Walls for G-Hilb via Reid's Recipe

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Abstract. The three-dimensional McKay correspondence seeks to relate the geometry of crepant resolutions of Gorenstein 3-fold quotient singularities $\mathbb{A}^3/G$ with the representation theory of the group G. The first crepant resolution studied in depth was the G-Hilbert scheme G-Hilb $\mathbb{A}^3$, which is also a moduli space of θ-stable representations of the McKay quiver associated to G. As the stability parameter θ varies, we obtain many other crepant resolutions. In this paper we focus on the case where G is abelian, and compute explicit inequalities for the chamber of the stability space defining G-Hilb $\mathbb{A}^3$ in terms of a marking of exceptional subvarieties of G-Hilb $\mathbb{A}^3$ called Reid's recipe. We further show which of these inequalities define walls. This procedure depends only on the combinatorics of the exceptional fibre and has applications to the birational geometry of other crepant resolutions.

Key words: wall-crossing; McKay correspondence; Reid's recipe; quivers

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1 Introduction

Let $G \subset \mathrm{SL}_n(\mathbb{C})$ be a finite subgroup. When $n = 2$ there is a famous ADE classification of such subgroups that matches the classification of Du Val or modality zero singularities by taking a subgroup G to the quotient singularity $0 \in \mathbb{A}^2/G$. This observation and the surrounding deep interactions of the geometry of $\mathbb{A}^2/G$ and its resolutions, and the representation theory of G are known as the two-dimensional McKay correspondence [1, 2, 13, 16, 17, 21, 23]. In this case, the unique minimal or crepant resolution has a modular interpretation as the G-Hilbert scheme G-Hilb $\mathbb{A}^2$. The moduli space G-Hilb M for M a variety and $G \subset \mathrm{Aut}(M)$ a finite subgroup parameterises G-clusters in M: zero-dimensional G-invariant subschemes M of $\mathbb{A}^2$ with $H^0(\mathcal{O}_Z) \cong \mathbb{C}[G]$ as G-modules. This was generalised to three dimensions for finite abelian subgroups of $\mathrm{SL}_3(\mathbb{C})$ by Nakamura [22] who showed that G-Hilb $\mathbb{A}^3$ is a crepant resolution of $\mathbb{A}^3/G$ and then to all subgroups G by the celebrated work of Bridgeland–King–Reid [5]. They moreover established an equivalence of categories

$$D^b\big(G\text{-Hilb }\mathbb{A}^3\big) \simeq D_G^b\big(\mathbb{A}^3\big), \tag{1.1}$$

which also holds if G-Hilb $\mathbb{A}^3$ is replaced by any projective crepant resolution of $\mathbb{A}^3/G$. Compare also the results of Bridgeland [4].

Using the GIT approach of King [19] to constructing moduli of quiver representations the G-Hilbert scheme can also be realised as a moduli space of θ-stable quiver representations $\mathcal{M}_\theta(Q, \underline{d})$, where Q is the McKay quiver of G and $\underline{d} = (d_i)$ is a given dimension vector. In this situation the stability parameter θ lives in the stability space

$$\Theta := \left\{ \theta \in \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}^{Q_0}, \mathbb{Q}\big) \colon \sum_{i \in Q_0} d_i \theta(i) = 0 \right\},$$

where Q_0 is the set of vertices of Q. By definition, the vertices of the McKay quiver biject with the irreducible representations $\mathrm{Irr}(G)$ of G and so one can view $\mathbb{Z}^{Q_0}$ as the abelian group underlying the representation ring of G. As θ varies, it is possible that one obtains many different crepant resolutions of $\mathbb{A}^3/G$; in the case that G is abelian, Craw–Ishii [10] show that all projective crepant resolutions arise in this way. The stability space Θ has a wall-and-chamber structure such that the moduli space $\mathcal{M}_\theta(Q, \underline{d})$ is constant so long as θ remains inside a given chamber. We denote the moduli space $\mathcal{M}_{\mathfrak{C}} := \mathcal{M}_\theta(Q, \underline{d})$ for any generic θ in a chamber $\mathfrak{C}$. Denote the chamber corresponding to G-Hilb $\mathbb{A}^3$ by $\mathfrak{C}_0$. The positive orthant

$$\Theta^+ := \big\{ \theta \in \Theta \colon \theta(\rho) > 0 \text{ for all nontrivial } \rho \in \mathrm{Irr}(G) \big\}$$

lies inside $\mathfrak{C}_0$ however it is not usually equal to it. The primary purpose of this paper is to provide explicit combinatorial inequalities defining $\mathfrak{C}_0$ and identify precisely which of these define walls of $\mathfrak{C}_0$. We remark that such equations were computed for a group of order 11 in [10, Example 9.6].

Assume that G is abelian. In this context [10, Theorem 9.5] gives an abstract description of such inequalities, however it is difficult to perform explicit calculations or deduce general statements from their presentation. One can view some of the results herein as a combinatorialisation of [10, Theorem 9.5], which turn out to be very amenable to applications.

We briefly outline the context and notation of [10] that we will also use. For a chamber $\mathfrak{C} \subset \Theta$ the equivalence from (1.1) induces an isomorphism $\varphi_{\mathfrak{C}} \colon K_0(\mathcal{M}_{\mathfrak{C}}) \to K_G(\mathbb{A}^3) = \mathrm{Rep}(G)$. Here $K_0(\mathcal{M}_{\mathfrak{C}})$ denotes the K-group of sheaves supported on the preimage of the G-orbit for the origin under the resolution $\mathcal{M}_{\mathfrak{C}} \to \mathbb{A}^3/G$. Walls in Θ are cut out by hyperplanes $\big(\sum_i \alpha_i \cdot \theta(\chi_i) = 0 \big)$ for some characters $\chi_i \in \mathrm{Irr}(G)$ and integers $\alpha_i \in \mathbb{Z}$, though in general not all θ on such a hyperplane will be non-generic. The inequalities in [10] have three different forms, each coming from exceptional subvarieties. Firstly, each exceptional curve $C \subset G$-Hilb $\mathbb{A}^3$ gives an inequality of the form

$$\theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)\big) > 0.$$

The characters appearing in these inequalities are packaged in collections of monomials associated to exceptional curves that were named by Nakamura [22] in a different context as G-*igsaw pieces*. Our first result is to pin down which characters lie in G-igsaw pieces. In general there are several G-igsaw pieces corresponding to a single curve C; the union of all the pieces that do not include the trivial character is the set of characters that appear in the inequality for C. We call this union the *total G-igsaw piece*.

As G is abelian the singularity $\mathbb{A}^3/G$ and its crepant resolutions are toric. There is a method of marking the exceptional subvarieties of G-Hilb – the edges and vertices in the triangulation – by characters of G known as "Reid's recipe". This was used to explicitly describe the McKay correspondence in the classical terms of providing a basis of $H^*\big(G\text{-Hilb}\,\mathbb{A}^3, \mathbb{Z}\big)$ indexed by characters by Craw [9]. It was later categorified by Logvinenko [20], Cautis–Logvinenko [7], and Cautis–Craw–Logvinenko [6], who expressed the locus labelled by a character χ in terms of the support of an object associated to χ in the derived category of G-Hilb. We will discuss this in more detail in Sections 2.2 and 2.4.

Theorem 1.1 (Algorithm 3.3)**.** *There is a combinatorial procedure that we call the unlocking procedure for computing the characters appearing in the total G-igsaw piece for an exceptional curve in G-Hilb $\mathbb{A}^3$. The input of this procedure is the data of Reid's recipe and the combinatorics of the exceptional fibre.*

To briefly illustrate how the procedure works, we consider the example of $G = \frac{1}{30}(25, 2, 3)$. This notation means that G is the subgroup of $\mathrm{SL}_3(\mathbb{C})$ generated by

$$g = \begin{pmatrix} \varepsilon^{25} & & \\ & \varepsilon^2 & \\ & & \varepsilon^3 \end{pmatrix},$$

where ε is a primitive 30th root of unity. Crepant resolutions correspond to triangulations of the simplex at height 1 – the "junior simplex" – with vertices in the lattice $\mathbb{Z}^3 + \mathbb{Z} \cdot \left(\frac{25}{30}, \frac{2}{30}, \frac{3}{30}\right)$. The triangulation for G-Hilb is shown in Fig. 1 along with Reid's recipe. We often denote the character χ_a defined by $\chi_a(g) = \varepsilon^a$ by the integer a.
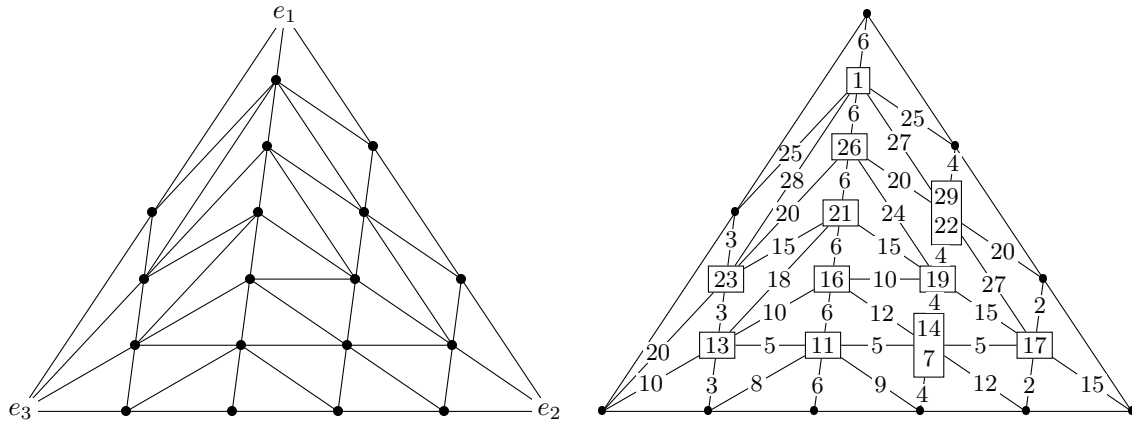

Figure 1. G-Hilb and Reid's recipe for $G = \frac{1}{30}(25, 2, 3)$.

Throughout this paper we use the convention of ordering vertices as shown in Fig. 1. We will demonstrate the unlocking procedure for the curve C shown on the left side of Fig. 2 marked with the character 5; that is, the character taking $g \mapsto \varepsilon^5$. On the right side of Fig. 2 we illustrate the unlocking procedure. Roughly, we consider all the curves (or edges) marked with 5, add one character marking each divisor containing two such curves (or vertices between two edges marked with 5), and finally add the characters appearing in G-igsaw pieces for certain curves cohabiting a divisor with a curve marked with 5. In this case, the only such extra curve is marked with 9 and the G-igsaw piece for this curve consists just of the character 9 itself. Some recursion will be necessary in general to compute the smaller G-igsaw pieces of such curves. It follows that the total G-igsaw piece for C has characters 5, 7, 9, 11. Observe that this total G-igsaw piece only picked one of the two characters 7, 14 marking a divisor containing two 5-curves. We will elaborate in Section 3.1 how the unlocking procedure identifies which of the two characters should be added.

