}}$-fibres.

Proof. The symplectic orthogonality of $\widetilde{\mathcal{I}}$-fibres and $\widetilde{\mathcal{J}}$-fibres follows from (3.10). Due to the fact that both momentum maps are constant on the trajectories of $\{\sigma_t^{\lambda,\alpha}\}_{t\in\mathbb{R}}$ the surjective epimorphisms $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{J}}$ are defined by $\mathcal{I}$ and $\mathcal{J}$ respectively. ∎

Remark 3.4. In general the fibres of $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{J}}$ are neither connected nor simply connected. So, symplectic manifolds Δ_s and Ω_s are not Morita equivalent in sense of [5, Chapter IV, Section 9.3].

Since $\Omega_s \subset \mathfrak{so}(5)$ is invariant with respect to the coadjoint action of $\mathrm{SO}_{\lambda,\alpha}(5)$ we will investigate the decomposition of Ω_s into the orbits of this action. For this reason we note that $(\epsilon\mathcal{I})^{-1}(\vec{d}) \subset T^*_{\mathrm{reg}}\mathbb{R}^5$, where $(\epsilon\mathcal{I})(q,p) := \epsilon\mathcal{I}(q,p)$, is invariant with respect to the action (3.1). From Proposition 3.2(i) one has

$$
A \begin{pmatrix} d_1(q,p) & d_3(q,p) \\ d_3(q,p) & d_2(q,p) \end{pmatrix} A^\top = \begin{pmatrix} d_1(\Psi_A(q,p)) & d_3(\Psi_A(q,p)) \\ d_3(\Psi_A(q,p)) & d_2(\Psi_A(q,p)) \end{pmatrix}
\qquad (3.22)
$$

for $A \in \mathrm{SL}(2,\mathbb{R})$. Since both maps in diagram (3.20) are surjective submersion we find that $\Omega_s = \mathcal{J}(\Gamma_s) = \mathcal{J}((\epsilon\mathcal{I})^{-1}(\Delta_s))$. It follows from (3.22) that $\mathcal{J}((\epsilon\mathcal{I})^{-1}(\vec{d})) = \mathcal{J}((\epsilon\mathcal{I})^{-1}(\vec{d'}))$ iff for $\vec{d}, \vec{d'} \in \Delta_s \subset \mathrm{SL}(2,\mathbb{R})$ there exists $A \in \mathrm{SL}(2,\mathbb{R})$ such

$$
\begin{pmatrix} d_1' & d_3' \\ d_3' & d_2' \end{pmatrix} = A \begin{pmatrix} d_1 & d_3 \\ d_3 & d_2 \end{pmatrix} A^\top .
$$

So, in order to describe invariant subsets $\mathcal{J}((\epsilon\mathcal{I})^{-1}(\vec{d}) \cap T^*_{\text{reg}}\mathbb{R}^5) \subset \Omega_s$, where $\vec{d} \in \Delta_s$, we formulate

Proposition 3.5. *For any $\vec{d} \in \Delta_s$ there is $A \in \mathrm{SL}(2,\mathbb{R})$ such that:*

(i) *if $s < 0$*

$$\begin{pmatrix} d_1 & d_3 \\ d_3 & d_2 \end{pmatrix} = A \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} A^\top,$$

(ii) *if $s \geq 0$*

$$\begin{pmatrix} d_1 & d_3 \\ d_3 & d_2 \end{pmatrix} = \pm A \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} A^\top.$$

If $(q,p) \in (\epsilon\mathcal{I})^{-1}(\vec{d}) \cap T^*_{\text{reg}}\mathbb{R}^5$ then the signature of the symmetric form $\begin{pmatrix} d_1 & d_3 \\ d_3 & d_2 \end{pmatrix}$ is the same as the signature of the restriction $\eta_{\lambda,\alpha}\big|_V$ of $\eta_{\lambda,\alpha}$ to the 2-dimensional subspace $V \subset \mathbb{R}^5$ spaned by q and $\eta_{\lambda,\alpha}^{-1}p$. The action (3.1) preserves the signature of $\eta_{\lambda,\alpha}\big|_V$.

Let us note that if $V_1, V_2 \in G_+(2,5)$ have identical signatures with respect to $\eta_{\lambda,\alpha}$ then they belong to the same orbit of $\mathrm{SO}_{\lambda,\alpha}(5)$. Thus and from Proposition 3.5 we conclude that the following proposition is valid.

Proposition 3.6.

