Symmetry, Integrability and Geometry: Methods and Applications SIGMA **13** (2017), 064, 6 pages

A Generalization of the Doubling Construction for Sums of Squares Identities

Chi ZHANG † *and Hua-Lin HUANG* ‡

† *School of Mathematics, Shandong University, Jinan 250100, China*
E-mail: *chizhang@mail.sdu.edu.cn*

‡ *School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China*
E-mail: *hualin.huang@hqu.edu.cn*

Received May 16, 2017, in final form August 13, 2017; Published online August 16, 2017
https://doi.org/10.3842/SIGMA.2017.064

Abstract. The doubling construction is a fast and important way to generate new solutions to the Hurwitz problem on sums of squares identities from any known ones. In this short note, we generalize the doubling construction and obtain from any given admissible triple $[r, s, n]$ a series of new ones $[r + \rho(2^{m-1}), 2^m s, 2^m n]$ for all positive integer m, where ρ is the Hurwitz–Radon function.

Key words: Hurwitz problem; square identity

2010 Mathematics Subject Classification: 11E25

1 Introduction

In his seminal paper [2], Hurwitz addressed the famous problem: Determine all the sums of squares identities

$$\left(x_1^2 + x_2^2 + \cdots + x_r^2\right)\left(y_1^2 + y_2^2 + \cdots + y_s^2\right) = z_1^2 + z_2^2 + \cdots + z_n^2, \tag{1.1}$$

where $X = (x_1, x_2, \ldots, x_r)$ and $Y = (y_1, y_2, \ldots, y_s)$ are systems of indeterminants and every z_k is a bilinear form of X and Y with coefficients in some given field. If there does exist such an identity, we call $[r, s, n]$ an *admissible triple*. This problem of Hurwitz has close connections to various topics in algebra, arithmetic, combinatorics, geometry, topology, etc. Many mathematicians have studied this Hurwitz problem during the past century. See [7] for an overview.

Though a complete solution to the Hurwitz problem is still far out of reach at present, many admissible triples have been obtained in the literature. In particular, the admissible triples of form $[r, n, n]$ was settled independently by Hurwitz in [3] and by Radon in [6]. The celebrated Hurwitz–Radon theorem states that $[r, n, n]$ is admissible if and only if $r \leq \rho(n)$ where ρ is the Hurwitz–Radon function defined by $\rho(n) = 8\alpha + 2^\beta$ if $n = 2^{4\alpha+\beta}(2\gamma + 1)$ with $0 \leq \beta \leq 3$. In the early 1980s, Yuzvinsky introduced the novel idea of orthogonal pairings [8] and proposed in [9] the following three families of admissible triples in the neighborhood of the Hurwitz–Radon triples

$$\left[2n + 2, 2^n - \varphi(n), 2^n\right], \qquad \text{where} \quad \varphi(n) = \begin{cases} \binom{n}{n/2}, & n \equiv 0 \mod 4, \\ 2\binom{n-1}{(n-1)/2}, & n \equiv 1 \mod 4, \\ 4\binom{n-2}{(n-2)/2}, & n \equiv 2 \mod 4. \end{cases} \tag{1.2}$$

The first two families are confirmed in [4] and the third in [1]. Moreover, some new families of admissible triples are constructed in [1, 5].

Another natural idea of constructing admissible triples is to find some general procedures to generate new ones from known ones. Among which, the doubling construction, i.e., generating an admissible triple $[r+1, 2s, 2n]$ from any given triple $[r, s, n]$, is a fast and important program. In the present note we consider arbitrarily iterated doubling constructions and the aim is to optimize the obvious triples $[r+m, 2^m s, 2^m n]$. The well-known Hurwitz–Radon triples suggest that the first item might be properly increased as the form given below via the function ρ.

Main Theorem. *If $[r, s, n]$ is admissible, then so is $[r+\rho(2^{m-1}), 2^m s, 2^m n]$ for all positive integer m.*

Though our observation arises from the idea used in [1], it turns out that a more elementary approach of matrices will suffice for a proof.

2 Proof of the main theorem

First, we introduce the so-called admissible matrices to reformulate the Hurwitz problem. Then, as a trial we provide a proof via admissible matrices for the classical doubling construction. Finally, we extend the idea to iterated doubling constructions and complete the proof for the main theorem.