Following [25], walls inside Θ are of various types denoted 0-III depending on the birational geometry of the moduli spaces near the wall. Walls of Type I correspond to flops induced by curves. By [10, Theorem 9.12], every flop in a single exceptional $(-1, -1)$-curve can be realised by a wall-crossing of Type I directly from $\mathfrak{C}_0$, which is very much not true for other resolutions; see [10, Example 9.13]. Walls of Type III – that contract a divisor to a curve – arise from exceptional $(-2, 0)$-curves corresponding to certain "boundary" edges in the triangulation for G-Hilb. We will define this terminology in Section 2.2. The final possibility is that an exceptional curve is a $(1, -3)$-curve that, if the corresponding inequality was irredundant, would produce a wall of Type II where by definition a divisor is contracted to a point. [10, Proposition 3.8] shows that there are in fact no Type II walls in Θ.

We denote the set of characters appearing in the total G-igsaw piece for an exceptional curve C by G-ig(C). The following result is implied by [10, Corollary 5.2, Proposition 9.7 and Theorem 9.12] and Theorem 1.1.

Proposition 1.2 (Propositions 4.1 and 4.2). *Suppose $C \subset G$-Hilb $\mathbb{A}^3$ is an exceptional curve marked with character χ by Reid's recipe. The inequality corresponding to C is given by*

$$\theta(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)) = \sum_{\chi \in G\text{-ig}(C)} \deg(\mathcal{R}_\chi|_C)\theta(\chi) > 0.$$
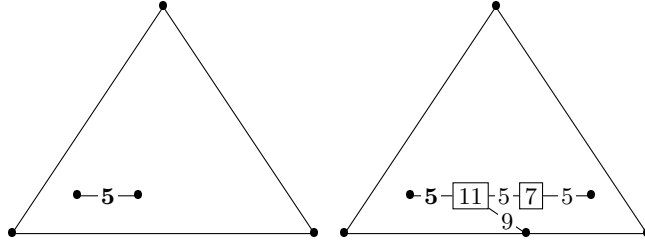
Figure 2. Unlocking for a 5-curve.

If C is a $(-1,-1)$-curve then the necessary inequality corresponding to C that defines a Type I wall of $\mathfrak{C}_0$ is given by

$$\theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)\big) = \sum_{\chi \in G\text{-ig}(C)} \theta(\chi) > 0.$$

If C is a $(1,-3)$-curve then the inequality corresponding to C is given by

$$\theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)\big) = 2 \cdot \theta\big(\chi^{\otimes 2}\big) + \sum_{\chi \in G\text{-ig}(C)\setminus\{\chi^{\otimes 2}\}} \theta(\chi) = 0.$$

In all three cases $G\text{-ig}(C)$ is computed by the unlocking procedure.

We can also use the unlocking procedure to compute inequalities that do not come from exceptional curves. The other two kinds of inequality come from exceptional divisors. For each character ψ marking a divisor, we obtain an inequality $\theta(\psi) > 0$. The second kind of inequality coming from divisors is more complicated.

Proposition 1.3 (Proposition 2.6)**.** *Suppose D' is a (not necessarily prime) exceptional divisor in G-Hilb $\mathbb{A}^3$. Then any $\theta \in \mathfrak{C}_0$ satisfies*

$$\theta\big(\varphi_{\mathfrak{C}_0}(\omega_{D'}^{\vee})\big) = \sum_{C \subset D'} \sum_{\chi \in G\text{-ig}(C)} \theta(\chi) > 0,$$

where C ranges over exceptional curves inside D'.

As a result of Propositions 1.2 and 1.3 we can immediately deduce the conclusion [10, Proposition 3.8] for $\mathfrak{C}_0$.

Corollary 1.4 (Corollary 4.3)**.** $\mathfrak{C}_0$ *has no Type II walls.*

We can similarly reprove [10, Theorem 9.12] by combinatorial means.

Proposition 1.5 (Proposition 4.4)**.** *Each flop in a $(-1,-1)$-curve in G-Hilb $\mathbb{A}^3$ is induced by a wall-crossing from $\mathfrak{C}_0$.*

We can use these formulae to show exactly which inequalities are necessary to define $\mathfrak{C}_0$.

Theorem 1.6 (Theorem 4.17)**.** *Suppose $G \subset \mathrm{SL}_3(\mathbb{C})$ is a finite abelian subgroup. The walls of the chamber $\mathfrak{C}_0$ for G-Hilb $\mathbb{A}^3$ and their types are as follows:*

- *a Type I wall for each exceptional $(-1,-1)$-curve,*
- *a Type III wall for each generalised long side,*
- *a Type 0 wall for each irreducible exceptional divisor,*

- *the remaining walls are of Type 0 coming from divisors as in Proposition* 1.3. *We discuss which of these are necessary and how to reconstruct the divisor* D' *in Section* 4.7.

We will define the term "generalised long side" in Definition 4.12, which is an entirely combinatorial notion. We complete the description of wall-crossing behaviours from [25] by calling a wall Type 0 if the corresponding contraction is a isomorphism.

This explicit and malleable description of the walls for $\mathfrak{C}_0$ has applications to studying the birational geometry of other crepant resolutions of $\mathbb{A}^3/G$. In forthcoming work [18], the author and Y. Ito use this description of $\mathfrak{C}_0$ to study the geometry of another Hilbert scheme-like resolution introduced in [14] called the "iterated G-Hilbert scheme" or "Hilb of Hilb".

2   Resolutions of $\mathbb{A}^3/G$

2.1   Setup

Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite abelian subgroup. We will assume that G is cyclic for notational simplicity during this preliminary section however we will shed this assumption from Section 3 onwards. We will denote by $\frac{1}{r}(a, b, c)$ the cyclic subgroup of $\mathrm{SL}_3(\mathbb{C})$ generated by the matrix

$$g = \begin{pmatrix} \varepsilon^a & & \\ & \varepsilon^b & \\ & & \varepsilon^c \end{pmatrix},$$

where ε is a primitive rth root of unity. The first resolution of the quotient singularity $\mathbb{A}^3/G$ to be studied was the G-Hilbert scheme G-Hilb $\mathbb{A}^3$, the fine moduli space of G-clusters: zero-dimensional G-invariant subschemes $Z \subset \mathbb{A}^3$ with $H^0(\mathcal{O}_Z) \cong \mathbb{C}[G]$ as G-modules. G-Hilb was shown to be smooth for abelian G by Nakamura [22], and subsequently shown to be smooth – and hence a resolution – for all finite $G \subset \mathrm{SL}_3(\mathbb{C})$ by Bridgeland–King–Reid [5].

From the work of King [19], Ito–Nakajima [16], and Craw [8] one can reinterpret G-Hilb as a moduli space of quiver representations. The quiver in question is the *McKay quiver* with vertices indexed by irreducible representations of G and the number of arrows between ρ and ρ' defined to be

$$\dim \mathrm{Hom}_G(\rho' \otimes \rho_{\mathrm{std}}, \rho),$$

where ρ_{std} is the standard representation of G acting on $\mathbb{C}^3$. We choose the dimension vector $\underline{d} = (\dim \rho)_\rho$ and a stability parameter $\theta \in \Theta$ as defined above. We define $\mathcal{M}_\theta := \mathcal{M}_\theta(Q, \underline{d})$ to be the fine moduli space of θ-stable representations of the McKay quiver with dimension vector $\underline{d}$ subject to certain relations coming from the associated preprojective algebra. When $\theta(\rho) > 0$ for all nontrivial ρ one has that $\mathcal{M}_\theta(Q, \underline{d}) = G$-Hilb $\mathbb{A}^3$.

It was apparent from [5] that their smoothness result and equivalence of categories (1.1) holds for any generic θ and so one obtains potentially many different resolutions of the form $\mathcal{M}_\theta(Q, \underline{d})$ and also the corresponding equivalences of categories

$$\Phi_\theta \colon \ D^b(\mathcal{M}_\theta) \to D^b_G(\mathbb{A}^3).$$

As discussed above, the stability space Θ has a wall-and-chamber structure such that any θ, θ' from the same open chamber $\mathfrak{C} \subset \Theta$ produce isomorphic moduli spaces: $\mathcal{M}_\theta \cong \mathcal{M}_{\theta'}$. For simplicity we denote by $\mathcal{M}_\mathfrak{C}$ and $\Phi_\mathfrak{C}$ the moduli space and equivalence of categories for any generic $\theta \in \mathfrak{C}$.

When G is abelian, each resolution $\mathcal{M}_\mathfrak{C}$ is toric. Fix the lattice $N = \mathbb{Z}^3 + \mathbb{Z} \cdot \left(\frac{a}{r}, \frac{b}{r}, \frac{c}{r} \right)$. The singularity $\mathbb{A}^3/G$ is described by the cone $\sigma = \mathrm{Cone}(e_1, e_2, e_3)$ inside $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R} = \mathbb{R}^3_{\langle x_1, x_2, x_3 \rangle}$

and crepant resolutions correspond to triangulations of the slice $\sigma \cap (x_1 + x_2 + x_3 = 1)$ – usually referred to as the "junior simplex" – such that the vertices of each triangle lies in N, and each triangle is smooth (its vertices form a $\mathbb{Z}$-basis of N). In pictures we will always draw only the slice to produce two-dimensional pictures.

2.2 Reid's recipe

Let us focus on the case $\mathcal{M}_{\mathfrak{C}} = G$-Hilb $\mathbb{A}^3$. We will denote the universal G-cluster by $\mathcal{Z}$ and the chamber of Θ corresponding to G-Hilb by $\mathfrak{C}_0$. Craw–Reid [12] present an entertaining algorithm to construct the triangulation for G-Hilb that, after commenting on some of the salient features, we will use without comment.

We call the triangulation for G-Hilb the *Craw–Reid triangulation*. It divides the junior simplex into so-called "regular triangles" of equal side length that fall into one of two cases:

- *corner triangles*, which have one of the vertices e_1, e_2, e_3 of the junior simplex as a vertex,
- *meeting of champions*, for which none of the vertices of the junior simplex are vertices.

Craw–Reid show that there is at most one meeting of champions triangle (possibly of side length zero, in which case it is a point). After dividing the junior simplex into such triangles, one subdivides them further into smooth triangles as depicted in Fig. 3: the resulting unimodal triangulation describes the resolution G-Hilb. We call line segments on the boundary of a regular triangle *boundary edges* and the corresponding toric curves *boundary curves*.
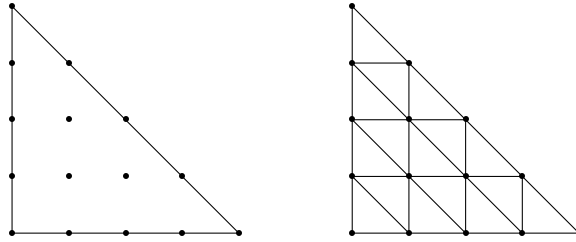

Figure 3. A regular triangle and its triangulation.

In early versions of the McKay correspondence [23] one of the chief aims was to supply a bijection from irreducible characters of G to a basis of cohomology on a crepant resolution. This was explicitly computed for G-Hilb by Craw [9] when G is abelian using "Reid's recipe": a labelling of exceptional subvarieties by characters of G. Reid's recipe is one of the main tools we will use to compute walls and so we will describe it in some detail.