(i) *If $s < 0$ then Ω_s is six-dimensional $\mathrm{Ad}^*(\mathrm{SO}_{\lambda,\alpha}(5))$-orbit which is isomorphic (as a homogeneous space) to the Grassmannian $G_+^{+-}(2,5)$ of the 2-dimensional oriented subspaces $V \subset \mathbb{R}^5$ such that $\mathrm{sign}\,\eta_{\lambda,\alpha}\big|_V = (+-)$.*

(ii) *If $s = 0$ then Ω_0 is decomposed into the six-dimensional $\mathrm{Ad}^*(\mathrm{SO}_{\lambda,\alpha}(5))$-orbits Ω_0^{+0} and Ω_0^{-0} which are isomorphic to the Grassmannians $G_+^{+0}(2,5)$ and $G_+^{-0}(2,5)$ of the 2-dimensional oriented subspaces $V \subset \mathbb{R}^5$ such that $\mathrm{sign}\,\eta_{\lambda,\alpha}\big|_V$ are $(+0)$ and (-0), respectively.*

(iii) *If $s > 0$ then Ω_s is decomposed into the six-dimensional $\mathrm{Ad}^*(\mathrm{SO}_{\lambda,\alpha}(5))$-orbits Ω_s^{++} and Ω_s^{--} which are isomorphic to the Grassmannians $G_+^{++}(2,5)$ and $G_+^{--}(2,5)$ of the 2-dimensional oriented subspaces $V \subset \mathbb{R}^5$ such that $\mathrm{sign}\,\eta_{\lambda,\alpha}\big|_V$ are $(++)$ and $(--)$, respectively.*

(iv) *If $\vec{d} = \vec{0}$ then $\mathcal{J}((\epsilon\mathcal{I})^{-1}(\vec{0}))$ is a four-dimensional $\mathrm{Ad}^*(\mathrm{SO}_{\lambda,\alpha}(5))$-orbit isomorphic to the Grassmannian $G_+^{00}(2,5)$ of the 2-dimensional oriented subspaces $V \subset \mathbb{R}^5$ such that $\mathrm{sign}\,\eta_{\lambda,\alpha}\big|_V = (00)$.*

In the case $\alpha\lambda \neq 0$ the group $\mathrm{SO}_{\lambda,\alpha}(5)$ is isomorphic to one of the following groups: $\mathrm{SO}(5)$, $\mathrm{SO}(1,4)$ and $\mathrm{SO}(2,4)$. Let us describe all these subcases separately.

Proposition 3.7.

(i) *For special orthogonal group $\mathrm{SO}(5)$ one has $s > 0$ and $\Omega_s \cong G_+^{++}(2,5)$.*

(ii) *For de Sitter group $\mathrm{SO}(1,4)$ one has: $\Omega_s \cong G_+^{++}(2,5)$ for $s > 0$, $\Omega_s \cong G_+^{+-}(2,5)$ for $s < 0$; $\Omega_0 \cong G_+^{+0}(2,5)$ for $s = 0$.*

(iii) *For anti-de Sitter group $\mathrm{SO}(2,3)$ one has: $\Omega_s \cong G_+^{++}(2,5)$ or $\Omega_s \cong G_+^{--}(2,5)$ for $s > 0$; $\Omega_s \cong G_+^{+-}(2,5)$ for $s < 0$ and $\Omega_0 \cong G_+^{+0}(2,5) \cup G_+^{-0}(2,5)$ for $s = 0$. The case $\mathcal{J}((\epsilon\mathcal{I})^{-1}(\vec{0})) \cong G_+^{00}(2,5)$ described in Proposition 3.6(iv) is admissible for the anti-de Sitter group.*

Completing this section let us shortly discuss the case $T^*_{\text{sing}}\mathbb{R}^5 = \mathcal{J}^{-1}(0)$. If $(q,p) \in \mathcal{J}^{-1}(0)$ then one has $b_1 q + b_2 \eta_{\lambda,\alpha}^{-1} p = 0$ for some $0 \neq \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$. Thus, we find that $\mathcal{I}(q,p)\epsilon \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0$ and hence $\mathcal{I}\left(\mathcal{J}^{-1}(0)\right) \subset \Delta_0$. Summing up the above facts we obtain bundle $\mathcal{I} : \mathcal{J}^{-1}(0) \setminus \{0\} \to \Delta_0 \setminus \{0\}$.

