Eigenvalue Estimates of the spin^c Dirac Operator and Harmonic Forms on Kähler–Einstein Manifolds

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Abstract. We establish a lower bound for the eigenvalues of the Dirac operator defined on a compact Kähler–Einstein manifold of positive scalar curvature and endowed with particular spin^c structures. The limiting case is characterized by the existence of Kählerian Killing spin^c spinors in a certain subbundle of the spinor bundle. Moreover, we show that the Clifford multiplication between an effective harmonic form and a Kählerian Killing spin^c spinor field vanishes. This extends to the spin^c case the result of A. Moroianu stating that, on a compact Kähler–Einstein manifold of complex dimension $4\ell+3$ carrying a complex contact structure, the Clifford multiplication between an effective harmonic form and a Kählerian Killing spinor is zero.

Key words: spin^c Dirac operator; eigenvalue estimate; Kählerian Killing spinor; parallel form; harmonic form

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1 Introduction

The geometry and topology of a compact Riemannian spin manifold (M^n, g) are strongly related to the existence of special spinor fields and thus, to the spectral properties of a fundamental operator called the Dirac operator D [1, 27]. A. Lichnerowicz [27] proved, under the weak condition of the positivity of the scalar curvature, that the kernel of the Dirac operator is trivial. Th. Friedrich [6] gave the following lower bound for the first eigenvalue λ of D on a compact Riemannian spin manifold (M^n, g):

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M S, \tag{1.1}$$

where S denotes the scalar curvature, assumed to be nonnegative. Equality holds if and only if the corresponding eigenspinor φ is parallel (if $\lambda = 0$) or a Killing spinor of Killing constant $-\frac{\lambda}{n}$ (if $\lambda \neq 0$), i.e., if $\nabla_X \varphi = -\frac{\lambda}{n} X \cdot \varphi$, for all vector fields X, where "$\cdot$" denotes the Clifford multiplication and ∇ is the spinorial Levi-Civita connection on the spinor bundle ΣM (see also [10]). Killing (resp. parallel) spinors force the underlying metric to be Einstein (resp. Ricci flat). The classification of complete simply-connected Riemannian spin manifolds with real

Killing (resp. parallel) spinors was done by C. Bär [2] (resp. M.Y. Wang [38]). Useful geometric information has been also obtained by restricting parallel and Killing spinors to hypersurfaces [3, 13, 14, 15, 16, 17]. O. Hijazi proved that the Clifford multiplication between a harmonic k-form β ($k \neq 0, n$) and a Killing spinor vanishes. In particular, the equality case in (1.1) cannot be attained on a Kähler spin manifold, since the Clifford multiplication between the Kähler form and a Killing spinor is never zero. Indeed, on a Kähler compact manifold (M^{2m}, g, J) of complex dimension m and complex structure J, K.-D. Kirchberg [19] showed that the first eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \begin{cases} \dfrac{m+1}{4m} \inf\limits_M S, & \text{if } m \text{ is odd,} \\ \dfrac{m}{4(m-1)} \inf\limits_M S, & \text{if } m \text{ is even.} \end{cases} \tag{1.2}$$

Kirchberg's estimates rely essentially on the decomposition of ΣM under the action of the Kähler form Ω. In fact, we have $\Sigma M = \oplus_{r=0}^{m} \Sigma_r M$, where $\Sigma_r M$ is the eigenbundle corresponding to the eigenvalue $i(2r-m)$ of Ω. The limiting manifolds of (1.2) are also characterized by the existence of spinors satisfying a certain differential equation similar to the one fulfilled by Killing spinors. More precisely, in odd complex dimension $m = 2\ell + 1$, it is proved in [11, 20, 21] that the metric is Einstein and the corresponding eigenspinor φ of λ is a Kählerian Killing spinor, i.e., $\varphi = \varphi_\ell + \varphi_{\ell+1} \in \Gamma(\Sigma_\ell M \oplus \Sigma_{\ell+1} M)$ and it satisfies

$$\begin{aligned} \nabla_X \varphi_\ell &= -\frac{\lambda}{2(m+1)} (X + iJX) \cdot \varphi_{\ell+1}, \\ \nabla_X \varphi_{\ell+1} &= -\frac{\lambda}{2(m+1)} (X - iJX) \cdot \varphi_\ell, \end{aligned} \tag{1.3}$$

for any vector field X. We point out that the existence of spinors of the form $\varphi = \varphi_{\ell'} + \varphi_{\ell'+1} \in \Gamma(\Sigma_{\ell'} M \oplus \Sigma_{\ell'+1} M)$ satisfying (1.3), implies that m is odd and they lie in the middle, i.e., $l' = \frac{m-1}{2}$. If the complex dimension is even, $m = 2\ell$, the limiting manifolds are characterized by constant scalar curvature and the existence of so-called anti-holomorphic Kählerian twistor spinors $\varphi_{\ell-1} \in \Gamma(\Sigma_{\ell-1} M)$, i.e., satisfying for any vector field X: $\nabla_X \varphi_{\ell-1} = -\frac{1}{2m}(X + iJX) \cdot D\varphi_{\ell-1}$. The limiting manifolds for Kirchberg's inequalities (1.2) have been geometrically described by A. Moroianu in [28] for m odd and in [30] for m even. In [36], this result is extended to limiting manifolds of the so-called refined Kirchberg inequalities, obtained by restricting the square of the Dirac operator to the eigenbundles $\Sigma_r M$. When m is even, the limiting manifold cannot be Einstein. Thus, on compact Kähler–Einstein manifolds of even complex dimension, K.-D. Kirchberg [22] improved (1.2) to the following lower bound

$$\lambda^2 \geq \frac{m+2}{4m} S. \tag{1.4}$$

Equality is characterized by the existence of holomorphic or anti-holomorphic spinors. When m is odd, A. Moroianu extended the above mentioned result of O. Hijazi to Kähler manifolds, by showing that the Clifford multiplication between a harmonic effective form of nonzero degree and a Kählerian Killing spinor vanishes. We recall that the manifolds of complex dimension $m = 4\ell + 3$ admitting Kählerian Killing spinors are exactly the Kähler–Einstein manifolds carrying a complex contact structure (cf. [23, 28, 33]).

In the present paper, we extend this result of A. Moroianu to Kählerian Killing spin^c spinors (see Theorem 4.2). In this more general setting difficulties occur due to the fact that the connection on the spin^c bundle, hence its curvature, the Dirac operator and its spectrum, do not only depend on the geometry of the manifold, but also on the connection of the auxiliary line bundle associated with the spin^c structure.

Spin^c geometry became an active field of research with the advent of Seiberg–Witten theory, which has many applications to 4-dimensional geometry and topology [5, 8, 25, 26, 37, 39]. From an intrinsic point of view, almost complex, Sasaki and some classes of CR manifolds carry a canonical spin^c structure. In particular, every Kähler manifold is spin^c but not necessarily spin. For example, the complex projective space $\mathbb{C}P^m$ is spin if and only if m is odd. Moreover, from the extrinsic point of view, it seems that it is more natural to work with spin^c structures rather than spin structures [12, 34, 35]. For instance, on Kähler–Einstein manifolds of positive scalar curvature, O. Hijazi, S. Montiel and F. Urbano [12] constructed spin^c structures carrying Kählerian Killing spin^c spinors, i.e., spinors satisfying (1.3), where the covariant derivative is the spin^c one. In [9], M. Herzlich and A. Moroianu extended Friedrich's estimate (1.1) to compact Riemannian spin^c manifolds. This new lower bound involves only the conformal geometry of the manifold and the curvature of the auxiliary line bundle associated with the spin^c structure. The limiting case is characterized by the existence of a spin^c Killing or parallel spinor, such that the Clifford multiplication of the curvature form of the auxiliary line bundle with this spinor is proportional to it.