An exceptional curve C in G-Hilb corresponds to an edge in the Craw–Reid triangulation, which in turn corresponds to a two-dimensional cone in the fan for G-Hilb. A primitive normal vector (α, β, γ) to this cone defines a G-invariant ratio of monomials

$$x^\alpha y^\beta z^\gamma = m_1/m_2,$$

where x, y, z are eigencoordinates on $\mathbb{C}^3$ for G. Mark the curve C with the character by which G acts on m_1 (or m_2). We define the χ-*chain* to be the collection of all exceptional curves (or edges in the Craw–Reid triangulation) marked by the character χ. We say that a triangle in the Craw–Reid triangulation is a χ-*triangle* if one of its edges is marked with the character χ.

After marking all curves, there is a procedure for labelling the compact exceptional divisors, or interior vertices of the triangulation. Let D be such a divisor corresponding to a vertex v. There are three cases:

- v is trivalent: $D \cong \mathbb{P}^2$ and the three exceptional curves in D are all marked with the same character χ. *Mark D with $\chi^{\otimes 2}$.*

- v is 4- or 5-valent, or 6-valent and not inside a regular triangle: D is a Hirzebruch surface blown up in valency-4 points. There are two pairs of exceptional curves in D each marked with the same character χ and χ'. *Mark D with $\chi \otimes \chi'$.*

- v is 6-valent and lies in the interior of a regular triangle: D is a del Pezzo surface of degree 6, and there are three pairs of exceptional curves each marked with the same character χ, χ', χ''. D has two G-invariant maps to $\mathbb{P}^2$, *mark D by the two characters arising from the monomials constituting these two maps.* These two characters ϕ_1, ϕ_2 satisfy

$$\chi \otimes \chi' \otimes \chi'' = \phi_1 \otimes \phi_2.$$

For more detail see [9, Lemmas 3.1–3.4]. We will frequently refer to a curve or a divisor marked with a character χ as a χ-curve or a χ-divisor.

Example 2.1. In Fig. 1 with $G = \frac{1}{30}(25, 2, 3)$, the leftmost curve marked with the character 20 has normal $(-2, 25, 0)$ giving the G-invariant ratio y^{25}/x^2. G acts on the numerator and denominator by the character $\varepsilon \mapsto \varepsilon^{20}$, hence the marking. The divisor marked with 23 incident to the previous curve marked with 20 has two pairs of curves with characters 20 and 3 and a fifth curve with character 15. Thus, the divisor is correctly marked by $20 + 3 = 23$.

We refer to divisors of the first two types – that is, all divisors not isomorphic to a del Pezzo surface of degree 6 – as *Hirzebruch divisors*, and to divisors isomorphic to a del Pezzo surface of degree 6 as *del Pezzo divisors*. We ask the reader to have grace on the slight abuse of terminology as $\mathbb{P}^2$ is also a del Pezzo surface. For a character χ marking a curve, we denote by $\mathrm{Hirz}(\chi)$ the set of characters marking Hirzebruch divisors in the interior of the χ-chain and by $\mathrm{dP}(\chi)$ the set of characters marking del Pezzo divisors in the interior of the χ-chain. We will often say "along the χ-chain" in place of "in the interior of the χ-chain".

2.3 G-igsaw pieces

Consider the G-clusters at torus-fixed points of G-Hilb, or triangles in the Craw–Reid triangulation. The ideal defining such a cluster is a monomial ideal and one can draw a Newton polygon in the hexagonal lattice $\mathbb{Z}^3/\mathbb{Z} \cdot (1, 1, 1)$ to illustrate the monomial basis. An example of a torus-fixed G-cluster for the group $G = \frac{1}{6}(1, 2, 3)$ is shown in Fig. 4. Notice that there is exactly one monomial in each character space for G as desired.
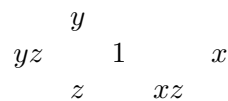
$$
\begin{array}{ccccc}
 & y & & & \\
yz & & 1 & & x \\
 & z & & xz &
\end{array}
$$

Figure 4. A G-cluster for $G = \frac{1}{6}(1, 2, 3)$ corresponding to a torus-fixed point of G-Hilb.

The monomial ideal in $\mathbb{C}[x, y, z]$ defining this cluster is

$$\langle x^2, y^2, z^2, xy \rangle.$$

G-clusters corresponding to adjacent triangles separated by an exceptional curve C differ by taking a subset of the monomials basing one G-cluster and moving them to other monomials in the same character space; that is, multiplying by G-invariant ratios of monomials. This process was studied in [22] and called a *G-igsaw transformation*.

Definition 2.2. Let $C \subset G$-Hilb $\mathbb{A}^3$ be an exceptional curve corresponding to the common edge of two adjacent triangles τ, τ'. Denote by $Z_\tau, Z_{\tau'}$ the two G-clusters corresponding to the torus-fixed points of G-Hilb $\mathbb{A}^3$ for τ, τ'. We call the set of characters labelling monomials in Z_τ (or in $Z_{\tau'}$) that partake in the G-igsaw transformation the *total G-igsaw piece* for C and denote it by G-ig(C).

We will often also refer to the set of monomials underlying G-ig(C) for one of the triangles either side of C as a total G-igsaw piece. There is a single monomial that divides all others in the total G-igsaw piece, and this is the monomial in the G-cluster in the character space for the character marking C. Indeed, an alternative definition of a total G-igsaw piece for C is the set of all monomials in the G-graph for one of the triangles adjacent to C that are divisible by the monomial in this eigenspace.

Example 2.3. We continue the example of $G = \frac{1}{6}(1,2,3)$. The Craw–Reid triangulation and Reid's recipe for this group is shown in Fig. 5. The triangle labelled by $\star$ is the triangle corresponding to the G-cluster from Fig. 4.
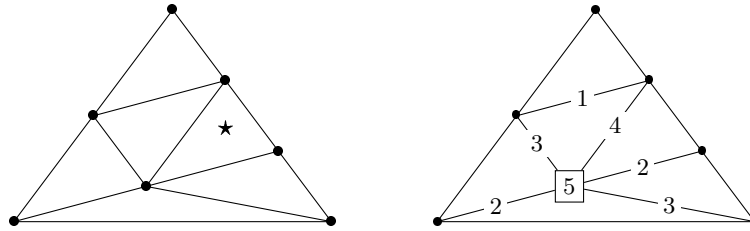

Figure 5. Triangulation and Reid's recipe for $\frac{1}{6}(1,2,3)$.

Passing through the 4-curve C adjacent to the triangle $\star$ performs a G-igsaw transformation with total G-igsaw piece centred on the monomial with character 4, which in this case is xz. The G-igsaw transformation switches xz for y^2 – since the G-invariant ratio for C is xz/y^2 – producing the new G-cluster

$$
\begin{array}{cccc}
y^2 & & & \\
 & y & & \\
yz & & 1 & x \\
 & z & &
\end{array}
$$

If we pass through the 2-curve bordering $\star$ then the total G-igsaw piece contains the monomials y, yz and produces the G-cluster

$$
\begin{array}{cccc}
 & 1 & x & x^2 \\
z & & xz & x^2z
\end{array}
$$

As the two total G-igsaw pieces for a given curve are related by multiplying by G-invariant ratios it is clear that they each have the same set of characters represented by their monomials. We denote the set of characters in either total G-igsaw piece for a curve C by G-ig(C). For convenience we will also denote by $\chi(m)$ the character by which G acts on a monomial m.

2.4   Tautological bundles

The sheaf $\mathcal{R} = \pi_* \mathcal{O}_{\mathcal{Z}}$ is locally free with fibre $H^0(\mathcal{O}_Z)$ above $Z \in G$-Hilb $\mathbb{A}^3$. It splits into eigensheaves

$$
\mathcal{R} = \bigoplus_{\chi \in \mathrm{Irr}\, G} \mathcal{R}_\chi
$$

and these summands are called *tautological bundles*. Since G is abelian, the $\mathcal{R}_\chi$ are line bundles. [9] gives relations between these line bundles in the Picard group, which translate to divisibility relations between eigenmonomials. For a triangle τ in the Craw–Reid triangulation, denote the generator of $\mathcal{R}_\chi$ on the affine piece corresponding to τ by $r_{\chi,\tau}$. We usually omit reference to τ so long as the context is clear

Theorem 2.4 ([9, Theorem 6.1]). *The relations between (generators of) tautological line bundles are described by Reid's recipe in the following way.*

- *If three lines marked with the same character χ meet at a vertex marked with $\psi = \chi^{\otimes 2}$ then*

  $$r_\chi^2 = r_\psi.$$

- *If four or five or six lines consisting of two pairs marked by characters χ, χ' and zero or one or two extra lines marked with further characters meet at a vertex marked with $\psi = \chi \otimes \chi'$ then*

  $$r_\chi \cdot r_{\chi'} = r_\psi.$$

- *If six lines consisting of three pairs marked by characters χ, χ', χ'' meet at a vertex marked with ϕ, ϕ' then*

  $$r_\chi \cdot r_{\chi'} \cdot r_{\chi''} = r_\phi \cdot r_{\phi'}.$$

The claim is that these relations hold and generate all relations between tautological bundles. We will make heavy use of these divisibility relations between eigenmonomials to study G-igsaw pieces for exceptional curves.

As alluded to, the work of Logvinenko [20], Cautis–Logvinenko [7], and Craw–Cautis–Logvinenko [6] categorifies Reid's recipe via the tautological bundles. Many of the constructions in [6, Sections 3–4] resemble constructions made in Section 3 below, however the computations they make are for the inverse equivalence of (1.1) to that utilised in [10] and here.

Evident from [6, 10] and below, characters marking a divisor or a single curve are special. They are termed "essential characters" and have been further examined in [11, 24].

2.5 Abstract inequalities for $\mathfrak{C}_0$

In [10, Section 9] Craw–Ishii provide an abstract description of sufficiently many inequalities to carve out the chamber $\mathfrak{C}_0$. These inequalities arise from the linearisation map

$$L_{\mathfrak{C}_0} \colon \ \Theta \to \mathrm{Pic}(\mathcal{M}_{\mathfrak{C}_0}),$$

which takes θ to the ample $\mathbb{Q}$-divisor on $\mathcal{M}_\theta$ arising from GIT; note that we have identified the Picard groups of different resolutions. By construction $L_{\mathfrak{C}_0}(\mathfrak{C}_0) \subset \mathrm{Amp}(\mathcal{M}_{\mathbb{C}_0})$ and so one obtains inequalities

$$\theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)\big) > 0$$

for all exceptional curves $C \subset G$-Hilb. If such an inequality is necessary to define $\mathfrak{C}_0$, the geometry of C determines the type of the corresponding wall as follows:

- If C is a $(-1,-1)$-curve – that is, it corresponds to an interior edge inside a regular triangle – then $(\theta(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)) = 0) \cap \overline{\mathfrak{C}}_0$ is a Type I wall.

- If C is a $(1, -3)$-curve – that is, it corresponds to one of the edges incident to a trivalent vertex – then $(\theta(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)) = 0) \cap \overline{\mathfrak{C}}_0$ is a Type II wall.

- If C is contained in a Hirzebruch divisor but it is not in either of the previous cases, then $(\theta(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)) = 0) \cap \overline{\mathfrak{C}}_0$ is a Type III wall.