The canonical form γ after restriction to $\mathcal{J}^{-1}(0) \setminus \{0\}$ is given by

$$\gamma\big|_{\mathcal{J}^{-1}(0)} = \frac{1}{2} d_3 \mathrm{d}\ln|d_1| = -\frac{1}{2} d_3 \mathrm{d}\ln|d_2| + \mathrm{d}d_3.$$

So, $\mathrm{d}\gamma\big|_{\mathcal{J}^{-1}(0)}$ is equal to the lifting $\mathcal{I}^*\omega_0$ of the $\mathrm{SL}(2,\mathbb{R})$-invariant symplectic form ω_0 of the symplectic leaf $\Delta_0 \subset \mathrm{SL}(2,\mathbb{R})$. We will not consider this case in what follows. The reason is that the Hamiltonian $H \circ \mathcal{J}$ after restriction to $\mathcal{J}^{-1}(0)$ vanishes, so it generates trivial dynamics.

In the next section we will use fibration (3.20) to integrate Hamiltonian equations defined by Hamiltonian $H \circ \mathcal{J}$ for regular case $\mathcal{I}^{-1}(\vec{d}) \cap T^*_r \mathbb{R}^5$.

4   Solutions and their physical interpretations

Our goal is to use results of two previous section for solving Hamilton equations

$$\frac{dq}{dt} = \frac{\partial h}{\partial p} \qquad \text{and} \qquad \frac{dp}{dt} = -\frac{\partial h}{\partial q} \tag{4.1}$$

on $T^*\mathbb{R}^5$ with Hamiltonian

$$\begin{aligned}
h := H \circ \mathcal{J} = {} & \gamma\big(\alpha\big(\alpha\lambda^2 q_{-1}^2 + \lambda^2 q_0^2 + \epsilon\vec{q}^{\,2}\big)\vec{p}^{\,2} + \big(p_{-1}^2 + \alpha p_0^2\big)\big(\vec{q}^{\,2} + \epsilon q_0^2 + \alpha\epsilon q_{-1}^2\big) \\
& - \alpha\epsilon\big(q_{-1}p_{-1} + q_0 p_0 + \vec{q}\cdot\vec{p}\big)^2 - 2\alpha(\lambda - \epsilon)(q_{-1}p_{-1} + q_0 p_0)(\vec{q}\cdot\vec{p}) \\
& + \nu(\lambda - \epsilon)^2(\alpha q_{-1}p_0 - q_0 p_{-1})^2(\vec{q}\times\vec{p})^2,
\end{aligned} \tag{4.2}$$

where H is defined in (2.11). After substituting (4.2) into (4.1) we obtain

$$\begin{aligned}
\frac{dq_{-1}}{dt} = {} & 2\gamma\big((\vec{q}^{\,2} + \epsilon q_0^2)p_{-1} - \alpha(\epsilon q_0 p_0 + \lambda\vec{q}\cdot\vec{p})q_{-1}\big) \\
& - 2\nu(\lambda - \epsilon)^2(\alpha q_{-1}p_0 - q_0 p_{-1})(\vec{q}\times\vec{p})^2 q_0,
\end{aligned}$$

$$\begin{aligned}
\frac{dq_0}{dt} = {} & 2\gamma\alpha\big((\vec{q}^{\,2} + \alpha\epsilon q_{-1}^2)p_0 - (\epsilon q_{-1}p_{-1} + \lambda\vec{q}\cdot\vec{p})q_0\big) \\
& + 2\nu\alpha(\lambda - \epsilon)^2(\alpha q_{-1}p_0 - q_0 p_{-1})(\vec{q}\times\vec{p})^2 q_{-1},
\end{aligned}$$

$$\begin{aligned}
\frac{dp_{-1}}{dt} = {} & -2\gamma\alpha\big((\alpha\lambda^2\vec{p}^{\,2} + \alpha\epsilon p_0^2)q_{-1} - (\epsilon q_0 p_0 + \lambda\vec{q}\cdot\vec{p})p_{-1}\big) \\
& - 2\alpha\nu(\lambda - \epsilon)^2(\alpha q_{-1}p_0 - q_0 p_{-1})(\vec{q}\times\vec{p})^2 p_0,
\end{aligned}$$

$$\begin{aligned}
\frac{dp_0}{dt} = {} & -2\gamma\big((\alpha\lambda^2\vec{p}^{\,2} + \epsilon p_{-1}^2)q_0 - \alpha(\epsilon q_{-1}p_{-1} + \lambda\vec{q}\cdot\vec{p})p_0\big) \\
& + 2\nu(\lambda - \epsilon)^2(\alpha q_{-1}p_0 - q_0 p_{-1})(\vec{q}\times\vec{p})^2 p_{-1},
\end{aligned}$$