In this paper, we give an estimate for the eigenvalues of the spin^c Dirac operator, by restricting ourselves to compact Kähler–Einstein manifolds endowed with particular spin^c structures. More precisely, we consider (M^{2m}, g, J) a compact Kähler–Einstein manifold of positive scalar curvature S and of index $p \in \mathbb{N}^*$. We endow M with the spin^c structure whose auxiliary line bundle is a tensorial power $\mathcal{L}^q$ of the *p-th* root $\mathcal{L}$ of the canonical bundle K_M of M, where $q \in \mathbb{Z}$, $p+q \in 2\mathbb{Z}$ and $|q| \leq p$. Our main result is the following:

Theorem 1.1. *Let (M^{2m}, g) be a compact Kähler–Einstein manifold of index p and positive scalar curvature S, carrying the* spin^c *structure given by $\mathcal{L}^q$ with $q+p \in 2\mathbb{Z}$, where $\mathcal{L}^p = K_M$. We assume that $p \geq |q|$ and the metric is normalized such that its scalar curvature equals $4m(m+1)$. Then, any eigenvalue λ of D^2 is bounded from below as follows*

$$\lambda \geq \left(1 - \frac{q^2}{p^2}\right)(m+1)^2. \tag{1.5}$$

Equality is attained if and only if $b := \frac{q}{p} \cdot \frac{m+1}{2} + \frac{m-1}{2} \in \mathbb{N}$ and there exists a Kählerian Killing spin^c *spinor in $\Gamma(\Sigma_b M \oplus \Sigma_{b+1} M)$.*

Indeed, this is a consequence of more refined estimates for the eigenvalues of the square of the spin^c Dirac operator restricted to the eigenbundles $\Sigma_r M$ of the spinor bundle (see Theorem 3.5). The proof of this result is based on a refined Schrödinger–Lichnerowicz spin^c formula (see Lemma 3.4) written on each such eigenbundle $\Sigma_r M$, which uses the decomposition of the covariant derivative acting on spinors into its holomorphic and antiholomorphic part. This formula has already been used in literature, for instance by K.-D. Kirchberg [22]. The limiting manifolds of (1.5) are characterized by the existence of Kählerian Killing spin^c spinors in a certain subbundle $\Sigma_r M$. In particular, this gives a positive answer to the conjectured relationship between spin^c Kählerian Killing spinors and a lower bound for the eigenvalues of the spin^c Dirac operator, as stated in [12, Remark 16].

Let us mention here that the Einstein condition in Theorem 1.1 is important in order to establish the estimate (1.5), since otherwise there is no control over the estimate of the term given by the Clifford action of the curvature form of the auxiliary line bundle of the spin^c structure (see (3.1)).

2 Preliminaries and notation

In this section, we set the notation and briefly review some basic facts about spin^c and Kähler geometries. For more details we refer to the books [4, 7, 24, 32].

Let (M^n, g) be an n-dimensional closed Riemannian spin^c manifold and denote by ΣM its complex spinor bundle, which has complex rank equal to $2^{[\frac{n}{2}]}$. The bundle ΣM is endowed with a Clifford multiplication denoted by "$\cdot$" and a scalar product denoted by $\langle\cdot,\cdot\rangle$. Given a spin^c structure on (M^n, g), one can check that the determinant line bundle $\det(\Sigma M)$ has a root L of index $2^{[\frac{n}{2}]-1}$. This line bundle L over M is called the auxiliary line bundle associated with the spin^c structure. The connection ∇^A on ΣM is the twisted connection of the one on the spinor bundle (induced by the Levi-Civita connection) and a fixed connection A on L. The spin^c Dirac operator D^A acting on the space of sections of ΣM is defined by the composition of the connection ∇^A with the Clifford multiplication. For simplicity, we will denote ∇^A by ∇ and D^A by D. In local coordinates:

$$D = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j},$$

where $\{e_j\}_{j=1,\dots,n}$ is a local orthonormal basis of TM. D is a first-order elliptic operator and is formally self-adjoint with respect to the L^2-scalar product. A useful tool when examining the spin^c Dirac operator is the Schrödinger–Lichnerowicz formula

$$D^2 = \nabla^*\nabla + \frac{1}{4}S + \frac{1}{2}F_A\cdot, \tag{2.1}$$

where ∇^* is the adjoint of ∇ with respect to the L^2-scalar product and F_A is the curvature (imaginary-valued) 2-form on M associated to the connection A defined on the auxiliary line bundle L, which acts on spinors by the extension of the Clifford multiplication to differential forms.

We recall that the complex volume element $\omega_{\mathbb{C}} = i^{[\frac{n+1}{2}]}e_1\wedge\cdots\wedge e_n$ acts as the identity on the spinor bundle if n is odd. If n is even, $\omega_{\mathbb{C}}^2 = 1$. Thus, under the action of the complex volume element, the spinor bundle decomposes into the eigenspaces $\Sigma^{\pm}M$ corresponding to the ±1 eigenspaces, the *positive* (resp. *negative*) spinors.

Every spin manifold has a trivial spin^c structure, by choosing the trivial line bundle with the trivial connection whose curvature F_A vanishes. Every Kähler manifold (M^{2m}, g, J) has a canonical spin^c structure induced by the complex structure J. The complexified tangent bundle decomposes into $T^{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M$, the i-eigenbundle (resp. $(-i)$-eigenbundle) of the complex linear extension of J. For any vector field X, we denote by $X^{\pm} := \frac{1}{2}(X \mp iJX)$ its component in $T_{1,0}M$, resp. $T_{0,1}M$. The spinor bundle of the canonical spin^c structure is defined by

$$\Sigma M = \Lambda^{0,*}M = \bigoplus_{r=0}^{m} \Lambda^r(T^*_{0,1}M),$$

and its auxiliary line bundle is $L = (K_M)^{-1} = \Lambda^m(T^*_{0,1}M)$, where $K_M = \Lambda^{m,0}M$ is the canonical bundle of M. The line bundle L has a canonical holomorphic connection, whose curvature form is given by $-i\rho$, where ρ is the Ricci form defined, for all vector fields X and Y, by $\rho(X,Y) = \mathrm{Ric}(JX, Y)$ and Ric denotes the Ricci tensor. Let us mention here the sign convention we use to define the Riemann curvature tensor, respectively the Ricci tensor: $R_{X,Y} := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ and $\mathrm{Ric}(X,Y) := \sum_{j=1}^{2m} R(e_j, X, Y, e_j)$, for all vector fields X, Y on M, where $\{e_j\}_{j=1,\dots,2m}$ is a local orthonormal basis of the tangent bundle. Similarly, one defines the so called anti-canonical spin^c structure, whose spinor bundle is given by $\Lambda^{*,0}M = \oplus_{r=0}^{m}\Lambda^r(T^*_{1,0}M)$ and the auxiliary line bundle by K_M. The spinor bundle of any other spin^c structure on M can be written as

$$\Sigma M = \Lambda^{0,*}M \otimes \mathbb{L},$$

where $\mathbb{L}^2 = K_M \otimes L$ and L is the auxiliary line bundle associated with this spin^c structure. The Kähler form Ω, defined as $\Omega(X,Y) = g(JX,Y)$, acts on ΣM via Clifford multiplication and this action is locally given by

$$\Omega \cdot \psi = \frac{1}{2}\sum_{j=1}^{2m} e_j \cdot Je_j \cdot \psi, \tag{2.2}$$

for all $\psi \in \Gamma(\Sigma M)$, where $\{e_1, \ldots, e_{2m}\}$ is a local orthonormal basis of TM. Under this action, the spinor bundle decomposes as follows