One can express the inequality $\theta(\varphi_{\mathfrak{C}}(\mathcal{O}_C)) > 0$ abstractly via [10, Corollary 5.2], a consequence of which is

$$\theta\big(\varphi_{\mathfrak{C}}(\mathcal{O}_C)\big) = \sum_\rho \deg(\mathcal{R}_\rho|_C)\theta(\rho).$$

Any character ρ not in G-ig(C) has $\mathcal{R}_\rho|_C = \mathcal{O}_C$ and so it doesn't appear in the sum above. It follows that

$$\theta\big(\varphi_{\mathfrak{C}}(\mathcal{O}_C)\big) = \sum_{\rho \in G\text{-ig}(C)} \deg(\mathcal{R}_\rho|_C)\theta(\rho).$$

To complete the description by Craw–Ishii, their remaining inequalities – which if they are necessary inequalities will define walls of Type 0 – are obtained from divisors in two ways outlined in Lemma 2.5 and Proposition 2.6.

Lemma 2.5 ([10, Corollary 5.6 and Theorem 9.3]). *Suppose that $D \subset G$-Hilb $\mathbb{A}^3$ is an irreducible exceptional divisor marked with a character ψ. Then all $\theta \in \mathfrak{C}_0$ satisfy the inequality*

$$\theta\big(\varphi_{\mathfrak{C}_0}\big(\mathcal{R}_\psi^{-1}|_D\big)\big) = \theta(\psi) > 0.$$

Moreover, the inequalities of this form are necessary and hence define walls of $\mathfrak{C}_0$.

We call the inequalities coming from Lemma 2.5 *subsheaf inequalities*. The reason for this is the following. Suppose that $\mathcal{O}_Z$ is a G-cluster parameterised by a point in an irreducible exceptional divisor $D \subset G$-Hilb. Let ψ be a character marking D. It follows from [10, Corollary 4.6 and Lemma 9.1] that r_ψ is in the socle of $H^0(\mathcal{O}_Z)$ and is constant on D, hence defines a subsheaf $S = \mathcal{O}_0 \otimes \psi$ of $\mathcal{O}_Z$ for all $Z \in D$. By definition we must have $\theta(S) = \theta(\psi) > 0$ for all $\theta \in \mathfrak{C}_0$ and [10, Proposition 9.3] shows that this is indeed a wall of $\mathfrak{C}_0$.

There is a dual version of this using quotients instead of subsheaves. [10, Lemma 5.7 and Theorem 9.5] shows that

$$\theta\big(\varphi_{\mathfrak{C}_0}(\omega_{D'})\big) < 0$$

for all $\theta \in \mathfrak{C}_0$ and for every possibly reducible but connected exceptional divisor D'. From [10, Theorem 9.5] this inequality corresponds to evaluating θ on the minimal rigid quotient Q of $\mathcal{O}_Z$ for all $Z \in D'$.

Proposition 2.6. *Suppose $D' \subset G$-Hilb $\mathbb{A}^3$ is a possibly reducible but connected exceptional divisor. Then all $\theta \in \mathfrak{C}_0$ satisfy the inequality*

$$\sum_{C \subset D'} \sum_{\chi \in G\text{-ig}(C)} \theta(\chi) > 0.$$

We call the inequalities from Proposition 2.6 *quotient inequalities*.

Proof. For the sheaf Q to be trivial on D' it means that $\mathcal{R}_\rho|_{D'}$ is trivial for all $\rho \subset Q$. Equivalently, all torus-invariant G-clusters in D' share the same eigenmonomial r_ρ for each $\rho \subset Q$ or, also equivalently, $\rho \notin G$-ig(C) for any $C \subset D'$. Reversing the inequality $\theta(Q) < 0$ gives $\theta(\mathbb{C}[G]/Q) > 0$ and $\mathbb{C}[G]/Q$ contains exactly the characters in the statement of the result since G is abelian and so all irreducible representations have multiplicity 1 in $\mathbb{C}[G]$. ∎

The value of Proposition 2.6 is that it reduces computing inequalities from divisors to computing various total G-igsaw pieces, which is the topic of the next section.

3   Computing characters in total G-igsaw pieces

Our motivating question for this section is the following: *how can one determine the characters that appear in a total G-igsaw piece for a curve purely from the data of Reid's recipe?* As we shall see, the answer depends somewhat on how C sits inside G-Hilb though it is still completely combinatorial.

3.1   Combinatorial definitions

We start by making some combinatorial definitions. For two points $u, v \in \mathbb{R}^3$ we denote by $[u, v]$, (u, v), $[u, v)$, $(u, v]$ the closed, open, and half-open line segments with endpoints u and v. For example, $u \in [u, v)$ but $v \notin [u, v)$.

Let C be a $(-1, -1)$-curve marked with χ. Let α be the edge corresponding to C in the Craw–Reid triangulation. Pick a point u_0 in the interior of α. We will use u_0 to view the χ-chain as an (embedded) quiver Ξ_{χ, u_0} as follows. Let the vertices of the quiver be the vertices corresponding to Hirzebruch divisors incident to the χ-chain (including those at the ends if applicable) and include one extra vertex corresponding to u_0. Let the edges be the parts of the χ-chain between these divisors with the edge containing α split into two with one on either side of u_0. We orient an edge $\beta \subset \Xi_{\chi, u_0}$ by declaring that the tail is the boundary vertex of β closest to u_0. Note that the χ-chain is a tree and so this makes sense. We show an example of Ξ_{χ, u_0} for the group $G = \frac{1}{30}(25, 2, 3)$ with the character χ_3 and the point $*$ in Fig. 6. Note that as an abstract quiver Ξ_{χ, u_0} depends only on the edge α or, equivalently, the curve C and so we slightly abuse notation by subsequently denoting it Ξ_C.
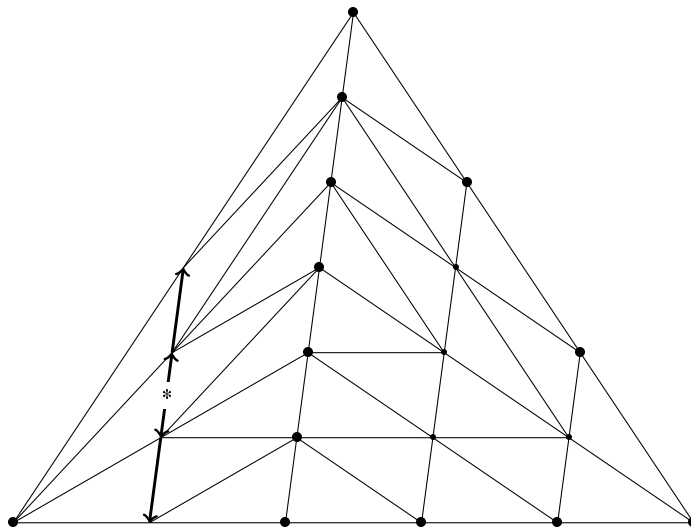

Figure 6. The embedded quiver $\Xi_{\chi_3, *} = \Xi_{C_1}$.

Definition 3.1. Let C be a χ-curve. We say that a divisor D along the χ-chain is *downstream* of C if it is a Hirzebruch divisor. We say that a $(-1, -1)$-curve E incident to such a divisor D is *downstream* of C if the edge for E meets the tail of the arrow in Ξ_C corresponding to the χ-curve in the same regular triangle as E.

This is illustrated schematically in Fig. 7 and concretely on the left side of Fig. 10 for the 3-curve C_1 inside $\frac{1}{30}(25, 2, 3)$-Hilb whose edge includes the point $*$ from Fig. 6. In Fig. 7 the bold edges indicate the boundary of a regular triangle, hence the central vertex is a Hirzebruch

divisor, and only the two dotted curves are downstream according to the given orientation of the χ-chain. In Fig. 10 the curves downstream of C_1 are dotted.
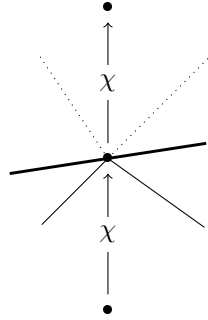

Figure 7. Schematic of curves downstream from a $(-1, -1)$-curve.

Now suppose that C is a boundary curve marked with χ. By the construction of the Craw–Reid triangulation there is a vertex e_i of the junior simplex and an interior vertex v of the junior simplex such that the edge for C is contained in the line segment $[e_i, v]$. Let v_C be the vertex furthest from e_i such that this is true. At v_C the χ-chain will change slope. There exists a vertex v'_C such that all curves in the χ-chain between v_C and v' are $(-1, -1)$-curves and hence lie in the interior of various regular triangles. Note that $v_C = v'_C$ is possible, in which case there are no $(-1, -1)$-curves along the χ-chain. At v'_C there are two possibilities for the χ-chain: either the χ-chain terminates, or it continues into a line segment $[v'_C, e_j]$ from v'_C to a vertex e_j of the junior simplex where it then terminates. We illustrate the situation where the χ-chain terminates at e_j in Fig. 8. The dashed segment of the χ-chain represents the part between v_C and v'_C, which consists of $(-1, -1)$-curves, and the vertices shown there correspond to Hirzebruch divisors, which occur where the χ-chain passes between two different regular triangles.
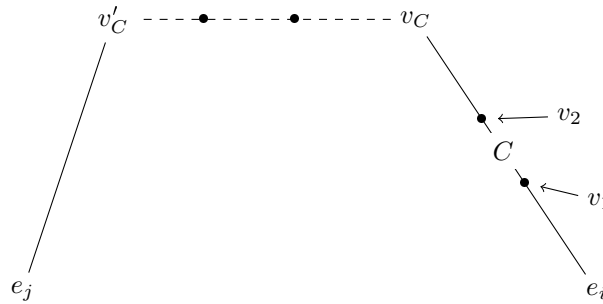

Figure 8. χ-chain for a boundary curve.

Similarly to the case where C was a $(-1, -1)$-curve, we create an embedded quiver Ξ_C supported on the part of the χ-chain between the vertices v_2 and v'_C. The vertices are the Hirzebruch divisors incident to this part of the χ-chain and the edges are the parts of the χ-chain between these divisors. We orient the edges by declaring that v_2 is the unique source of the quiver and v'_C is the unique sink.

We use the notation of Fig. 8 in the following definitions.

Definition 3.2. Let C be a boundary curve as depicted in Fig. 8. We say that a Hirzebruch divisor D is *downstream* of C if either:

- the vertex for D lies in $[v_2, v_C]$,
- the vertex for D lies in the part of the χ-chain between v_C and v'_C (excluding v'_C).

Let E be a $(-1, -1)$-curve marked with a character ρ and contained in a Hirzebruch divisor D downstream of C. We say that E is *downstream* of C if either:

- the vertex for D lies in the line segment $[v_2, v_C]$,

- the vertex for D lies between v_C and v'_C and the edge for E meets the tail of the arrow in Ξ_C corresponding to the χ-curve in the same regular triangle as E,

and if either

- the ρ-chain terminates at D,

- the edges in the ρ-chain incident to D have different slopes.