2.1 Admissible matrices

The notion of admissible matrices arises naturally from an attempt to reformulate the sums of squares identity (1.1) by a system of polynomial equations. Indeed, in (1.1) if we write

$$z_k = \sum_{\substack{1\le i\le r\\1\le j\le s}} c_{i,j,k} x_i y_j,$$

then it is easy to see that the identity (1.1) is equivalent to the following system of algebraic equations

$$\begin{aligned}
&\sum_{k=1}^{n} c_{i,j,k}^2 = 1, && 1\le i\le r, \quad 1\le j\le s,\\
&\sum_{k=1}^{n} c_{i_1,j,k}c_{i_2,j,k} = 0, && 1\le i_1< i_2\le r, \quad 1\le j\le s,\\
&\sum_{k=1}^{n} c_{i,j_1,k}c_{i,j_2,k} = 0, && 1\le i\le r, \quad 1\le j_1< j_2\le s,\\
&\sum_{k=1}^{n} (c_{i_1,j_1,k}c_{i_2,j_2,k} + c_{i_1,j_2,k}c_{i_2,j_1,k}) = 0, && 1\le i_1< i_2\le r, \quad 1\le j_1< j_2\le s.
\end{aligned} \tag{2.1}$$

In the rest of the note, we always regard the resulting cuboid $A := (c_{i,j,k})_{r\times s\times n}$ as an $r\times s$ matrix with (i,j)-entry the n-dimensional vector $A_{i,j} := (c_{i,j,1}, c_{i,j,2}, \dots, c_{i,j,n})$. Taking the formal inner product on n-dimensional vectors, namely $\langle (u_1, u_2, \dots, u_n), (v_1, v_2, \dots, v_n)\rangle := u_1v_1 + u_2v_2 + \cdots + u_nv_n$, then (2.1) can be rewritten as

$$\begin{aligned}
&(1)\ \ \langle A_{i,j}, A_{i,j}\rangle = 1, && 1\le i\le r, \quad 1\le j\le s,\\
&(2)\ \ \langle A_{i_1,j}, A_{i_2,j}\rangle = 0, && 1\le i_1< i_2\le r, \quad 1\le j\le s,\\
&(3)\ \ \langle A_{i,j_1}, A_{i,j_2}\rangle = 0, && 1\le i\le r, \quad 1\le j_1< j_2\le s,\\
&(4)\ \ \langle A_{i_1,j_1}, A_{i_2,j_2}\rangle + \langle A_{i_1,j_2}, A_{i_2,j_1}\rangle = 0, && 1\le i_1< i_2\le r, \quad 1\le j_1< j_2\le s.
\end{aligned} \tag{2.2}$$

Obviously, the existence of such a matrix A is equivalent to the existence of an admissible triple of size $[r, s, n]$. In keeping the terminologies coherent, such matrices are said to be *admissible*.

2.2 The doubling construction revisited

For a better explanation of our method, firstly we provide a proof by admissible matrices for the classical doubling construction. Some preparing definitions and notations are necessary. Let $\Bbbk$ be a field of characteristic not 2.

Definition 2.1. Fix two integers n and m. A vector in $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{mn}) \in \Bbbk^{mn}$ is said to be in *level* $k \in \{1, 2, \ldots, m\}$ if its arguments α_l are 0 unless $(k-1)n + 1 \leq l \leq kn$. Let $\beta \in \Bbbk^n$. We call $\gamma \in \Bbbk^{mn}$ a *positive* copy of β in level k if $\gamma_{(k-1)n+i} = \beta_i$ $(1 \leq i \leq n)$ and other arguments of γ are 0. Similarly, we call γ a *negative* copy of β in level k if $\gamma_{(k-1)n+i} = -\beta_i$ $(1 \leq i \leq n)$ and other arguments of γ are 0.

Remark 2.2. Let $\alpha_1, \alpha_2 \in \Bbbk^{mn}$ and $\beta_1, \beta_2 \in \Bbbk^n$.