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M, \tag{2.3}$$

where $\Sigma_r M$ denotes the eigenbundle to the eigenvalue $i(2r-m)$ of Ω, of complex rank $\binom{m}{k}$. It is easy to see that $\Sigma_r M \subset \Sigma^+ M$ (resp. $\Sigma_r M \subset \Sigma^- M$) if and only if r is even (resp. r is odd). Moreover, for any $X \in \Gamma(TM)$ and $\varphi \in \Gamma(\Sigma_r M)$, we have $X^+ \cdot \varphi \in \Gamma(\Sigma_{r+1} M)$ and $X^- \cdot \varphi \in \Gamma(\Sigma_{r-1} M)$, with the convention $\Sigma_{-1} M = \Sigma_{m+1} M = M \times \{0\}$. Thus, for any spin^c structure, we have $\Sigma_r M = \Lambda^{0,r} M \otimes \Sigma_0 M$. Hence, $(\Sigma_0 M)^2 = K_M \otimes L$, where L is the auxiliary line bundle associated with the spin^c structure. For example, when the manifold is spin, we have $(\Sigma_0 M)^2 = K_M$ [18, 19]. For the canonical spin^c structure, since $L = (K_M)^{-1}$, it follows that $\Sigma_0 M$ is trivial. This yields the existence of parallel spinors (the constant functions) lying in $\Sigma_0 M$, cf. [31].

Associated to the complex structure J, one defines the following operators

$$D^+ = \sum_{j=1}^{2m} e_j^+ \cdot \nabla_{e_j^-}, \qquad D^- = \sum_{j=1}^{2m} e_j^- \cdot \nabla_{e_j^+},$$

which satisfy the relations

$$D = D^+ + D^-, \qquad (D^+)^2 = 0, \qquad (D^-)^2 = 0, \qquad D^+D^- + D^-D^+ = D^2.$$

When restricting the Dirac operator to $\Sigma_r M$, it acts as

$$D = D^+ + D^- \colon\ \Gamma(\Sigma_r M) \to \Gamma(\Sigma_{r-1} M \oplus \Sigma_{r+1} M).$$

Corresponding to the decomposition $TM \otimes \Sigma_r M \cong \Sigma_{r-1} M \oplus \Sigma_{r+1} M \oplus \mathrm{Ker}_r$, where Ker_r denotes the kernel of the Clifford multiplication by tangent vectors restricted to $\Sigma_r M$, we have, as in the spin case (for details see, e.g., [36, equation (2.7)]), the following Weitzenböck formula relating the differential operators acting on sections of $\Sigma_r M$:

$$\nabla^*\nabla = \frac{1}{2(r+1)} D^-D^+ + \frac{1}{2(m-r+1)} D^+D^- + T_r^* T_r,$$

where T_r is the so-called Kählerian twistor operator and is defined by

$$T_r\varphi := \nabla\varphi + \frac{1}{2(m-r+1)} e_j \otimes e_j^+ \cdot D^-\varphi + \frac{1}{2(r+1)} e_j \otimes e_j^- \cdot D^+\varphi.$$

This decomposition further implies the following identity for $\varphi \in \Gamma(\Sigma_r M)$, by the same argument as in [36, Lemma 2.5],

$$|\nabla\varphi|^2 = \frac{1}{2(r+1)}|D^+\varphi|^2 + \frac{1}{2(m-r+1)}|D^-\varphi|^2 + |T_r\varphi|^2. \tag{2.4}$$

Hence, we have the inequality

$$|\nabla\varphi|^2 \geq \frac{1}{2(r+1)}|D^+\varphi|^2 + \frac{1}{2(m-r+1)}|D^-\varphi|^2. \tag{2.5}$$

Equality in (2.5) is attained if and only if $T_r\varphi = 0$, in which case φ is called a Kählerian twistor spinor. The Lichnerowicz–Schrödinger formula (2.1) yields the following:

Lemma 2.1. *Let (M^{2m}, g, J) be a compact Kähler manifold endowed with any* spin^c *structure. If φ is an eigenspinor of D^2 with eigenvalue λ, $D^2\varphi = \lambda\varphi$, and satisfies*

$$|\nabla\varphi|^2 \geq \frac{1}{j}|D\varphi|^2, \tag{2.6}$$

for some real number $j > 1$, and $(S + 2F_A)\cdot\varphi = c\varphi$, where c is a positive function, then

$$\lambda \geq \frac{j}{4(j-1)}\inf_M c. \tag{2.7}$$

Moreover, equality in (2.7) holds if and only if the function c is constant and equality in (2.6) holds at all points of the manifold.

Let $\{e_1, \ldots, e_{2m}\}$ be a local orthonormal basis of M^{2m}. We implicitly use the Einstein summation convention over repeated indices. We have the following formulas for contractions that hold as endomorphisms of $\Sigma_r M$:

$$e_j^+ \cdot e_j^- = -2r, \qquad e_j^- \cdot e_j^+ = -2(m-r), \tag{2.8}$$

$$e_j \cdot \mathrm{Ric}(e_j) = -S, \qquad e_j^- \cdot \mathrm{Ric}(e_j^+) = -\frac{S}{2} - i\rho, \qquad e_j^+ \cdot \mathrm{Ric}(e_j^-) = -\frac{S}{2} + i\rho. \tag{2.9}$$

The identities (2.8) follow directly from (2.2), which gives the action of the Kähler form and has $\Sigma_r M$ as eigenspace to the eigenvalue $i(2r-m)$, implying that $ie_j \cdot Je_j = 2i\Omega = -2(2r-m)$, and from the fact that $e_j \cdot e_j = -2m$. The identities (2.9) are obtained from the following identities

$$e_j \cdot \mathrm{Ric}(e_j) = e_j \wedge \mathrm{Ric}(e_j) - g(\mathrm{Ric}(e_j), e_j) = -S,$$
$$ie_j \cdot \mathrm{Ric}(Je_j) = ie_j \wedge \mathrm{Ric}(Je_j) - ig(\mathrm{Ric}(Je_j), e_j) = 2i\rho.$$

The spin^c Ricci identity, for any spinor φ and any vector field X, is given by

$$e_i \cdot \mathcal{R}^A_{e_i,X}\varphi = \frac{1}{2}\mathrm{Ric}(X)\cdot\varphi - \frac{1}{2}(X \lrcorner F_A)\cdot\varphi, \tag{2.10}$$

where $\mathcal{R}^A$ denotes the spin^c spinorial curvature, defined with the same sign convention as above, namely $\mathcal{R}^A_{X,Y} := \nabla^A_X\nabla^A_Y - \nabla^A_Y\nabla^A_X - \nabla^A_{[X,Y]}$. For a proof of the spin^c Ricci identity we refer to [7, Section 3.1]. For any vector field X parallel at the point where the computation is done, the following commutator rules hold

$$[\nabla_X, D] = -\frac{1}{2}\mathrm{Ric}(X)\cdot + \frac{1}{2}(X \lrcorner F_A)\cdot, \tag{2.11}$$

$$[\nabla_X, D^+] = -\frac{1}{2}\mathrm{Ric}(X^+)\cdot + \frac{1}{2}\big(X^+ \lrcorner F_A^{1,1}\big)\cdot + \frac{1}{2}\big(X^- \lrcorner F_A^{0,2}\big)\cdot, \tag{2.12}$$

$$[\nabla_X, D^-] = -\frac{1}{2}\mathrm{Ric}(X^-)\cdot + \frac{1}{2}\big(X^- \lrcorner F_A^{1,1}\big)\cdot + \frac{1}{2}\big(X^+ \lrcorner F_A^{2,0}\big)\cdot, \tag{2.13}$$

where the 2-form F_A is decomposed as $F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2}$, into forms of type $(2,0)$, $(1,1)$, respectively $(0,2)$. The identity (2.11) is obtained from the following straightforward computation

$$\begin{aligned}\nabla_X(D\varphi) &= \nabla_X(e_j \cdot \nabla_{e_j}\varphi) = e_j \cdot \mathcal{R}^A_{X,e_j}\varphi + e_j \cdot \nabla_{e_j}\nabla_X\varphi \\ &\overset{(2.10)}{=} -\frac{1}{2}\mathrm{Ric}(X)\cdot\varphi + \frac{1}{2}(X \lrcorner F_A)\cdot\varphi + D(\nabla_X\varphi).\end{aligned}$$