We show a schematic for the downstream curves relative to C in Fig. 9. The bold arrows represent Ξ_C and the additional edges correspond to sides of the various regular triangles that the χ-chain passes through. The dotted edges are the curves downstream of C, and the dashed edges represent two $(-1, -1)$-curves marked with the same character and whose edges have the same slope, hence the dashed curves are not downstream of C.
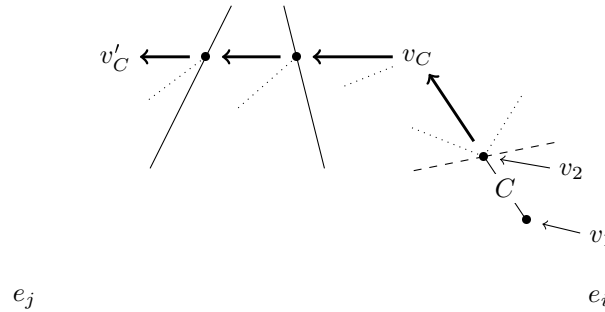

Figure 9. Schematic of curves downstream of a boundary curve.

Note that in Fig. 8 the divisors for v_2 and v_C are downstream of C but the divisor for v_1 is not. On the right of Fig. 10, when $G = \frac{1}{30}(25, 2, 3)$ the divisors D_2 and D_3 are all the divisors downstream of the 15-curve C_2 whereas D_1 is the only divisor along the 15-chain that is not. We have bolded the sides of regular triangles in the triangulation to make it clear that C_2 is a boundary curve, and to clarify which divisors are Hirzebruch divisors. We also show the curves downstream of C_2 dotted.
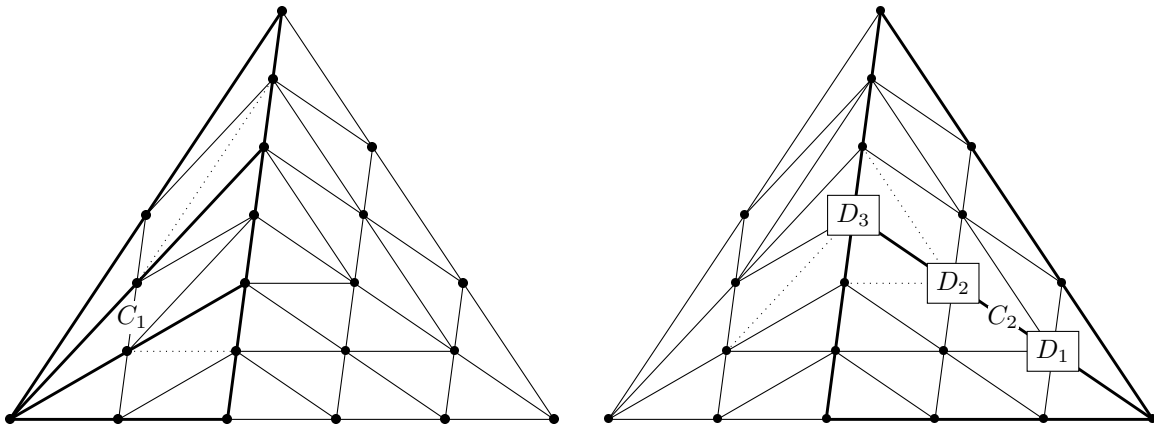

Figure 10. Downstream curves and divisors in $\frac{1}{30}(25, 2, 3)$-Hilb.

For any exceptional curve C and a Hirzebruch divisor D downstream of C, we denote the set of curves in D downstream of C by $\mathcal{C}_C(D)$.

We will say that the ρ-chain for some character ρ is *broken* at a vertex v (or the corresponding divisor) if either the ρ-chain terminates at v or if the edges in the ρ-chain incident to v have different slopes as in the second part of Definition 3.2.

Lastly, we define a character $\chi_{\mathrm{dP}}(C, D)$ associated to a χ-curve C and a del Pezzo divisor D along the χ-chain. Let Δ be the regular triangle containing the vertex for D, let v be the vertex corresponding to D, and let α be the edge of the triangulation corresponding to C.

Let $\{p, q, m\} = \{1, 2, 3\}$. We denote $x_1 = x$, $x_2 = y$, $x_3 = z$ for convenience. We assume that Δ is a corner triangle with e_m as vertex and one side coming from a ray out of e_p; we will treat the meeting of champions case shortly. Here ϕ_1, ϕ_2 denote the characters marking the del Pezzo divisor at v, and a, b, c, d, e, f are positive integers coming from the edges in the Craw–Reid triangulation defining out Δ. More precisely, the two sides incident to e_m have the ratios $x_p^d : x_q^b$ and $x_q^e : x_p^a$ marking them, and the side coming from a ray out of e_p has ratio $x_m^f : x_q^c$. We denote by $r = f$ the side length of the regular triangle. Each of the indices i, j, k ranges from $0, \ldots, r$. Consider the local picture for $p = 1$, $q = 2$, $m = 3$ shown in Fig. 11 adapted from the proof of [9, Theorem 6.1], specifically [9, Fig. 12], for eigenmonomials near v inside Δ.
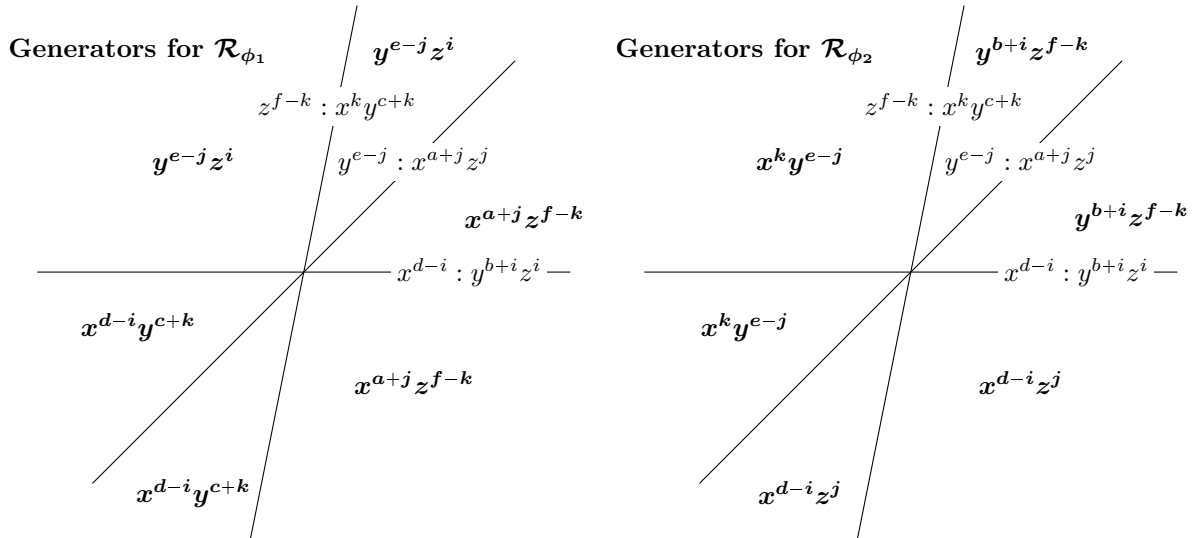

Figure 11. Generators for tautological bundles near v.

Suppose $\chi = \chi\big(x_p^{d-i}\big)$; that is, if $p = 1$, $q = 2$, $m = 3$ then χ marks the horizontal chain of curves in Fig. 11. Following [9] we denote by $e_i v e_j$ the convex part of the junior simplex enclosed by the line segments from e_i to v and from e_j to v. Define

$$\chi_{\mathrm{dP}}(C, D) := \begin{cases} \chi \otimes \chi\big(x_m^j\big) & \text{if } \alpha \subset e_p v e_q, \\ \chi \otimes \chi\big(x_q^{c+k}\big) & \text{if } \alpha \subset e_p v e_m. \end{cases}$$

Observe that $\alpha \subset e_q v e_m$ is not a possibility from considering slopes. Similarly, if $\chi = \chi\big(x_q^{e-j}\big)$ define

$$\chi_{\mathrm{dP}}(C, D) := \begin{cases} \chi \otimes \chi\big(x_m^i\big) & \text{if } \alpha \subset e_q v e_p, \\ \chi \otimes \chi\big(x_p^k\big) & \text{if } \alpha \subset e_q v e_m, \end{cases}$$

and if $\chi = \chi\big(x_m^{f-k}\big)$ define

$$\chi_{\mathrm{dP}}(C,D) := \begin{cases} \chi \otimes \chi\big(x_p^{a+j}\big) & \text{if } \alpha \subset e_m v e_q, \\ \chi \otimes \chi\big(x_q^{b+i}\big) & \text{if } \alpha \subset e_m v e_p. \end{cases}$$

When Δ is a meeting of champions triangle with side ratios $x^d : y^b$, $y^e : z^c$, $x^a : z^f$ we make slight modifications to the above as follows. In this setting, when $\chi = \chi\big(x^{d-i}\big)$ define

$$\chi_{\mathrm{dP}}(C,D) := \begin{cases} \chi \otimes \chi\big(z^j\big) & \text{if } \alpha \subset e_1 v e_2, \\ \chi \otimes \chi\big(y^k\big) & \text{if } \alpha \subset e_1 v e_3. \end{cases}$$

If $\chi = \chi\big(y^{e-j}\big)$ define

$$\chi_{\mathrm{dP}}(C,D) := \begin{cases} \chi \otimes \chi\big(z^i\big) & \text{if } \alpha \subset e_2 v e_1, \\ \chi \otimes \chi\big(x^{a+k}\big) & \text{if } \alpha \subset e_2 v e_3, \end{cases}$$

and if $\chi = \chi\big(z^{f-k}\big)$ define

$$\chi_{\mathrm{dP}}(C,D) := \begin{cases} \chi \otimes \chi\big(x^j\big) & \text{if } \alpha \subset e_3 v e_2, \\ \chi \otimes \chi\big(y^{b+i}\big) & \text{if } \alpha \subset e_3 v e_1. \end{cases}$$

We remark that it follows from Case 4 of the proof of [9, Theorem 6.1] that $\chi_{\mathrm{dP}}(C,D)$ is one of the characters marking D, and is moreover the unique such character ϕ with $r_\chi \, | \, r_\phi$. It also follows from the construction that $\chi_{\mathrm{dP}}(C,D)$ takes the same value on χ-curves in each of the two connected components of the χ-chain minus the vertex v.

3.2 Unlocking procedure

In this subsection we outline the algorithm that we use to compute G-ig(C). We will spend the remainder of this section proving its validity.

Algorithm 3.3 (unlocking procedure)**.** Input: *An exceptional curve $C \subset G$-Hilb $\mathbb{A}^3$ marked with a character χ by Reid's recipe.*

Ch *Let $S = \{\chi\}$.*

dP *For each del Pezzo divisor D along the χ-chain, add $\chi_{dP}(C,D)$ to S.*

H1 *For each Hirzebruch divisor along the χ-chain, add the character marking it to S.*

Re *For each Hirzebruch divisor D downstream of C and for each $E \in \mathcal{C}_C(D)$, compute G-ig(E) by running the unlocking procedure with E as input.*

H2 *For each Hirzebruch divisor D downstream of C, add the characters in $\bigcup_{E \in \mathcal{C}_C(D)} G$-ig$(E)$ to S.*

Output: G-ig$(C) = S$.