$$\begin{aligned}
\frac{d\vec{q}}{dt} = {} & 2\gamma\alpha\big((\epsilon\vec{q}^{\,2} + \lambda^2 q_0^2 + \alpha\lambda^2 q_{-1}^2)\vec{p} - (\lambda q_{-1}p_{-1} + \lambda q_0 p_0 + \epsilon\vec{q}\cdot\vec{p})\vec{q}\big) \\
& + 2\nu(\lambda - \epsilon)^2(\alpha q_{-1}p_0 - q_0 p_{-1})^2(\vec{q}^{\,2}\vec{p} - (\vec{q}\cdot\vec{p})\vec{q}),
\end{aligned}$$

$$\begin{aligned}
\frac{d\vec{p}}{dt} = {} & -2\gamma\big((\alpha\epsilon\vec{p}^{\,2} + \alpha p_0^2 + p_{-1}^2)\vec{q} - \alpha(\lambda q_{-1}p_{-1} + \lambda q_0 p_0 + \epsilon\vec{q}\cdot\vec{p})\vec{p}\big) \\
& - 2\nu(\lambda - \epsilon)^2(\alpha q_{-1}p_0 - q_0 p_{-1})^2(\vec{p}^{\,2}\vec{q} - (\vec{q}\cdot\vec{p})\vec{p}).
\end{aligned} \tag{4.3}$$

Using (3.6) and (3.14)–(3.16) we transform above system of equations to the following one

$$\frac{d}{dt}\begin{pmatrix} \sqrt{\alpha}q_{-1}(t) \\ q_0(t) \\ p_{-1}(t) \\ \sqrt{\alpha}p_0(t) \end{pmatrix} = \begin{pmatrix} -2\gamma\alpha\lambda d_3 & \sqrt{\alpha}B & 2\gamma\sqrt{\alpha}d_1 & 0 \\ -\sqrt{\alpha}B & -2\gamma\alpha\lambda d_3 & 0 & 2\gamma\sqrt{\alpha}d_1 \\ -2\gamma\alpha\sqrt{\alpha}\lambda^2 d_2 & 0 & 2\gamma\alpha\lambda d_3 & \sqrt{\alpha}B \\ 0 & -2\gamma\alpha\sqrt{\alpha}\lambda^2 d_2 & -\sqrt{\alpha}B & 2\gamma\alpha\lambda d_3 \end{pmatrix}\begin{pmatrix} \sqrt{\alpha}q_{-1}(t) \\ q_0(t) \\ p_{-1}(t) \\ \sqrt{\alpha}p_0(t) \end{pmatrix}, \quad (4.4)$$

$$\frac{d}{dt}\begin{pmatrix} \vec{q}(t) \\ \vec{p}(t) \end{pmatrix} = 2\begin{pmatrix} -\gamma\alpha\lambda d_3 + C\vec{p}(t)\cdot\vec{q}(t) & \gamma\alpha\lambda d_1 - C\vec{q}^2(t) \\ -\gamma\alpha\lambda d_2 + C\vec{p}^2(t) & \gamma\alpha\lambda d_3 - C\vec{p}(t)\cdot\vec{q}(t) \end{pmatrix}\begin{pmatrix} \vec{q}(t) \\ \vec{p}(t) \end{pmatrix}, \quad (4.5)$$

where

$$B := 2(\lambda - \epsilon)a\left(\gamma - \nu(\lambda - \epsilon)\vec{\mu}^2\right), \qquad C := (\lambda - \epsilon)\left(\gamma\alpha - \nu(\lambda - \epsilon)a^2\right).$$

Integrating linear system given in (4.4) we obtain

$$\begin{pmatrix} \sqrt{\alpha}q_{-1}(t) \\ q_0(t) \\ p_{-1}(t) \\ \sqrt{\alpha}p_0(t) \end{pmatrix} = \begin{pmatrix} D(t) & E(t) & F(t) & G(t) \\ -E(t) & D(t) & -G(t) & F(t) \\ I(t) & J(t) & K(t) & L(t) \\ -J(t) & I(t) & -L(t) & K(t) \end{pmatrix}\begin{pmatrix} \sqrt{\alpha}q_{-1}(0) \\ q_0(0) \\ p_{-1}(0) \\ \sqrt{\alpha}p_0(0) \end{pmatrix}, \quad (4.6)$$