The identity (2.12) follows from the identities

$$\begin{aligned}\nabla_{X^+}(D^+\varphi) &= \nabla_{X^+}(e_i^+ \cdot \nabla_{e_i^-}\varphi) = e_i^+ \cdot \mathcal{R}^A_{X^+,e_i^-}\varphi + e_i^+ \cdot \nabla_{e_i^-}\nabla_{X^+}\varphi \\ &= -\frac{1}{2}\mathrm{Ric}(X^+)\cdot\varphi + \frac{1}{2}\big(X^+ \lrcorner F_A^{1,1}\big)\cdot\varphi + D^+(\nabla_{X^+}\varphi), \\ \nabla_{X^-}(D^+\varphi) &= \nabla_{X^-}\big(e_i^+ \cdot \nabla_{e_i^-}\varphi\big) = e_i^+ \cdot R_{X^-,e_i^-}\varphi + e_i^+ \cdot \nabla_{e_i^-}\nabla_{X^-}\varphi \\ &= \frac{1}{2}\big(X^- \lrcorner F_A^{0,2}\big)\cdot\varphi + D^+(\nabla_{X^-}\varphi).\end{aligned}$$

The identity (2.13) follows either by an analogous computation or by conjugating (2.12).

On a Kähler manifold (M, g, J) endowed with any spin^c structure, a spinor of the form $\varphi_r + \varphi_{r+1} \in \Gamma(\Sigma_r M \oplus \Sigma_{r+1}M)$, for some $0 \le r \le m$, is called a *Kählerian Killing* spin^c *spinor* if there exists a non-zero real constant α, such that the following equations are satisfied, for all vector fields X,

$$\nabla_X\varphi_r = \alpha X^- \cdot \varphi_{r+1}, \qquad \nabla_X\varphi_{r+1} = \alpha X^+ \cdot \varphi_r. \tag{2.14}$$

Kählerian Killing spinors lying in $\Gamma(\Sigma_m M \oplus \Sigma_{m+1}M) = \Gamma(\Sigma_m M)$ or in $\Gamma(\Sigma_{-1}M \oplus \Sigma_0 M) = \Gamma(\Sigma_0 M)$ are just parallel spinors. A direct computation shows that each Kählerian Killing spin^c spinor is an eigenspinor of the square of the Dirac operator. More precisely, the following equalities hold

$$D\varphi_r = -2(r+1)\alpha\varphi_{r+1}, \qquad D\varphi_{r+1} = -2(m-r)\alpha\varphi_r, \tag{2.15}$$

which further yield

$$D^2\varphi_r = 4(m-r)(r+1)\alpha^2\varphi_r, \qquad D^2\varphi_{r+1} = 4(m-r)(r+1)\alpha^2\varphi_{r+1}. \tag{2.16}$$

In [12], the authors gave examples of spin^c structures on compact Kähler–Einstein manifolds of positive scalar curvature, which carry Kählerian Killing spin^c spinors lying in $\Sigma_r M \oplus \Sigma_{r+1}M$, for $r \neq \frac{m\pm1}{2}$, in contrast to the spin case, where Kählerian Killing spinors may only exist for m odd in the middle of the decomposition (2.3). We briefly describe these spin^c structures here. If the first Chern class $c_1(K_M)$ of the canonical bundle of the Kähler M is a non-zero cohomology class, the greatest number $p \in \mathbb{N}^*$ such that

$$\frac{1}{p}c_1(K_M) \in H^2(M, \mathbb{Z}),$$

is called the (*Fano*) *index* of the manifold M. One can thus consider a p-th root of the canonical bundle K_M, i.e., a complex line bundle $\mathcal{L}$, such that $\mathcal{L}^p = K_M$. In [12], O. Hijazi, S. Montiel and F. Urbano proved the following:

Theorem 2.2 ([12, Theorem 14])**.** *Let M be a $2m$-dimensional Kähler–Einstein compact manifold with scalar curvature $4m(m+1)$ and index $p \in \mathbb{N}^*$. For each $0 \le r \le m+1$, there exists on M a* spin^c *structure with auxiliary line bundle given by $\mathcal{L}^q$, where $q = \frac{p}{m+1}(2r-m-1) \in \mathbb{Z}$, and carrying a Kählerian Killing spinor $\psi_{r-1} + \psi_r \in \Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$, i.e., it satisfies the first-order system*

$$\nabla_X\psi_r = -X^+ \cdot \psi_{r-1}, \qquad \nabla_X\psi_{r-1} = -X^- \cdot \psi_r,$$

for all $X \in \Gamma(TM)$.

For example, if M is the complex projective space $\mathbb{C}P^m$ of complex dimension m, then $p = m+1$ and $\mathcal{L}$ is just the tautological line bundle. We fix $0 \leq r \leq m+1$ and we endow $\mathbb{C}P^m$ with the spin^c structure whose auxiliary line bundle is given by $\mathcal{L}^q$ where $q = \frac{p}{m+1}(2r-m-1) = 2r-m-1 \in \mathbb{Z}$. For this spin^c structure, the space of Kählerian Killing spinors in $\Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$ has dimension $\binom{m+1}{r}$. A Kähler manifold carrying a complex contact structure necessarily has odd complex dimension $m = 2\ell+1$ and its index p equals $\ell+1$. We fix $0 \leq r \leq m+1$ and we endow M with the spin^c structure whose auxiliary line bundle is given by $\mathcal{L}^q$ where $q = \frac{p}{m+1}(2r-m-1) = r-\ell-1 \in \mathbb{Z}$. For this spin^c structure, the space of Kählerian Killing spinors in $\Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$ has dimension 1. In these examples, for $r = 0$ (resp. $r = m+1$), we get the canonical (resp. anticanonical) spin^c structure for which Kählerian Killing spinors are just parallel spinors.