We call this the *unlocking procedure* as passing through a Hirzebruch divisor "unlocks" simpler G-igsaw puzzles for the curves E downstream of C that one recursively solves in the step **Re** and then feeds into the total G-igsaw piece for C. It can be visualised as a flow through the triangulation emanating from the curve C with preferred paths defining its tributaries. We note that the convoluted definition of $\chi_{\mathrm{dP}}(C,D)$ is only important for explicit calculations and not for qualitative discussion; the step **dP** states that G-ig(C) contains exactly one of the characters

marking each del Pezzo divisor along the χ-chain. We will often refer to curves downstream of C as being "unlocked" by C.

As an example use case, if G-Hilb has a meeting of champions of side length 0 with the three champions marked with a character χ then for any curve C along the χ-chain the characters in the G-igsaw piece are given by the unlocking procedure applied to the branch of the χ-chain that C lies on, combined with all the characters from (Hirzebruch) divisors along the other two branches of the χ-chain. We will see an example of this in Section 3.8.

3.3 Monomials for divisors

We will begin by relating the characters marking divisors along the χ-chain to G-igsaw pieces for χ-curves.

Lemma 3.4. *Suppose C is a χ-curve. Then G-ig(C) includes exactly one character from each divisor that is along the χ-chain. Moreover, if D is a del Pezzo divisor along the χ-chain then $\chi_{\mathrm{dP}}(C, D)$ is the character marking D that appears in G-ig(C).*

That is, G-ig(C) contains the characters marking each Hirzebruch divisor along the χ-chain and precisely one of the two characters marking each del Pezzo surface along the χ-chain.

Proof. Let D be a Hirzebruch divisor along the χ-chain marked with a character ψ. Cases 2–3 of the proof of [9, Theorem 6.1] give that $\mathcal{R}_\psi$ has degree 1 on a given χ-curve, and hence it follows that $r_\chi \,|\, r_\psi$. It follows that any G-igsaw piece featuring r_χ – such as a G-igsaw piece for C – will also feature each r_ψ and so $\psi \in G$-ig(C). Now let D be a del Pezzo divisor along the χ-chain marked with characters ϕ_1, ϕ_2. It follows from Case 4 of the proof of [9, Theorem 6.1] that exactly one of $\mathcal{R}_{\phi_1}$, $\mathcal{R}_{\phi_2}$ has degree 1 on a given χ-curve and so $r_\chi \,|\, r_{\phi_1}$ or $r_\chi \,|\, r_{\phi_2}$ but not both. It follows that exactly one of ϕ_1, ϕ_2 lie in G-ig(C). Let $\phi = \chi_{\mathrm{dP}}(C, D) \in \{\phi_1, \phi_2\}$. As noted above, it follows from Case 4 of the proof of [9, Theorem 6.1] that $r_\chi \,|\, r_\phi$ and so r_ϕ is the unique character marking D that appears in G-ig(C). ∎

Lemma 3.4 gives an effective way of finding the characters in G-ig(C) coming from divisors. However, this does not usually supply all characters in G-ig$(C) \backslash \{\chi\}$.

3.4 Counting characters

Our method for showing that Algorithm 3.3 is valid for a curve C is to source many characters from divisors (as discussed in the previous subsection) and from curves (coming next) that feature in G-ig(C) and to then count how many characters are actually in G-ig(C) to verify that all characters in the total G-igsaw piece have been located. To move towards this second aim we cite a lemma of Craw–Ishii.

Lemma 3.5 ([10, Lemma 9.1])**.** *A character χ marks a torus-invariant compact divisor $D \subset G$-Hilb $\mathbb{A}^3$ iff r_χ is in the socle of every G-cluster corresponding to a torus-fixed point in D.*

Select a $(-1, -1)$-curve C marked with χ. This lies in two del Pezzo divisors from the endpoints of the corresponding line segment. From Lemma 3.4 we see that r_χ divides exactly two of the monomials in the character spaces labelling these two divisors. Suppose τ is a χ-triangle neighbouring C. By the shape of the ratios in Fig. 11 we can assume that r_χ is not a power of a single variable. The Unique Valley Lemma [22, Lemma 3.3] of Nakamura implies that r_χ divides exactly two elements of the socle of the torus-invariant G-cluster Z_τ corresponding to τ. Lemma 3.5 implies that the elements in the socle of Z_τ that r_χ divides correspond exactly to these two characters labelling the neighbouring del Pezzo divisors. These are the outermost

monomials in the G-igsaw piece for C on τ, so that knowing them will allow us to count how many characters appear in G-ig(C).

Using this observation we will first prove the validity of the unlocking procedure for curves inside regular triangles (i.e. those able to define flops, or $(-1, -1)$-curves) before justifying the procedure for the other exceptional curves.

3.5 $(-1, -1)$-curves

We consider four cases covering all $(-1, -1)$-curves in G-Hilb based on the different ratios labelling edges in Fig. 11. For this subsection denote $x_1 = x$, $x_2 = y$, $x_3 = z$. We will use indices $\{p, q, r\} = \{1, 2, 3\}$ to symmetrise the discussion. A (p, q)-*triangle* is an e_p-corner triangle with one edge coming from a straight line out of e_q.

- Type Ix: curves in the interior of a (r, p)-triangle with ratios $x_p^{d-i} : x_q^{b+i} x_r^i$.

- Type Iy: curves in the interior of an (r, p)-triangle with ratios $x_q^{e-j} : x_p^{a+j} x_r^j$.

- Type Iz: curves in the interior of an (r, p)-corner triangle with ratios $x_r^{f-k} : x_p^k x_q^{c+k}$.

- Type Ic: curves in the interior of the meeting of champions triangle (if existent).

We will treat each of these types of $(-1, -1)$-curves but will specialise to the case $p = 1$, $q = 2$, $m = 3$, which suffices to cover all possibilities by symmetry. Fix a $(3, 1)$-triangle Δ.

3.5.1   Type Iy curves

We consider the edges in the interior of Δ marked with ratios of the form $y^{e-j} : x^{a+j} z^j$; that is, of Type Iy. The analysis from Section 3.1 gives a precise description of the socle of the G-clusters corresponding to nearby torus-invariant points as depicted in Fig. 12 and hence we identify the total G-igsaw pieces for such χ-curves. The only additional calculation required is of the monomials r_{ϕ_0} and r_{ϕ_m} for the characters at the endpoints. Consider r_{ϕ_m}. The ratio marking the side of Δ containing the vertex marked with ϕ_m is $z^f : y^c$. It follows that $y^c \mid r_{\phi_m}$ on Δ but then it cannot be the case that r_χ divides r_{ϕ_m} since $r_\chi = x^{a+j} z^j$ for some basic triangles in Δ. It follows from Section 2.2 that r_{ϕ_m} is given by $x^{a+j} y^c$ in the basic triangle where it is displayed in Fig. 12. A similar argument applies to compute r_{ϕ_0}.
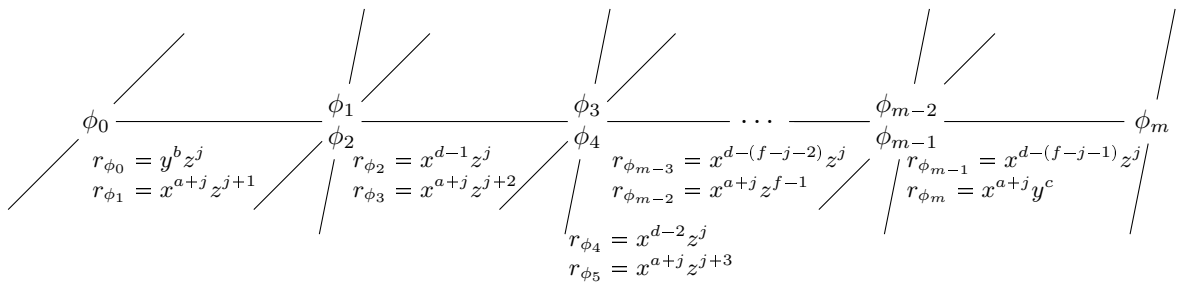

Figure 12. Generators of eigenspaces along a $\chi(x^{a+j} z^j)$-chain inside a regular triangle.

Lemma 3.6. *A total G-igsaw piece for a χ-curve of Type Iy on a χ-triangle chosen so that in the coordinates used above $r_\chi = x^{a+j} z^j$ is*

$$
\begin{array}{cccc}
 & r_\chi & x r_\chi & \cdots & x^{f-i-j-1} r_\chi \\
 & z r_\chi & & & \\
 & \cdot\!\cdot\!\cdot & & & \\
z^i r_\chi & & & &
\end{array}
$$

where the curve corresponds to the edge whose endpoints are the intersection of the χ-chain with the lines marked with $x^{d-i} : y^{b+i}z^i$ and $x^{d-i-1} : y^{b+i+1}z^{i+1}$. Moreover, the χ-chain does not continue outside of this regular triangle. In particular, $\mathrm{Hirz}(\chi) = \varnothing$.

Proof. The calculation of the total G-igsaw piece follows immediately from the description of the eigenmonomials in Fig. 12. As noted the χ-chain cannot continue outside of this regular triangle since neither r_{ϕ_0} nor r_{ϕ_m} are divisible by r_χ and so Theorem 2.4 implies that there cannot be two edges marked with χ incident to either boundary vertex. ∎

Notice that this means that there are $f - j - 1$ characters to account for, excluding χ. But this is exactly the number of del Pezzo surfaces along the χ-chain, each of which contributes one character.

Corollary 3.7. *For an exceptional curve C of Type Iy G-ig(C) consists exactly of χ and the characters $\chi_{\mathrm{dP}}(C,D)$ for each del Pezzo divisor D along the χ-chain.*

Observe that this is a situation in which there is no recursion necessary since $\mathrm{Hirz}(\chi) = \varnothing$. This is one of the base cases that we will reduce to.

3.5.2   Type Ix curves

Suppose now that C is a χ-curve inside Δ that is marked with the ratio $x^{d-i} : y^{b+i}z^i$; that is, C is of Type Ix. [9, Theorem 6.1] yields the identities in Fig. 13 for eigenmonomials on triangles neighbouring the χ-chain, which allow us to completely describe G-igsaw pieces inside regular triangles. In the following we continue the notation of Fig. 11 and let $\kappa = r - (i+1)$.
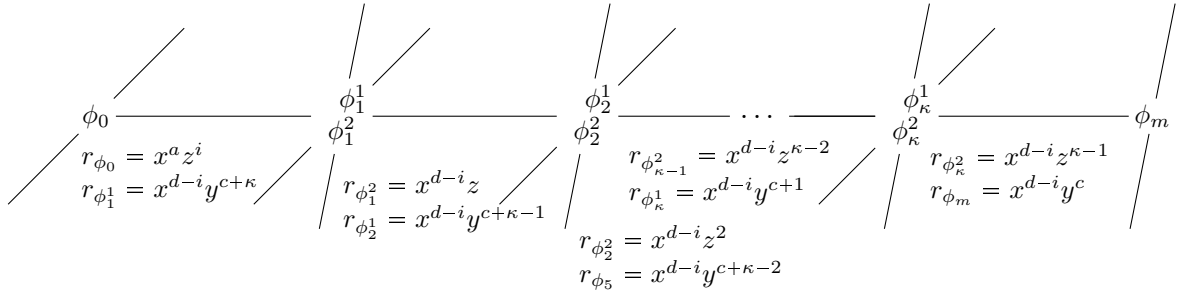

Figure 13. Generators of eigenspaces along a χ-chain inside a regular triangle.