where

$$D(t) = \left(-\frac{\alpha\lambda}{\sqrt{-\delta_{\lambda,\alpha}}}d_3\sinh(2\gamma\sqrt{-\delta_{\lambda,\alpha}}t) + \cosh(2\gamma\sqrt{-\delta_{\lambda,\alpha}}t)\right)\cos(\sqrt{\alpha}Bt),$$

$$E(t) = \left(-\frac{\alpha\lambda}{\sqrt{-\delta_{\lambda,\alpha}}}d_3\sinh(2\gamma\sqrt{-\delta_{\lambda,\alpha}}t) + \cosh(2\gamma\sqrt{-\delta_{\lambda,\alpha}}t)\right)\sin(\sqrt{\alpha}Bt),$$

$$F(t) = \frac{d_1\sqrt{\alpha}}{\sqrt{-\delta_{\lambda,\alpha}}}\sinh(2\gamma\sqrt{-\delta_{\lambda,\alpha}}t)\cos(\sqrt{\alpha}Bt),$$

$$G(t) = \frac{d_1\sqrt{\alpha}}{\sqrt{-\delta_{\lambda,\alpha}}}\sinh(2\gamma\sqrt{-\delta_{\lambda,\alpha}}t)\sin(\sqrt{\alpha}Bt),$$

$$I(t) = \frac{-\alpha\sqrt{\alpha}\lambda^2 d_2}{\sqrt{-\delta_{\lambda,\alpha}}}\sinh(2\gamma\sqrt{-\delta_{\lambda,\alpha}}t)\cos(\sqrt{\alpha}Bt),$$

$$J(t) = \frac{-\alpha\sqrt{\alpha}\lambda^2 d_2}{\sqrt{-\delta_{\lambda,\alpha}}}\sinh(2\gamma\sqrt{-\delta_{\lambda,\alpha}}t)\sin(\sqrt{\alpha}Bt),$$

$$K(t) = \left(\frac{\alpha\lambda}{\sqrt{-\delta_{\lambda,\alpha}}}d_3\sinh(2\gamma\sqrt{-\delta_{\lambda,\alpha}}t) + \cosh(2\gamma\sqrt{-\delta_{\lambda,\alpha}}t)\right)\cos(\sqrt{\alpha}Bt),$$

$$L(t) = \left(\frac{\alpha\lambda}{\sqrt{-\delta_{\lambda,\alpha}}}d_3\sinh(2\gamma\sqrt{-\delta_{\lambda,\alpha}}t) + \cosh(2\gamma\sqrt{\delta_{\lambda,\alpha}}t)\right)\sin(\sqrt{\alpha}Bt).$$

Further, substituting solutions (4.6) into (4.5), see also (3.14)–(3.16), we come to a non-autonomous linear system of equations for functions $\vec{q}(t)$ and $\vec{p}(t)$. In order to solve this system let us consider separately two subcases $a \neq 0$ and $a = 0$.

If $a \neq 0$ then from (3.6) we get

$$\begin{pmatrix} \vec{q}(t) \\ \vec{p}(t) \end{pmatrix} = \frac{1}{\lambda a}\begin{pmatrix} \lambda q_0(t) & -\alpha\lambda q_{-1}(t) \\ p_0(t) & -p_{-1}(t) \end{pmatrix}\begin{pmatrix} \vec{x}(t) \\ \vec{y}(t) \end{pmatrix}, \quad (4.7)$$

where $q_{-1}(t)$, $q_0(t)$, $p_{-1}(t)$ and $p_0(t)$ are given by (4.6) and $(\vec{x}(t), \vec{y}(t))$ were found in Section 2, see (2.25), (2.26).

In the case $a = 0$ one has $\vec{x}(t) \times \vec{y}(t) = 0$. So, instead of (4.7) we consider the equations

$$p_0(t)\vec{\mu} = \vec{p}(t) \times \vec{y}(t), \qquad \alpha\lambda q_{-1}(t)\vec{\mu} = \vec{q}(t) \times \vec{x}(t), \tag{4.8}$$

which also follows from (3.6). From (3.6) and (3.14), (3.15) we have

$$\vec{\mu} \cdot \vec{q}(t) = 0, \qquad \vec{\mu} \cdot \vec{p}(t) = 0,$$
$$\vec{q}^2(t) = d_1 - \alpha\lambda q_{-1}^2(t) - \lambda q_0^2(t), \qquad \vec{p}^2(t) = d_2 - \frac{1}{\alpha\lambda}p_{-1}^2(t) - \frac{1}{\lambda}p_0^2(t). \tag{4.9}$$