3 Eigenvalue estimates for the spin^c Dirac operator on Kähler–Einstein manifolds

In this section, we give a lower bound for the eigenvalues of the spin^c Dirac operator on a Kähler–Einstein manifold endowed with particular spin^c structures. More precisely, let (M^{2m}, g, J) be a compact Kähler–Einstein manifold of index $p \in \mathbb{N}^*$ and of positive scalar curvature S, endowed with the spin^c structure given by $\mathcal{L}^q$, where $\mathcal{L}$ is the *p-th* root of the canonical bundle and $q+p \in 2\mathbb{Z}$ (among all powers $\mathcal{L}^q$, only those satisfying $p+q \in 2\mathbb{Z}$ provide us a spin^c structure, cf. [12, Section 7]). The curvature form F_A of the induced connection A on $\mathcal{L}^q$ acts on the spinor bundle as $\frac{q}{p} i\rho$. Since (M^{2m}, g, J) is Kähler–Einstein, it follows that $\rho = \frac{S}{2m}\Omega$, where Ω is the Kähler form. Hence, for each $0 \leq r \leq m$, we have

$$(S + 2F_A)\cdot\varphi_r = \left(1 - \frac{q}{p}\cdot\frac{2r-m}{m}\right) S\varphi_r, \qquad \forall\, \varphi_r \in \Gamma(\Sigma_r M). \tag{3.1}$$

Let us denote by $c_r := 1 - \frac{q}{p}\cdot\frac{2r-m}{m}$ and

$$a_1 \colon \{0,\dots,m\} \to \mathbb{R}, \qquad a_1(r) := \frac{r+1}{2r+1}c_r,$$
$$a_2 \colon \{0,\dots,m\} \to \mathbb{R}, \qquad a_2(r) := \frac{m-r+1}{2m-2r+1}c_r.$$

With the above notation, the following result holds:

Proposition 3.1. *Each eigenvalue λ_r of D^2 restricted to $\Sigma_r M$ with associated eigenspinor φ_r satisfies the inequality*

$$\lambda_r \geq \max\big(\min\big(a_1(r), a_1(r-1)\big), \min\big(a_2(r), a_2(r+1)\big)\big)\cdot\frac{S}{2}. \tag{3.2}$$

Moreover, the equality case is characterized as follows:

a) $D^2\varphi_r = a_1(r)\frac{S}{2}\varphi_r \Longleftrightarrow T_r\varphi_r = 0,\ D^-\varphi_r = 0$;
b) $D^2\varphi_r = a_1(r-1)\frac{S}{2}\varphi_r \Longleftrightarrow T_{r-1}(D^-\varphi_r) = 0$;
c) $D^2\varphi_r = a_2(r)\frac{S}{2}\varphi_r \Longleftrightarrow T_r\varphi_r = 0,\ D^+\varphi_r = 0$;
d) $D^2\varphi_r = a_2(r+1)\frac{S}{2}\varphi_r \Longleftrightarrow T_{r+1}(D^+\varphi_r) = 0$.

Proof. For $0 \leq r \leq m$ we have: $(S+2F_A)\cdot\varphi_r = c_r S\varphi_r$, $\forall\varphi_r \in \Gamma(\Sigma_r M)$. Let $r \in \{0,\dots,m\}$ be fixed, λ_r be an eigenvalue of $D^2|_{\Sigma_r M}$ and $\varphi_r \in \Gamma(\Sigma_r M)$ be an eigenspinor: $D^2\varphi_r = \lambda_r\varphi_r$. We distinguish two cases.

i) If $D^-\varphi_r = 0$, then $|D\varphi_r|^2 = |D^+\varphi_r|^2$ and (2.5) implies

$$|\nabla\varphi_r|^2 \geq \frac{1}{2(r+1)}|D^+\varphi_r|^2 = \frac{1}{2(r+1)}|D\varphi_r|^2.$$

By Lemma 2.1, it follows that

$$\lambda_r \geq \frac{r+1}{2(2r+1)}c_r S.$$

ii) If $D^-\varphi_r \neq 0$, then we consider $\varphi_r^- := D^-\varphi_r$, which satisfies $D^2\varphi_r^- = \lambda_r\varphi_r^-$ and $D^-\varphi_r^- = 0$, so in particular $|D\varphi_r^-|^2 = |D^+\varphi_r^-|^2$. We now apply the argument in i) to $\varphi_r^- \in \Gamma(\Sigma_{r-1}M)$. By (2.5), it follows that

$$|\nabla\varphi_r^-|^2 \geq \frac{1}{2r}|D^+\varphi_r^-|^2 = \frac{1}{2r}|D\varphi_r^-|^2.$$

Applying again Lemma 2.1, we obtain $\lambda_r \geq \frac{r}{2(2r-1)}c_{r-1}S$.

Hence, we have showed that $\lambda_r \geq \min\big(a_1(r), a_1(r-1)\big)\frac{S}{2}$. The same argument applied to the cases when $D^+\varphi_r = 0$ and $D^+\varphi_r \neq 0$ proves the inequality $\lambda_r \geq \min\big(a_2(r), a_2(r+1)\big)\frac{S}{2}$. Altogether we obtain the estimate in Proposition 3.1. The characterization of the equality cases is a direct consequence of Lemma 2.1, identity (2.4) and the description of the limiting case of inequality (2.5). ∎

Remark 3.2. The inequality (3.2) can be expressed more explicitly, by determining the maximum according to several possible cases. However, since in the sequel we will refine this eigenvalue estimate, we are only interested in the characterization of the limiting cases, which will be used later in the proof of the equality case of the estimate (1.5).

In order to refine the estimate (3.2), we start by the following two lemmas.

Lemma 3.3. *Let (M^{2m}, g, J) be a compact Kähler–Einstein manifold of index p and of positive scalar curvature S, endowed with a* spin^c *structure given by $\mathcal{L}^q$, where $q + p \in 2\mathbb{Z}$. For any spinor field φ and any vector field X, the* spin^c *Ricci identity is given by*

$$e_j \cdot \mathcal{R}^A_{e_j,X}\varphi = \frac{1}{2}\mathrm{Ric}(X)\cdot\varphi - \frac{S}{4m}\frac{q}{p}(X\lrcorner i\Omega)\cdot\varphi, \tag{3.3}$$

and it can be refined as follows

$$e_j^- \cdot \mathcal{R}^A_{e_j^+,X^-}\varphi = \frac{1}{2}\mathrm{Ric}(X^-)\cdot\varphi - \frac{S}{4m}\frac{q}{p}X^-\cdot\varphi, \tag{3.4}$$

$$e_j^+ \cdot \mathcal{R}^A_{e_j^-,X^+}\varphi = \frac{1}{2}\mathrm{Ric}(X^+)\cdot\varphi + \frac{S}{4m}\frac{q}{p}X^+\cdot\varphi. \tag{3.5}$$

Proof. Since the curvature form F_A of the spin^c structure acts on the spinor bundle as $\frac{q}{p}i\rho = \frac{q}{p}\frac{S}{2m}i\Omega$, (3.3) follows directly from the Ricci identity (2.10). The refined identities (3.4) and (3.5) follow by replacing X in (3.3) with X^-, respectively X^+, which is possible since both sides of the identity are complex linear in X, and by taking into account that when decomposing $e_j = e_j^+ + e_j^-$, the following identities (and their analogue for X^+) hold: $e_j \cdot \mathcal{R}^A_{e_j^-,X^-} = 0$ and $e_j^+ \cdot \mathcal{R}^A_{e_j^+,X^-} = 0$. These last two identities are a consequence of the J-invariance of the curvature tensor, i.e., $\mathcal{R}^A_{JX,JY} = \mathcal{R}^A_{X,Y}$, for all vector fields X, Y, as this implies $\mathcal{R}^A_{e_j^-,X^-} = \mathcal{R}^A_{Je_j^-,JX^-} = (-i)^2\mathcal{R}^A_{e_j^-,X^-}$ and also $e_j^+ \cdot \mathcal{R}^A_{e_j^+,X^-} = Je_j^+ \cdot \mathcal{R}^A_{Je_j^+,X^-} = i^2 e_j^+ \cdot \mathcal{R}^A_{e_j^+,X^-}$, so they both vanish. In order to obtain the second term on the right hand side of (3.4) and (3.5), we use the following identities of endomorphisms of the spinor bundle: $X^-\lrcorner i\Omega = X^-$ and $X^+\lrcorner i\Omega = -X^+$. ∎