Lemma 3.8. *The G-igsaw piece for a χ-curve C of Type Ix on a χ-triangle chosen so that in the coordinates used above $r_\chi = x^{d-i}$ is*

$$
\begin{matrix}
y^{c+k-1}r_\chi & & & & \\
 & \ddots & & & \\
 & & yr_\chi & & \\
 & & & r_\chi & \\
 & & zr_\chi & & \\
 & \iddots & & & \\
z^j r_\chi & & & &
\end{matrix}
$$

where C corresponds to the edge whose endpoints are the intersection of the χ-chain with the lines marked with $y^{e-j} : x^{a+j}z^j$ and $y^{e-j-1} : x^{a+j+1}z^{j+1}$, and where $i + j + k = r$. Moreover, the χ-chain continues to the right and does not continue to the left of Fig. 13.

Proof. The same argument as for Lemma 3.6 applies, except that r_χ does divide r_{ϕ_m} and so by Theorem 2.4 the χ-chain must continue past the rightmost vertex. ∎

Notice that the only characters in any such G-igsaw piece that are unaccounted for by divisors along the χ-chain in the same regular triangle are those for the monomials

$$yr_\chi, \ \ldots, \ y^c r_\chi$$

though $y^c r_\chi = r_{\phi_m}$, which we have seen corresponds to a Hirzebruch divisor appearing along the χ-chain.

Lemma 3.9. *Suppose C is a χ-curve of Type* `Ix` *such that the χ-chain continues into a boundary edge of a corner triangle. Then G-ig(C) consists of χ, one character from every del Pezzo divisor along the χ-chain, and the characters marking Hirzebruch divisors along the χ-chain.*

This follows since the corner triangle has side length c and so there are exactly c Hirzebruch divisors along the boundary part of the χ-chain that contribute the remaining c characters to the G-igsaw piece. We say that the curves from Lemma 3.9 are of Type `Ixb`. This is the other base case to which the unlocking procedure reduces. Note that the character in G-ig(C) from a del Pezzo divisor D along the χ-chain is by definition $\chi_{\mathrm{dP}}(C, D)$.

We consider the remaining possibilities where the χ-chain merges into the interior of an e_2-corner triangle or the interior of an e_1-corner triangle.

Lemma 3.10. *Suppose C is a χ-curve of Type* `Ix` *and suppose that the χ-chain continues into the interior of an e_2-corner triangle Δ'. Then G-ig(C) consists of χ, one character from each del Pezzo surface along the χ-chain, the character marking the Hirzebruch divisor D between the two regular triangles, and the characters from the total G-igsaw piece of the* `Iy` *curve also incident to D inside Δ'.*

As above, the character in G-ig(C) from a del Pezzo divisor D along the χ-chain is $\chi_{\mathrm{dP}}(C, D)$.

Proof. Let C' be the Type `Iy` curve incident to D in Δ'. Denote its character by χ'. From Lemma 3.6 the characters in the total G-igsaw piece for C' are χ' and one character from each del Pezzo divisor along the χ'-chain inside Δ'. Let the ratios marking sides of Δ' be

$$x^{d'} : z^{b'}, \quad z^{e'} : x^{a'}, \quad z^f : y^c,$$

where $a' < d'$ and where $z^f : y^c$ marks the common side with Δ. Let the part of the χ-chain in Δ' be marked by the ratio $x^{d'-i'} : z^{b'+i'} y^{i'}$ and so $d' - i' = d - i$. It follows that the χ'-chain is marked by $z^{e'-j'} : x^{a'+j'} y^{j'}$ where $i' + j' = f$. Using the relation $d' - a' = c$ we see that $a' + j' = d - i$.

Examining the situation explicitly, we see that on the lower χ-triangle neighbouring C one has $r_\chi = x^{d-i}$ and $r_{\chi'} = x^{a'+j'} y^{j'}$ so that $r_\chi \mid r_{\chi'}$ near C. Moreover, one can see that the zone where $r_{\chi'}$ divides one character from each del Pezzo divisor along the χ'-chain includes this χ-triangle containing C and so these divisibility relations remain. Hence, the G-igsaw piece for C' is contained in the G-igsaw piece for C. The divisibility relations are depicted in Fig. 14.

From Lemma 3.8 the total G-igsaw piece for C is missing c characters after counting the characters in Δ. There are $c - i'$ new characters along the χ-chain corresponding to the del Pezzo divisors along the χ-chain and the boundary Hirzebruch divisor D. There are $c - (c - i') - 1 = i' - 1$ divisors along the χ'-chain, making a contribution of i' characters in total including χ' itself. Thus these account for all of the c missing characters. ∎
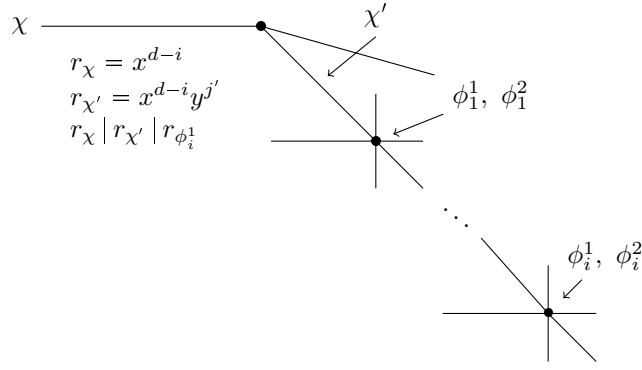
Figure 14. Unlocking for a Type `Ix` curve merging into a $(2, 1)$-triangle.

Note that this vindicates the unlocking procedure for such curves, where only one recursion was required to unlock the single Type `Iy` curve downstream of C. The final case to consider is when the χ-chain merges into an e_1-corner triangle.

Suppose the χ-chain passes through n e_1-corner triangles before entering an e_2-corner triangle Δ'.

Let the ratio $x^{d_m} : y^{b_m}$ mark the edge opposite e_1 for the mth e_1-corner triangle Δ_m from the left and so Δ_m has side length d_m. Suppose the χ-chain enters Δ_m at height i_m. This means that the χ-chain picks up $d_m - i_m$ divisors from del Pezzo divisors and a single Hirzebruch divisor inside Δ_m. From analysing local divisibility relations as above, it is clear that r_χ divides all of the monomials in the G-igsaw pieces for the Type `Ix` curve incident to the χ-chain and the leftmost Hirzebruch divisor inside each of these regular triangles. See Fig. 15 for a schematic. We denote $D_m := \sum_{q=1}^{m} d_q$ and $BD_m := \sum_{q=1}^{m}(b_q + d_q)$.
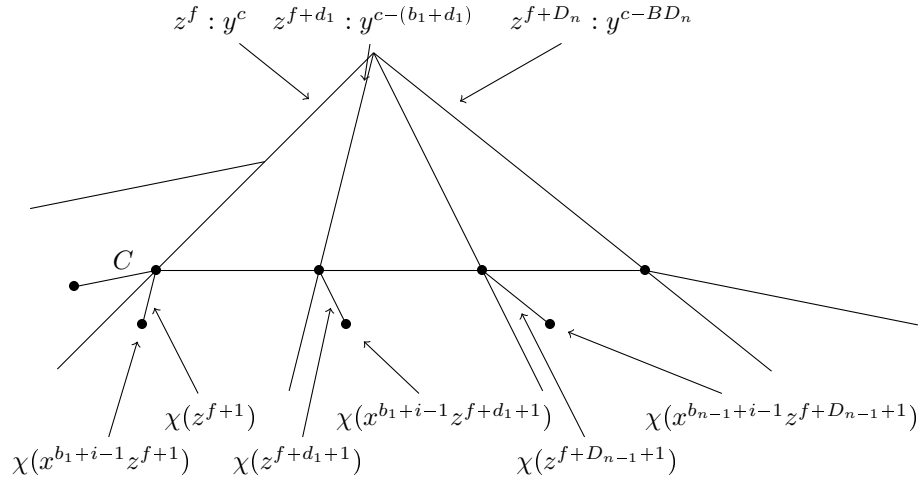

Figure 15. Unlocking for a Type `Ix` curve in a series of e_1-corner triangles.

By computing the characters on the nearby del Pezzo divisor, one can tell that these Type `Ix` curves each have $b_m + i_m$ characters in their G-igsaw pieces, making the total number of characters they contribute to the G-igsaw piece of C

$$\sum_{q=1}^{n}(d_q - i_q + b_q + i_q) = \sum_{q=1}^{n}(b_q + d_q).$$

From [9, Section 2] the ratios marking the edges from e_1 for the e_1-corner triangles are of the form

$$z^{f+\sum_{q=1}^{m} d_q} : y^{c-\sum_{q=1}^{m}(b_q+d_q)} \qquad \text{for } m = 0, \ldots, n$$

with the last edge marked by $z^{f+\sum_{q=1}^{n} d_q} : y^{c-\sum_{q=1}^{n}(b_q+d_q)}$. In particular, this means that the Δ' has side length $c - \sum_{q=1}^{n}(b_q + d_q)$. Assume the χ-chain continues into a chain of Type $\mathtt{Ix}$ curves in Δ'. By the same reasoning as for Lemma 3.10 this produces $c - \sum_{q=1}^{n}(b_q + d_q)$ new characters in G-ig(C) coming from Δ. But then we have

$$\sum_{q=1}^{n}(b_q + d_q) + c - \sum_{q=1}^{n}(b_q + d_q) = c$$

characters in total so far, which exhausts all characters in G-ig(C) by Lemma 3.8. Hence in this case Δ' is the rightmost regular triangle containing χ-curves. Note that the χ-chain cannot merge into a chain of Type $\mathtt{Iy}$ curves in Δ' as such curves cannot escape a single regular triangle. Also, it is clear from convex geometric considerations that the χ-chain cannot continue into a chain of Type $\mathtt{Iz}$ curves. The only remaining option is that the χ-chain continues into a chain of boundary curves, thus again producing $c - \sum_{q=1}^{n}(b_q + d_q)$ new characters from the Hirzebruch divisors along the side of Δ'. In either case the number of characters coming from Δ' is exactly the number of characters in G-ig(C) not accounted for by del Pezzo divisors in Δ by Lemma 3.8. This completes the proof of validity of the unlocking procedure for curves of Type $\mathtt{Ix}$.