1. If α_1 is a copy of β_1 in level k, α_2 is a copy of β_2 in level l and they have the same sign, then $\langle \alpha_1, \alpha_2 \rangle = \langle \beta_1, \beta_2 \rangle$ if $k = l$, $\langle \alpha_1, \alpha_2 \rangle = 0$ if $k \neq l$.
2. If α_1 is a copy of β_1 in level k, α_2 is a copy of β_2 in level l and they have different signs, then $\langle \alpha_1, \alpha_2 \rangle = -\langle \beta_1, \beta_2 \rangle$ if $k = l$, $\langle \alpha_1, \alpha_2 \rangle = 0$ if $k \neq l$.

Given an admissible triple of size $[r, s, n]$, we have a corresponding $r \times s$ admissible matrix A. We shall construct an $(r+1) \times 2s$ admissible matrix B whose entries are $2n$-dimensional vectors as follows:

1) $B_{i,j}$ is a positive copy of $A_{i,j}$ for $1 \leq i \leq r$, $1 \leq j \leq s$ in level 1,

2) $B_{i,s+j}$ is a positive copy of $A_{i,j}$ for $2 \leq i \leq r$, $1 \leq j \leq s$ in level 2,

3) $B_{r+1,j}$ is a positive copy of $A_{1,j}$ for $1 \leq j \leq s$ in level 2,

4) $B_{r+1,s+j}$ is a positive copy of $A_{1,j}$ for $1 \leq j \leq s$ in level 1,

5) $B_{1,s+j}$ is a negative copy of $A_{1,j}$ for $1 \leq j \leq s$ in level 2.

We give a detailed verification of the admissibility of B and hope this will shed some light on the study of iterated doubling constructions.

1. Every $B_{i,j}$ is a copy of some entry $A_{k,l}$ of A , so $\langle B_{i,j}, B_{i,j} \rangle = \langle A_{k,l}, A_{k,l} \rangle = 1$, hence (1) of (2.2) holds.
2. $\langle B_{i,j_1}, B_{i,j_2} \rangle = 0$ $(j_1 < j_2)$. Indeed, if $1 \leq j_1 \leq s$ and $s + 1 \leq j_2 \leq 2s$, then the two vectors are in different levels; if $1 \leq i \leq r + 1$, $1 \leq j_1 < j_2 \leq s$, $\langle B_{i,j_1}, B_{i,j_2} \rangle = \langle A_{i,j_1}, A_{i,j_2} \rangle = 0$. A similar argument works for $s+1 \leq j_1 < j_2 \leq 2s$. So (2) of (2.2) holds. In the same way, one can show that (3) of (2.2) holds.
3. For (4) of (2.2), we need to verify $\langle B_{i_1,j_1}, B_{i_2,j_2} \rangle + \langle B_{i_1,j_2}, B_{i_2,j_1} \rangle = 0$ $(i_1 < i_2,\ j_1 < j_2)$.

   (a) If $1 \leq j_1 \leq s < j_2 \leq 2s$ and $1 \leq i_1 < i_2 \leq r$, B_{i_1,j_1} and B_{i_2,j_1} are in level 1 and B_{i_1,j_2} and B_{i_2,j_2} are in level 2. Hence the equation is obvious.

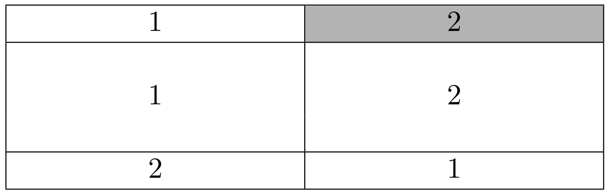
   (b) If $1 \leq j_1 \leq s$, $s + 1 \leq j_2 \leq 2s$ and $1 \leq i_1 \leq r$, $i_2 = r + 1$, B_{i_1,j_1} and B_{i_2,j_2} are in level 1 and B_{i_1,j_2} and B_{i_2,j_1} are in level 2. $\langle B_{i_1,j_1}, B_{i_2,j_2} \rangle + \langle B_{i_1,j_2}, B_{i_2,j_1} \rangle = \langle A_{i_1,j_1}, A_{1,j_2} \rangle + \langle A_{i_1,j_2}, A_{1,j_1} \rangle = 0$. If $i_1 = 1$ and $j_2 = j_1 + s$, then $\langle B_{i_1,j_1}, B_{i_2,j_2} \rangle + \langle B_{i_1,j_2}, B_{i_2,j_1} \rangle = \langle A_{1,j_1}, A_{1,j_1} \rangle - \langle A_{1,j_1}, A_{1,j_1} \rangle = 1 - 1 = 0$.