Lemma 3.4. *Under the same assumptions as in Lemma 3.3, the refined Schrödinger–Lichnerowicz formula for* spin^c *Kähler manifolds for the action on each eigenbundle $\Sigma_r M$ is given by*

$$2\nabla^{1,0^*}\nabla^{1,0} = D^2 - \frac{S}{4} - \frac{i}{2}\rho - \frac{m-r}{2m}\frac{q}{p}S, \tag{3.6}$$

$$2\nabla^{0,1^*}\nabla^{0,1} = D^2 - \frac{S}{4} + \frac{i}{2}\rho + \frac{r}{2m}\frac{q}{p}S, \tag{3.7}$$

where $\nabla^{1,0}$ (resp. $\nabla^{0,1}$) is the holomorphic (resp. antiholomorphic) part of ∇, i.e., the projections of ∇ onto the following two components

$$\nabla\colon\ \Gamma(\Sigma_r M) \to \Gamma(\Lambda^{1,0}M \otimes \Sigma_r M) \oplus \Gamma(\Lambda^{0,1}M \otimes \Sigma_r M).$$

They are locally defined, for all vector fields X, by

$$\nabla^{1,0}_X = g(X, e_i^-)\nabla_{e_i^+} = \nabla_{X^+} \qquad \text{and} \qquad \nabla^{0,1}_X = g(X, e_i^+)\nabla_{e_i^-} = \nabla_{X^-},$$

where $\{e_1, \dots, e_{2m}\}$ is a local orthonormal basis of TM.

Proof. Let $\{e_1, \dots, e_{2m}\}$ be a local orthonormal basis of TM (identified with $\Lambda^1 M$ via the metric g), parallel at the point where the computation is made. We recall that the formal adjoints $\nabla^{1,0^*}$ and $\nabla^{1,0^*}$ are given by the following formulas (for a proof, see, e.g., [32, Lemma 20.1])

$$\nabla^{1,0^*}\colon\ \Gamma(\Lambda^{1,0}M \otimes \Sigma_r M) \longrightarrow \Gamma(\Sigma_r M), \qquad \nabla^{1,0^*}(\alpha \otimes \varphi) = (\delta\alpha)\varphi - \nabla_\alpha \varphi,$$
$$\nabla^{0,1^*}\colon\ \Gamma(\Lambda^{0,1}M \otimes \Sigma_r M) \longrightarrow \Gamma(\Sigma_r M), \qquad \nabla^{0,1^*}(\alpha \otimes \varphi) = (\delta\alpha)\varphi - \nabla_\alpha \varphi.$$

We thus obtain for the corresponding Laplacians

$$\nabla^{1,0^*}\nabla^{1,0}\varphi = \nabla^{1,0^*}(e_j^- \otimes \nabla_{e_j^+}\varphi) = -\nabla_{e_j^-}\nabla_{e_j^+}, \tag{3.8}$$

since $\delta e_j^- = 0$, as the basis is parallel at the given point, and $g(\cdot, e_j^-) \in \Lambda^{1,0}M$. Analogously, or by conjugation, we have $\nabla^{0,1^*}\nabla^{0,1}\varphi = -\nabla_{e_j^+}\nabla_{e_j^-}$. We now prove (3.6). By a similar computation, one obtains (3.7)

$$\begin{aligned}
2\nabla^{1,0^*}\nabla^{1,0} &\overset{(3.8)}{=} -2g(e_i, e_j)\nabla_{e_i^-}\nabla_{e_j^+} = (e_i \cdot e_j + e_j \cdot e_i)\cdot \nabla_{e_i^-}\nabla_{e_j^+} \\
&= D^+D^- + e_j \cdot e_i \cdot (\nabla_{e_j^+}\nabla_{e_i^-} - R_{e_j^+, e_i^-}) = D^+D^- + D^-D^+ + e_j^- \cdot e_i^+ \cdot R_{e_i^-, e_j^+} \\
&\overset{(3.5)}{=} D^2 + e_j^- \cdot \left(\frac{1}{2}\mathrm{Ric}(e_j^+) + \frac{S}{4m}\frac{q}{p}e_j^+\right) \\
&\overset{(2.8),\ (2.9)}{=} D^2 - \frac{1}{2}\left(\frac{S}{2} + i\rho\right) - \frac{m-r}{2m}\frac{q}{p}S.
\end{aligned}$$

■

Theorem 3.5. *Let (M^{2m}, g, J) be a compact Kähler–Einstein manifold of index p and positive scalar curvature S, carrying the* spin^c *structure given by $\mathcal{L}^q$ with $q + p \in 2\mathbb{Z}$, where $\mathcal{L}^p = K_M$. We assume that $p \geq |q|$. Then, for each $r \in \{0, \dots, m\}$, any eigenvalue λ_r of $D^2|_{\Gamma(\Sigma_r M)}$ satisfies the inequality*

$$\lambda_r \geq e(r)\frac{S}{2}, \tag{3.9}$$

where

$$e\colon\ [0, m] \to \mathbb{R}, \qquad e(x) = \begin{cases} e_1(x) = \dfrac{m-x}{m}\left(1 + \dfrac{q}{p}\right), & \text{if } x \leq \left(1 + \dfrac{q}{p}\right)\dfrac{m}{2}, \\ e_2(x) = \dfrac{x}{m}\left(1 - \dfrac{q}{p}\right), & \text{if } x \geq \left(1 + \dfrac{q}{p}\right)\dfrac{m}{2}. \end{cases}$$

Moreover, equality is attained if and only if the corresponding eigenspinor $\varphi_r \in \Gamma(\Sigma_r M)$ is an antiholomorphic spinor: $\nabla^{1,0}\varphi_r = 0$, if $r \leq \left(1+\frac{q}{p}\right)\frac{m}{2}$, respectively a holomorphic spinor: $\nabla^{0,1}\varphi_r = 0$, if $r \geq \left(1+\frac{q}{p}\right)\frac{m}{2}$.

Proof. First we notice that our assumption $|q| \leq p$ implies that the lower bound in (3.9) is non-negative and that $0 \leq \left(1+\frac{q}{p}\right)\frac{m}{2} \leq m$. The formulas (3.6) and (3.7) applied to φ_r yield, after taking the scalar product with φ_r and integrating over M, the following inequalities

$$\lambda_r \geq \frac{m-r}{m}\left(1+\frac{q}{p}\right)\frac{S}{2}, \qquad \lambda_r \geq \frac{r}{m}\left(1-\frac{q}{p}\right)\frac{S}{2},$$

and equality is attained if and only if the corresponding eigenspinor φ_r satisfies $\nabla^{1,0}\varphi_r = 0$, resp. $\nabla^{0,1}\varphi_r = 0$. Hence, for any $0 \leq r \leq m$ we obtain the following lower bound:

$$\lambda_r \geq \max\left(\frac{m-r}{m}\left(1+\frac{q}{p}\right), \frac{r}{m}\left(1-\frac{q}{p}\right)\right) = e(r)\frac{S}{2}. \qquad \blacksquare$$

Remark 3.6. Let us denote $\frac{q}{p}\cdot\frac{m+1}{2}+\frac{m-1}{2}$ by b. Comparing the estimate given by Theorem 3.5 with the estimate from Proposition 3.1, we obtain for $r \leq b$

$$e(r) - a_1(r) = \frac{(m+1)\frac{q}{p} + m - 1 - 2r}{m(2r+1)} = -\frac{2(r-b)}{m(2r+1)}.$$

Hence, for $r \leq b$, we have $e(r) \geq a_1(r)$ and $e(r) = a_1(r)$ iff $r = b \in \mathbb{N}$. Similarly, for $r \geq b+1$, we compute

$$e(r) - a_2(r) = \frac{2(m-r)(r-b-1)}{m(2m-2r+1)}.$$

Hence, for $r \geq b+1$, we have $e(r) \geq a_2(r)$ and $e(r) = a_2(r)$ iff $r = b+1 \in \mathbb{N}$.