The functions $\vec{x}(t)$ and $\vec{y}(t)$ we find solving equations (2.14), (2.15) which in considered case reduce to the linear system

$$\frac{d\vec{x}}{dt} = 2(\epsilon - \lambda)\gamma\alpha\vec{\mu} \times \vec{x}, \qquad \frac{d\vec{y}}{dt} = 2(\epsilon - \lambda)\gamma\alpha\vec{\mu} \times \vec{y}. \tag{4.10}$$

Solution of (4.10) is given by

$$\vec{x}(t) = O_{\vec{\mu}}(t)\vec{x}(0) \qquad \text{and} \qquad \vec{y}_{\vec{\mu}}(t) = O_{\vec{\mu}}(t)\vec{y}(0),$$

where $O_{\vec{\mu}}(t) \in \mathrm{SO}(3)$ is the rotation on the angle $2(\epsilon - \lambda)\gamma\alpha t$ around the constant angular momentum vector $\vec{\mu}$. Now, assuming $\mu_1 = \mu_2 = 0$ after solving algebraic system of equations given by (4.8), (4.9) we easily find $\vec{q}(t)$ and $\vec{p}(t)$.

Finally let us discuss a few possible physical interpretations of the above integrated Hamiltonian systems.

Firstly let us note that if $\gamma = 1$ and $\epsilon = \lambda$ then $h = \frac{1}{\alpha\lambda}c_1 \circ \iota^{-1} \circ \mathcal{J} = \delta_{\lambda,\alpha}$. In this case equations (4.1) take the form

$$\frac{d}{dt}\begin{pmatrix} \eta_{\lambda,\alpha}q \\ p \end{pmatrix} = -2\begin{pmatrix} d_3\mathbf{1_5} & -d_1\mathbf{1_5} \\ d_2\mathbf{1_5} & -d_3\mathbf{1_5} \end{pmatrix}\begin{pmatrix} \eta_{\lambda,\alpha}q \\ p \end{pmatrix}. \tag{4.11}$$

Since

$$\{h, \delta_{\lambda,\alpha}\} = 0 \qquad \text{and} \qquad \{h, \vec{d}\} = 0$$

solution of (4.11) is given by

$$\begin{pmatrix} \eta_{\lambda,\alpha}q(t) \\ p(t) \end{pmatrix} = \Psi\left(A_{\lambda,\alpha}(t)\right)\begin{pmatrix} \eta_{\lambda,\alpha}q(0) \\ p(0) \end{pmatrix}, \tag{4.12}$$

where

$$A_{\lambda,\alpha}(t) = \exp\left(-2t\begin{pmatrix} d_3\mathbf{1_5} & -d_1\mathbf{1_5} \\ d_2\mathbf{1_5} & -d_3\mathbf{1_5} \end{pmatrix}\right) \tag{4.13}$$

is a one-parameter subgroup of $\mathrm{SL}(2,\mathbb{R})_{\vec{d}}$. This allows us to restrict the Hamiltonian $\delta_{\lambda,\alpha}$ and the flow $\Psi(A_{\lambda,\alpha}(t))$ to symplectic submanifold of $T^*\mathbb{R}^5$ defined by the equations $d_1 = $ const and $d_3 = 0$. Such a submanifold is the bundle $T^*Q_{\lambda,\alpha}$ cotangent to the quadric $Q_{\lambda,\alpha} := \{q \in \mathbb{R}^5 : \alpha\lambda q_{-1}^2 + \lambda q_0^2 + \vec{q}^2 = d_1 = \text{const}\}$.

The Hamiltonian $\delta_{\lambda,\alpha}$ after restriction to $T^*Q_{\lambda,\alpha}$ represents kinetic energy

$$\delta_{\lambda,\alpha} = d_1 d_2 = d_1\left(\frac{1}{\alpha\lambda}p_{-1}^2 + \frac{1}{\lambda}p_0^2 + \vec{p}^2\right) = \frac{1}{2}m\left(\frac{dq}{dt}\right)^\top \eta_{\lambda,\alpha}\left(\frac{dq}{dt}\right)$$
$$= d_1\left(\alpha\lambda\left(\frac{dq_{-1}}{dt}\right)^2 + \lambda\left(\frac{dq_0}{dt}\right)^2 + \left(\frac{d\vec{q}}{dt}\right)^2\right) \tag{4.14}$$

of the free particle localized on the quadric $Q_{\lambda,\alpha}$. In (4.14) we identify $2d_1$ with the mass of the particle and express momentum p by velocity $\frac{dq}{dt}$ by means of metric tensor

$$p = \eta_{\lambda,\alpha} \frac{dq}{dt}.$$

Therefore (4.12) is the geodesic flow on the four-dimensional hypersurface $Q_{\lambda,\alpha}$ which is for example:

i) S^4 if $\alpha = \lambda = 1$,

ii) de Sitter spaces dS_4 if $\alpha = \lambda = -1$,

iii) anti-de Sitter spaces AdS_4 if $\alpha = 1$ and $\lambda = -1$.