   (c) If $1 \leq i_1 < i_2 \leq r$, $1 \leq j_1 < j_2 \leq s$ or $s + 1 \leq j_1 < j_2 \leq 2s$, the four vectors are in the same level. If $i_1 = 1$, $s + 1 \leq j_1 \leq j_2 \leq 2s$, then $\langle B_{i_1,j_1}, B_{i_2,j_2} \rangle + \langle B_{i_1,j_2}, B_{i_2,j_1} \rangle = -\langle A_{i_1,j_1}, A_{i_2,j_2} \rangle - \langle A_{i_1,j_2}, A_{i_2,j_1} \rangle = 0$. Otherwise, $\langle B_{i_1,j_1}, B_{i_2,j_2} \rangle + \langle B_{i_1,j_2}, B_{i_2,j_1} \rangle = \langle A_{i_1,j_1}, A_{i_2,j_2} \rangle + \langle A_{i_1,j_2}, A_{i_2,j_1} \rangle = 0$.

(d) If $1 \leq i_1 \leq r$, $i_2 = r+1$, $1 \leq j_1 < j_2 \leq s$ or $s+1 \leq j_1 < j_2 \leq 2s$, then B_{i_1,j_1} and B_{i_2,j_2} are in different levels and this is also the case for B_{i_1,j_2} and B_{i_2,j_1}. Now the equation is clear.

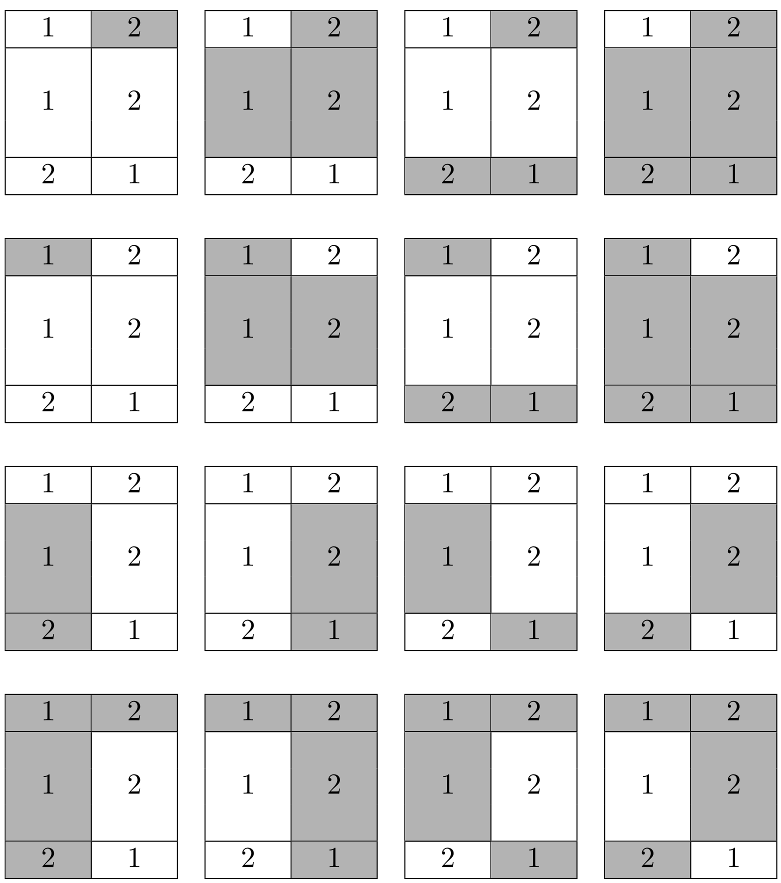
Thus, (4) of (2.2) holds.

Remark 2.3. The matrix B used in the above proof can be illustrated by the following table:



Here, cells in the first and the third rows are copies of the first row of A, and cells in the second rows are copies of the submatrix of A obtained by deleting the first row. The number given in the center of a cell represents the level of the vectors therein. The sign of a cell is indicated by its color: white means positive, gray means negative.