Theorem 3.5 implies the global lower bound for the eigenvalues of the spin^c Dirac operator acting on the whole spinor bundle in Theorem 1.1. We are now ready to prove this result.

Proof of Theorem 1.1. Since the lower bound established in Theorem 3.5 decreases on $\left(0, \left(1+\frac{q}{p}\right)\frac{m}{2}\right)$ and increases on $\left(\left(1+\frac{q}{p}\right)\frac{m}{2}, m\right)$, we obtain the following global estimate

$$\lambda \geq e\left(\left(1+\frac{q}{p}\right)\frac{m}{2}\right) = \frac{1}{2}\left(1-\frac{q^2}{p^2}\right)\frac{S}{2}.$$

However, this estimate is not sharp. Otherwise, this would imply that $\left(1+\frac{q}{p}\right)\frac{m}{2} \in \mathbb{N}$ and the limiting eigenspinor would be, according to the characterization of the equality case in Theorem 3.5, both holomorphic and antiholomorphic, hence parallel and, in particular, harmonic. This fact together with the Lichnerowicz–Schrödinger formula (2.1) and the fact that the scalar curvature is positive leads to a contradiction.

We now assume that there exists an $r \in \mathbb{N}$, such that $b < r < \left(1+\frac{q}{p}\right)\frac{m}{2}$ and the equality in (3.9) is attained. We obtain a contradiction as follows. Let φ_r be the corresponding eigenspinor: $D^2\varphi_r = e_1(r)\frac{S}{2}\varphi_r$ and $\nabla^{1,0}\varphi_r = 0$. Then $D^+\varphi_r \in \Sigma_{r+1}M$ is also an eigenspinor of D^2 to the eigenvalue $e_1(r)\frac{S}{2}$ (note that $D^+\varphi_r \neq 0$, otherwise φ_r would be a harmonic spinor and we could conclude as above). However, for all $r > b$, the strict inequality $e_2(r+1) > e_1(r)$ holds. Since $r+1 > \left(1+\frac{q}{p}\right)\frac{m}{2}$, this contradicts the estimate (3.9). The same argument as above shows that there exists no $r \in \mathbb{N}$, such that $\left(1+\frac{q}{p}\right)\frac{m}{2} < r < b+1$ and the equality in (3.9) is attained. Hence, we obtain the following global estimate

$$\lambda \geq e_1(b)\frac{S}{2} = e_2(b+1)\frac{S}{2} = \frac{m+1}{2m}\left(1-\frac{q^2}{p^2}\right)\frac{S}{2} = \left(1-\frac{q^2}{p^2}\right)(m+1)^2.$$

According to Theorem 3.5, the equality is attained if and only if $b \in \mathbb{N}$ and the corresponding eigenspinors $\varphi_b \in \Gamma(\Sigma_b M)$ and $\varphi_{b+1} \in \Gamma(\Sigma_{b+1} M)$ to the eigenvalue $\left(1-\frac{q^2}{p^2}\right)(m+1)^2$ are antiholomorphic resp. holomorphic spinors: $\nabla^{1,0}\varphi_b = 0$, $\nabla^{0,1}\varphi_{b+1} = 0$. In particular, this implies $D^-\varphi_b = 0$ and $D^+\varphi_{b+1} = 0$. By Remark 3.6, we have: $e_1(b) = a_1(b)$ and $e_2(b+1) = a_2(b+1)$. Hence, the characterization of the equality case in Proposition 3.1 yields $T_b\varphi_b = 0$ and $T_{b+1}\varphi_{b+1} = 0$, which further imply

$$\begin{aligned} \nabla_X \varphi_b &= -\frac{1}{2(b+1)} X^- \cdot D^+\varphi_b = -\frac{1}{2(b+1)} X^- \cdot D\varphi_b, \\ \nabla_X \varphi_{b+1} &= -\frac{1}{2(m-b)} X^+ \cdot D^-\varphi_{b+1} = -\frac{1}{2(m-b)} X^+ \cdot D\varphi_{b+1}. \end{aligned} \tag{3.10}$$

We now show that the spinors $\varphi_b + \frac{1}{(m+1)\left(1+\frac{q}{p}\right)} D\varphi_b \in \Gamma(\Sigma_b M \oplus \Sigma_{b+1} M)$ and $\varphi_{b+1} + \frac{1}{(m+1)\left(1-\frac{q}{p}\right)} D\varphi_{b+1} \in \Gamma(\Sigma_{b+1} M \oplus \Sigma_b M)$ are Kählerian Killing spin^c spinors. Note that for $q = 0$ (corresponding to the spin case), it follows that $\varphi_b + \frac{1}{m+1} D\varphi_b, \varphi_{b+1} + \frac{1}{m+1} D\varphi_{b+1} \in \Gamma(\Sigma_b M \oplus \Sigma_{b+1} M)$ are eigenspinors of the Dirac operator corresponding to the smallest possible eigenvalue $m+1$, i.e., Kählerian Killing spinors. From (3.10) it follows

$$\nabla_X \varphi_b = -X^- \cdot \frac{1}{(m+1)\left(1+\frac{q}{p}\right)} D\varphi_b. \tag{3.11}$$

Applying (2.12) to φ_b in this case for $\mathrm{Ric} = \frac{S}{2m} g = 2(m+1)g$ and $F_A = \frac{q}{p}\frac{S}{2m} i\Omega = 2(m+1)\frac{q}{p} i\Omega$, we get

$$\nabla_X (D^+\varphi_b) = -(m+1)\left(1+\frac{q}{p}\right) X^+ \cdot \varphi_b. \tag{3.12}$$

According to the defining equation (2.14) of a Kählerian Killing spin^c spinor, equations (3.11) and (3.12) imply that the spinor $\varphi_b + \frac{1}{(m+1)\left(1+\frac{q}{p}\right)} D\varphi_b \in \Gamma(\Sigma_b M \oplus \Sigma_{b+1} M)$ is a Kählerian Killing spin^c spinor. A similar computation yields that $\varphi_{b+1} + \frac{1}{(m+1)\left(1-\frac{q}{p}\right)} D\varphi_{b+1}$ is a Kählerian Killing spin^c spinor. Conversely, if $\varphi_b + \varphi_{b+1} \in \Gamma(\Sigma_b M \oplus \Sigma_{b+1} M)$ is a Kählerian Killing spin^c spinor, then according to (2.16), φ_b and φ_{b+1} are eigenspinors of D^2 to the eigenvalue $4(m-b)(b+1) = \left(1-\frac{q^2}{p^2}\right)(m+1)^2$. This concludes the proof. ∎