Hamiltonian (4.2) generalizes dynamics generated by Hamiltonian (4.14) in two aspects. Firstly, it contains interaction counterparts of the free energy Hamiltonian (4.14). Secondly, one can reduce the system (4.1), (4.2) to various invariant submanifolds of $T^*\mathbb{R}^5$. In particular, after reducing it to symplectic manifold, which is mapped by the momentum map $\mathcal{J}$ on the coadjoint orbit $\mathcal{J}((\mathcal{I}^{-1}(\vec{d}) \cap T^*_{\mathrm{reg}}\mathbb{R}^5)/\mathrm{SL}(2,\mathbb{R})_{\vec{d}})$, we come back to the system (2.12)–(2.15) restricted to this coadjoint orbit. Let us recall here that Hamiltonian flow σ_t^h defined by Hamiltonian (4.2) commutes with the action of $\mathrm{SL}(2,\mathbb{R})_{\vec{d}}$ and $\vec{d}$ is an integral of motion for this flow. Since for $\vec{d} \in \Delta_s$ one has $\mathcal{J}((\mathcal{I}^{-1}(\vec{d}) \cap T^*_{\mathrm{reg}}\mathbb{R}^5)/\mathrm{SL}(2,\mathbb{R})_{\vec{d}}) \subset \Omega_s$ we can consider $(\vec{x},\vec{y})$ as a local coordinates on $(\mathcal{I}^{-1}(\vec{d}) \cap T^*_{\mathrm{reg}}\mathbb{R}^5)/\mathrm{SL}(2,\mathbb{R})_{\vec{d}}$. The above follows from (3.17) and (2.3). Restricting integrals of motion I_1, I_2, I_3 and I_4 defined by (2.16) to $(\mathcal{I}^{-1}(\vec{d}) \cap T^*_{\mathrm{reg}}\mathbb{R}^5)/\mathrm{SL}(2,\mathbb{R})_{\vec{d}}$ we find three integrals of motion on $(\mathcal{I}^{-1}(\vec{d}) \cap T^*_{\mathrm{reg}}\mathbb{R}^5)/\mathrm{SL}(2,\mathbb{R})_{\vec{d}}$.

$$\tilde{I}_1 := \lambda \, (I_1 \circ \mathcal{J}) \, (I_2 \circ \mathcal{J}) = (\vec{x} \times \vec{y})_3 = x_1 y_2 - x_2 y_1,$$

$$\tilde{I}_2 := c_1 + \frac{\lambda}{\lambda - \epsilon} I_3 \circ \mathcal{J} = \vec{x}^{\,2} + \alpha \vec{y}^{\,2}, \qquad \tilde{I}_3 := I_4 \circ \mathcal{J} = \left(\frac{\lambda - \epsilon}{\lambda} \right)^2 (\vec{x} \times \vec{y})^2$$

being in involution. The integral of motion $I_1 \circ \mathcal{J} = a$, as it follows from equation

$$\lambda^2 a^4 + (\tilde{I}_2 - c_1) a^2 + \frac{\alpha \lambda}{(\lambda - \epsilon)^2} \tilde{I}_3 = 0,$$

is functionally dependent on $\tilde{I}_2$ and $\tilde{I}_3$. The rank of the 6×3 Jacobi matrix $D\tilde{I}(\vec{x},\vec{y})$ of the map $\tilde{I} : \mathbb{R}^6 \longrightarrow \mathbb{R}$ is equal three iff $(\vec{x},\vec{y}) \in \mathbb{R}^6 \setminus \Sigma$, where the closed subset $\Sigma \subset \mathbb{R}^6$ is defined as the intersection of zero levels of all 3×3 minors of $D\tilde{I}(\vec{x},\vec{y})$. Thus we conclude that $\tilde{I}_2$, $\tilde{I}_2$ and $\tilde{I}_3$ are functionally independent almost everywhere on $(\mathcal{I}^{-1}(\vec{d}) \cap T^*_{\mathrm{reg}}\mathbb{R}^5)/\mathrm{SL}(2,\mathbb{R})_{\vec{d}}$.