Above all, the table provides a visual admissibility of B. The conditions (1)–(3) of (2.2) are immediate, as the vectors are either in different levels, or essentially can be considered within A. The same reasoning also works for (4) of (2.2) in most cases. As for the case $i_1 = 1$, $i_2 = r+1$, $j_1 + s = j_2$, one further needs to take the signs into consideration. In fact, this also tells us in the very beginning how to manipulate the signs of the copies of cells so that (4) holds. Of course, the signing is far from unique. Just for such B, we have 16 kinds of correct schemes as follows. These tables are useful in the following for the verification of the admissibility of bigger matrices.



2.3 Iterated doubling constructions

Now we are ready to prove the main theorem. As before, let A be an admissible matrix corresponding to an admissible triple of size $[r, s, n]$. We will provide admissible matrices in terms of tables as in Remark 2.3 which will induce admissible triples of sizes $[r+2, 4s, 4n]$, $[r+4, 8s, 8n]$ and $[r+8, 16s, 16n]$.

1	2	3	4
1	2	3	4
2	1	4	3
3	4	1	2

the table of $[r+2, 4s, 4n]$

1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
5	6	7	8	1	2	3	4
8	7	6	5	4	3	2	1

the table of $[r+4, 8s, 8n]$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
5	6	7	8	1	2	3	4	13	14	15	16	9	10	11	12
8	7	6	5	4	3	2	1	16	15	14	13	12	11	10	9
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
12	11	10	9	16	15	14	13	4	3	2	1	8	7	6	5
14	13	16	15	10	9	12	11	6	5	8	7	2	1	4	3
15	16	13	14	11	12	9	10	7	8	5	6	3	4	1	2

the table of $[r+8, 16s, 16n]$

As the table of $[r+4, 8s, 8n]$ and the table of $[r+2, 4s, 4n]$ are both subtables of that of $[r+8, 16s, 16n]$, we just explain the last table:

1. Every cell in the first row of a table which stands for the first row of the corresponding admissible matrix is the copy of the first row of A.
2. Every cell in the second row of a table which stands for the rows from second to r-th of the corresponding admissible matrix is the copy of the rows from second to r-th of A.
3. Every cell in other rows of a table which stand for the k-th rows ($k \geq r+1$) of the corresponding admissible matrix is the copy of the first row of A.
4. For every cell, the number means the levels and the color means the signs.

Using the same discussion of Remark 2.3, it is easy to verify that (1)–(3) of (2.2) hold. For the verification of (4) of (2.2), it is enough to consider the 8 added rows and verify those entries which are in the same level. Then the admissibility follows by a direct and simple computation.

Acknowledgements

This research was supported by NSFC 11471186 and NSFC 11571199.

References

[1] Hu Y.-Q., Huang H.-L., Zhang C., $\mathbb{Z}_2^n$-graded quasialgebras and the Hurwitz problem on compositions of quadratic forms, *Trans. Amer. Math. Soc.*, to appear, arXiv:1506.01502.

[2] Hurwitz A., Über die Composition der quadratischen Formen von beliebig vielen Variabeln, *Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl.* (1898), 309–316.

[3] Hurwitz A., Über die Komposition der quadratischen Formen, *Math. Ann.* **88** (1922), 1–25.

[4] Lam T.Y., Smith T.L., On Yuzvinsky's monomial pairings, *Quart. J. Math. Oxford Ser. (2)* **44** (1993), 215–237.

[5] Lenzhen A., Morier-Genoud S., Ovsienko V., New solutions to the Hurwitz problem on square identities, *J. Pure Appl. Algebra* **215** (2011), 2903–2911, arXiv:1007.2337.

[6] Radon J., Lineare Scharen orthogonaler Matrizen, *Abh. Math. Sem. Univ. Hamburg* **1** (1922), 1–14.

[7] Shapiro D.B., Compositions of quadratic forms, *De Gruyter Expositions in Mathematics*, Vol. 33, Walter de Gruyter & Co., Berlin, 2000.

[8] Yuzvinsky S., Orthogonal pairings of Euclidean spaces, *Michigan Math. J.* **28** (1981), 131–145.

[9] Yuzvinsky S., A series of monomial pairings, *Linear and Multilinear Algebra* **15** (1984), 109–119.