Remark 3.7. If $q = 0$, which corresponds to the spin case, the assumption $p \geq |q| = 0$ is trivial and we recover from Theorem 3.5 and Theorem 1.1 Kirchberg's estimates on Kähler–Einstein spin manifolds: the lower bound (1.2) for m odd, namely $\lambda^2 \geq \frac{m+1}{4m} S = e\left(\frac{m+1}{2}\right)\frac{S}{2}$, and the lower bound (1.4) for m even, namely $\lambda^2 \geq \frac{m+2}{4m} S = e\left(\frac{m}{2}+1\right)\frac{S}{2}$. In the latter case, when m is even, the equality in (1.5) cannot be attained, as $b = \frac{m}{2} - \frac{1}{2} \notin \mathbb{N}$. Also for $r = \frac{m}{2}$ the inequality (3.9) is strict, since otherwise it would imply, according to the characterization of the equality case in Theorem 3.5, that the corresponding eigenspinor $\varphi \in \Sigma_{\frac{m}{2}} M$ is parallel, in contradiction to the positivity of the scalar curvature. Note that the same argument as in the proof of Theorem 1.1 shows that there cannot exist an eigenspinor $\varphi \in \Sigma_{\frac{m}{2}} M$ of D^2 to an eigenvalue strictly smaller than the lowest bound for $r = \frac{m}{2} \pm 1$, since otherwise $D^+\varphi$ and $D^-\varphi$ would either be eigenspinors or would vanish, leading in both cases to a contradiction. Hence, from the estimate (3.9) and the fact that the function e_1 decreases on $\left(0, \left(1+\frac{q}{p}\right)\frac{m}{2}\right)$ and e_2 increases on $\left(\left(1+\frac{q}{p}\right)\frac{m}{2}, m\right)$, it follows that the lowest possible bound for λ^2 in this case is given by $e_1\left(\frac{m}{2}-1\right)S = e_2\left(\frac{m}{2}+1\right)S = \frac{m+2}{4m}$. If $q = -p$ (resp. $q = p$), which corresponds to the canonical (resp. anti-canonical) spin^c structure, the lower bound in Theorem 1.1 equals 0 and is attained by the parallel spinors in $\Sigma_0 M$ (resp. $\Sigma_m M$), cf. [31].

4 Harmonic forms on limiting Kähler–Einstein manifolds

In this section we give an application for the eigenvalue estimate of the spin^c Dirac operator established in Theorem 1.1. Namely, we extend to spin^c spinors the result of A. Moroianu [29] stating that the Clifford multiplication between a harmonic effective form of nonzero degree and a Kählerian Killing spinor vanishes. As above, (M^{2m}, g) denotes a $2m$-dimensional Kähler–Einstein compact manifold of index p and normalized scalar curvature $4m(m+1)$, which carries the spin^c structure given by $\mathcal{L}^q$ with $q+p \in 2\mathbb{Z}$, where $\mathcal{L}^p = K_M$. We call M a *limiting manifold* if equality in (1.5) is achieved on M, which is by Theorem 1.1 equivalent to the existence of a Kählerian Killing spin^c spinor in $\Sigma_r M \oplus \Sigma_{r+1} M$ for $r = \frac{q}{p} \cdot \frac{m+1}{2} + \frac{m-1}{2} \in \mathbb{N}$. Let $\psi = \psi_{r-1} + \psi_r \in \Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$ be such a spinor, i.e., $\Omega \cdot \psi_{r-1} = i(2r-2-m)\psi_{r-1}$, $\Omega \cdot \psi_r = i(2r-m)\psi_r$ and the following equations are satisfied

$$\nabla_{X^+}\psi_r = -X^+ \cdot \psi_{r-1}, \qquad \nabla_{X^-}\psi_{r-1} = -X^- \cdot \psi_r.$$

By (2.15), we have

$$D\psi_r = 2(m-r+1)\psi_{r-1}, \qquad D\psi_{r-1} = 2r\psi_r.$$

Recall that a form ω on a Kähler manifold is called *effective* if $\Lambda\omega = 0$, where Λ is the adjoint of the operator $L\colon \Lambda^* M \longrightarrow \Lambda^{*+2}M$, $L(\omega) := \omega \wedge \Omega$. More precisely, Λ is given by the formula: $\Lambda = -2\sum_{j=1}^{2m} e_j^+ \lrcorner e_j^- \lrcorner$. Moreover, one can check that

$$(\Lambda L - L\Lambda)\omega = (m-t)\omega, \qquad \forall\, \omega \in \Lambda^t M.$$

Lemma 4.1. *Let $\psi = \psi_{r-1} + \psi_r \in \Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$ be a Kählerian Killing* spin^c *spinor and ω a harmonic effective form of type (k, k'). Then, we have*

$$D(\omega \cdot \psi_r) = 2(-1)^{k+k'}(m-r+1-k')\omega \cdot \psi_{r-1}, \tag{4.1}$$

$$D(\omega \cdot \psi_{r-1}) = 2(-1)^{k+k'}(r-k)\omega \cdot \psi_r. \tag{4.2}$$

Proof. The following general formula holds for any form ω of degree $\deg(\omega)$ and any spinor φ

$$D(\omega \cdot \varphi) = (d\omega + \delta\omega) \cdot \varphi + (-1)^{\deg(\omega)}\omega \cdot D\varphi - 2\sum_{j=1}^{2m}(e_j \lrcorner \omega) \cdot \nabla_{e_j}\varphi.$$

Applying this formula to an effective harmonic form ω of type (k, k') and to the components of the Kählerian Killing spin^c spinor ψ, we obtain

$$\begin{aligned} D(\omega \cdot \psi_r) &= (-1)^{k+k'}\omega \cdot D\psi_{r-1} - 2\sum_{j=1}^{2m}(e_j \lrcorner \omega) \cdot \nabla_{e_j}\psi_r \\ &= (-1)^{k+k'}2(m-r+1)\omega \cdot \psi_{r-1} + 2\sum_{j=1}^{2m}(e_j^- \lrcorner \omega) \cdot e_j^+ \cdot \psi_{r-1} \\ &= 2(-1)^{k+k'}\left[(m-r+1)\omega \cdot \psi_{r-1} + \left(\sum_{j=1}^{2m} e_j^+ \wedge (e_j^- \lrcorner \omega)\right) \cdot \psi_{r-1}\right]. \end{aligned}$$

Since ω is effective, we have for any spinor φ that

$$(e_j^- \lrcorner \omega) \cdot e_j^+ \cdot \varphi = (-1)^{k+k'-1}\big(e_j^+ \wedge (e_j^- \lrcorner \omega) + e_j^+ \lrcorner e_j^- \lrcorner \omega\big) \cdot \varphi.$$

Thus, we conclude $D(\omega \cdot \psi_r) = 2(-1)^{k+k'}(m-r+1-k')\omega \cdot \psi_{r-1}$. Analogously we obtain $D(\omega \cdot \psi_{r-1}) = 2(-1)^{k+k'}(r-k)\omega \cdot \psi_r$. ∎

Now, we are able to state the main result of this section, which extends the result of A. Moroianu mentioned in the introduction to the spin^c setting:

Theorem 4.2. *On a compact Kähler–Einstein limiting manifold, the Clifford multiplication of a harmonic effective form of nonzero degree with the corresponding Kählerian Killing* spin^c *spinor vanishes.*

Proof. Equations (4.1) and (4.2) imply that

$$D^2(\omega\cdot\psi)=4(r-k)(m-r+1-k')\omega\cdot\psi.$$

Note that for all values of $k,k'\in\{0,\dots,m\}$ and $r\in\{0,\dots,m+1\}$, either $4(r-k)(m-r+1-k')\leq 0$, or $4(r-k)(m-r+1-k')<4r(m-r+1)$, which for $r=b+1$ is exactly the lower bound obtained in Theorem 1.1 for the eigenvalues of D^2. This shows that $\omega\cdot\psi=0$. ■

Kähler–Einstein manifolds carrying a complex contact structure are examples of odd-dimensional Kähler manifolds with Kählerian Killing spin^c spinors in $\Sigma_{r-1}M\oplus\Sigma_rM$ for the spin^c structure (described in the introduction) whose auxiliary line bundle is given by $\mathcal{L}^q$ and $q=r-\ell-1$, where $m=2\ell+1$. Thus, the result of A. Moroianu is obtained as a special case of Theorem 4.2.

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