In order to obtain some other interpretation of Hamiltonian systems integrated above let us reduce the canonical one-form

$$\gamma = p_{-1} \mathrm{d}q_{-1} + p_0 \mathrm{d}q_0 + \vec{p} \cdot \mathrm{d}\vec{q}$$

of $T^*\mathbb{R}^5$ to $T^*Q_{\lambda,\alpha}$. From $d_1 = \mathrm{const}$ and $d_3 = 0$ we find that

$$q_{-1} = \pm \sqrt{\frac{1}{\alpha\lambda} \left(d_1 - \lambda q_0^2 - \vec{q}^{\,2} \right)}, \qquad p_{-1} = \frac{-1}{q_{-1}} (q_0 p_0 + \vec{q} \cdot \vec{p})$$

and thus

$$\gamma\big|_{T^*Q_{\lambda,\alpha}} = \pi_0 \mathrm{d}q_0 + \vec{\pi} \cdot \mathrm{d}\vec{q},$$

where

$$\pi_0 = p_0 + \lambda \frac{q_0 p_0 + \vec{q} \cdot \vec{p}}{d_1 - \lambda q_0^2 - \vec{q}^2} q_0, \qquad \vec{\pi} = \vec{p} + \frac{q_0 p_0 + \vec{q} \cdot \vec{p}}{d_1 - \lambda q_0^2 - \vec{q}^2} \vec{q}.$$

For $\alpha\lambda = 1$ Hamiltonian (4.2), after reduction to $T^*Q_{\lambda,\alpha}$, takes in the canonical coordinates $(q_0, \vec{q}, \pi_0, \vec{\pi})$ a form of polynomial of degree eight

$$
\begin{aligned}
h &= \gamma d_1 \lambda^{-1} \pi_0^2 + \gamma \lambda^{-1} (\epsilon - \lambda)(\vec{\pi} \times \vec{q})^2 + \gamma d_1 \vec{\pi}^2 - \gamma (\pi_0 q_0 + \vec{\pi} \cdot \vec{q})^2 \\
&\quad + (\lambda - \epsilon)\lambda^{-2} \big(\nu(\lambda - \epsilon)(\vec{\pi} \times \vec{q})^2 - \gamma\big)\big(d_1 - \lambda q_0^2 - \vec{q}^2\big)\pi_0^2.
\end{aligned}
\tag{4.15}
$$

Passing in (4.15) to complex coordinates ($z_0 = q_0 + i\pi_0$, $\vec{z} = \vec{q} + i\vec{\pi}$) we obtain Hamiltonian

$$
\begin{aligned}
h &= \frac{1}{4}\bigg(\gamma d_1 \big(2\vec{z} \cdot \bar{\vec{z}} - \vec{z}^2 - \bar{\vec{z}}^2\big) + \gamma d_1 \lambda^{-1}\big(2|z_0|^2 - z_0^2 - \bar{z}_0^2\big) \\
&\quad + \frac{\gamma(\lambda - \epsilon)}{\lambda}(\vec{z} \times \bar{\vec{z}})^2 + \frac{\gamma}{16}\big(z_0^2 - \bar{z}_0^2 + \vec{z}^2 - \bar{\vec{z}}^2\big)^2 \\
&\quad + \frac{\lambda - \epsilon}{16\lambda^2}\big(\nu(\lambda - \epsilon)(\vec{z} \times \bar{\vec{z}})^2 + 4\gamma\big)\big(z_0^2 + \bar{z}_0^2 - 2|z_0|^2\big) \\
&\quad \times \big(4d_1 - \lambda z_0^2 - \lambda \bar{z}_0^2 - 2\lambda|z_0|^2 - \vec{z}^2 - \bar{\vec{z}}^2 - 2\vec{z} \cdot \bar{\vec{z}}\big)\bigg),
\end{aligned}
\tag{4.16}
$$

which describes a system of four running plane waves, slowly varying in nonlinear dielectric medium. The terms in (4.16) higher than quadratic ones are responsible for such nonlinear optical effects as intensity-dependent phase shift (Kerr effect) and the conversion between the modes. In a similar way one can interpret Hamiltonian (4.2), rewritten in complex coordinates, to describe a system of five nonlinear running plane waves. We refer to [4] and [9] for the treatment of Hamiltonian formulation of propagation of optical traveling wave pulses. Also one can find this type of nonlinear Hamiltonian optical system integrated by quadratures in [16].

Acknowledgements

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