3.5.3 Type $\mathtt{Iz}$ curves

The third type of curve occurring inside regular triangles is Type $\mathtt{Iz}$: the curves marked by ratios of the form $z^{f-k} : x^k y^{c+k}$ in the coordinates we have been using for an e_3-corner triangle. We repeat the G-igsaw analysis for these curves, represented in Fig. 16 with the $\chi = \chi\left(z^{f-k}\right)$-chain dashed.
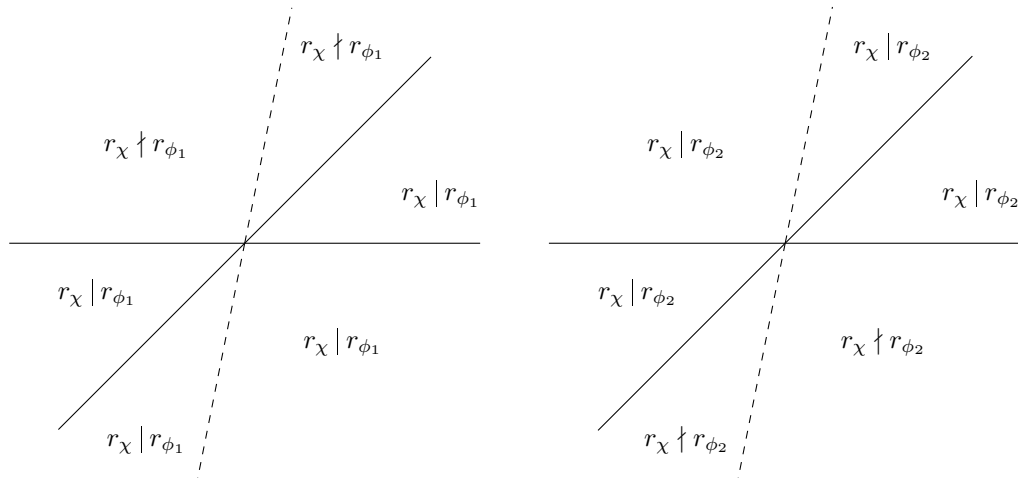

Figure 16. Divisibility relations near v.

As in all previous cases, exactly one character marking each incident del Pezzo surface has a monomial divisible by r_χ and so we can pin down the socle and hence the G-igsaw piece for such a curve.

Lemma 3.11. *The G-igsaw piece for a $(-1, -1)$-curve marked with χ on a χ-triangle chosen so that in the coordinates used above $r_\chi = z^{f-k}$ is*

$$
\begin{matrix}
y^{b+i-1}r_\chi & & & & \\
& \ddots & & & \\
& & yr_\chi & & \\
& & r_\chi & xr_\chi & \ldots & x^{d-i-k}r_\chi
\end{matrix}
$$

where the curve corresponds to the edge whose endpoints are the intersection of the χ-chain with the lines marked with $x^{d-i} : y^{b+i}z^i$ and $x^{d-i-1} : y^{b+i+1}z^{i+1}$.

This means that there are $b + d - k$ characters in the G-igsaw piece for such a `Iz` curve. We shift notation to match the setup of the final case for Type `Ix` curves shown in Fig. 15. In particular, we assume our Type `Iz` curve C lies in an e_1-corner triangle. Suppose it lies in the mth triangle from the left. From considering local divisibility relations near Hirzebruch divisors along the χ-chain this implies that C unlocks $m-1$ Type `Iy` curves to the left and $n-m$ Type `Ix` curves to the right. From the calculations for Type `Ix` curves, the $n-m$ Type `Ix` curves each feature $b_q + i_q$ characters in their G-igsaw pieces. From a similar calculation, one can verify that the Type `Iy` curves contain i_q characters in their G-igsaw pieces. These unlocked curves thus contribute

$$
\sum_{q=1}^{m-1} i_q + \sum_{q=m+1}^{n} (b_q + i_q) = \sum_{q=m+1}^{n} b_q + \sum_{q=1}^{n} i_q - i_m
$$

characters to G-ig(C). The part of the χ-chain in the e_3-corner triangle studied in the previous case contributes $f - i_0$ characters, and the part in the e_2-corner triangle contributes $c - \sum_{q=1}^{n}(b_q + d_q)$. If $i_0 \neq 0$ then we unlock another `Iy` curve with i_0 characters appearing in its G-igsaw piece. If $i_0 = 0$ then the χ-chain continues along the boundary of an e_3-corner triangle, contributing f characters. In either case there are f characters coming from the e_3-corner triangle. Lastly, there are $\sum_{q=1}^{n}(d_q - i_q)$ del Pezzo and Hirzebruch divisors along the part of the χ-chain inside e_1-corner triangles, giving in total

$$
\underbrace{f}_{e_3\text{-corner}} + \underbrace{\sum_{q=m+1}^{n} b_q + \sum_{q=1}^{n} i_q - i_m}_{\text{unlocked curves}} + \underbrace{\sum_{q=1}^{n}(d_q - i_q)}_{e_1\text{-corner}} + \underbrace{c - \sum_{q=1}^{n}(b_q + d_q)}_{e_2\text{-corner}} = f + c - \sum_{q=1}^{m} b_q - i_m
$$

characters. Compare to the quantity $b + d - k$ in Lemma 3.11, which in these coordinates is

$$
c - \sum_{q=1}^{m}(b_q + d_q) + f + \sum_{q=1}^{m} d_q - i_m = f + c - \sum_{q=1}^{m} b_q - i_m
$$

showing that every character in G-ig(C) is accounted for.

3.5.4   Type `Ic` curves

As in [9] the case of curves whose chains meet the interior of a meeting of champions triangle only requires minor notational changes for the arguments above to carry over verbatim. For brevity we omit it.

3.6   Boundary curves

Suppose now that C is a curve lying on the boundary of a regular triangle. We will see that the unlocking procedure computes G-ig(C) by a similar argument to the case of $(-1, -1)$-curves. We will again use neighbouring divisors to compute the socle and hence the total G-igsaw piece for C, and then assess local divisibility relations to evidence that all these characters come from the subvarieties in the unlocking procedure. We will sketch the novel elements of the proof below.

Choose coordinates so that C lies along a straight line from e_1. Assume for the moment that the edge for C is actually incident to e_1.

Suppose that D is a Hirzebruch divisor along the χ-chain. If D is at the boundary of two e_1-corner triangles or an e_1-corner triangle and a meeting of champions – as shown in Fig. 17 – then one can check that r_χ divides the monomials in the G-igsaw pieces for the Type Iy curves C_3 and C_4.
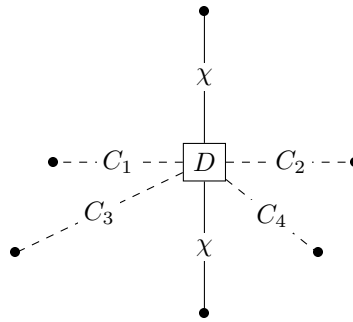
Figure 17. D bordering two e_1-corner triangles or meeting of champions.

Suppose now that D borders an e_2- and an e_3-corner triangle, or an e_1-corner triangle and an e_3-corner triangle. We illustrate this situation in Fig. 18, along with some of the ratios marking curves.
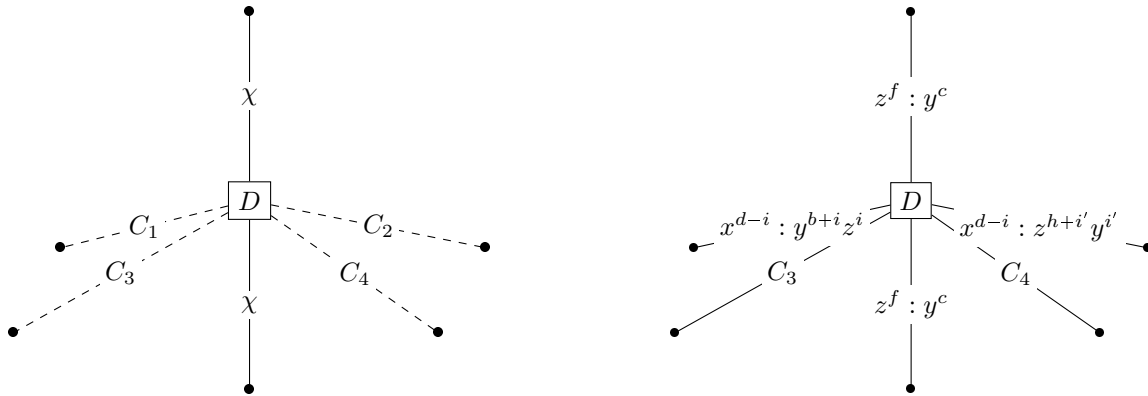

Figure 18. D bordering an e_1- or an e_2-corner triangle and an e_3-corner triangle.

The same argument as in the previous case gives that r_χ divides the G-igsaw pieces for C_3 and C_4.

To treat the remaining two curves C_1 and C_2 in each case, we use a generalised form of [12, Section 3.3.2]: an edge ℓ continues in a straight line past a boundary edge ℓ_0 if and only if the ratio marking ℓ features any common variables x, y, z raised to a strictly lower exponent than in the ratio marking ℓ_0. One can verify this by a case-by-case analysis using as its base the

original result from [12]. This implies that r_χ divides the G-igsaw pieces for "broken edges" that do not continue in a straight line past the χ-chain and that it does not divide any monomials in the G-igsaw pieces for "straight edges" that do continue past the χ-chain. This is captured exactly in the notion of downstream curves relative to boundary curves in Definition 3.2, and hence in the unlocking procedure.

For the case when C is not incident to e_1, consider two boundary curves C and C' shown in Fig. 19, where C is closer to e_1. One can verify using local divisibility relations that the only difference between the G-igsaw piece for C and for C' is that the latter loses the characters in the G-igsaw pieces for the dashed curves in broken chains; that is, exactly the curves that are downstream from C but not from C'. By retracing back to the edge incident to e_1 the first calculation performed above suffices to compute the G-igsaw piece for an arbitrary boundary curve.
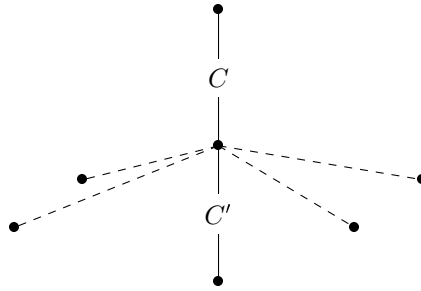

Figure 19. Two boundary curves.

Variations of the arguments above work just as well for the cases not depicted when some of the edges incident to D are also boundary edges of regular triangles. Counting up all these monomials and comparing them with a socle calculation shows that these are all the characters in the G-igsaw piece for C, which validates the unlocking procedure for boundary curves and hence for all exceptional curves in G-Hilb.

3.7   Example: $G = \frac{1}{30}(25, 2, 3)$

We will illustrate the unlocking procedure for G-Hilb in the case that $G = \frac{1}{30}(25, 2, 3)$. In the figure below, dashed lines are edges within a regular triangle and undashed lines are the result of the first stage of the Craw–Reid triangulation.

We will demonstrate the unlocking procedure for a few curves in G-Hilb. Consider the 15-curve C_{15} shown in Fig. 21. This curve is of Type Ixb since it is the only $(-1, -1)$-curve marked with 15 and feeds to the right into boundary edges only. This gives

$$G\text{-ig}(C_{15}) = \{15, 17, 19, 21\}.$$

Consider the 5-curve C_5 shown in Fig. 22. It passes into the right side of the junior simplex, unlocking the 9-curve of Type Iy and giving

$$G\text{-ig}(C_5) = \{5, 7, 9, 11\}.$$

Consider the 2-curve C_2 shown in Fig. 23. This is a curve of Type Iz. We first get the character 17 marking the divisor on the 2-chain, unlocking the 27-chain. The 27-chain contains a del Pezzo divisor contributing the character 22 in this case. Hence
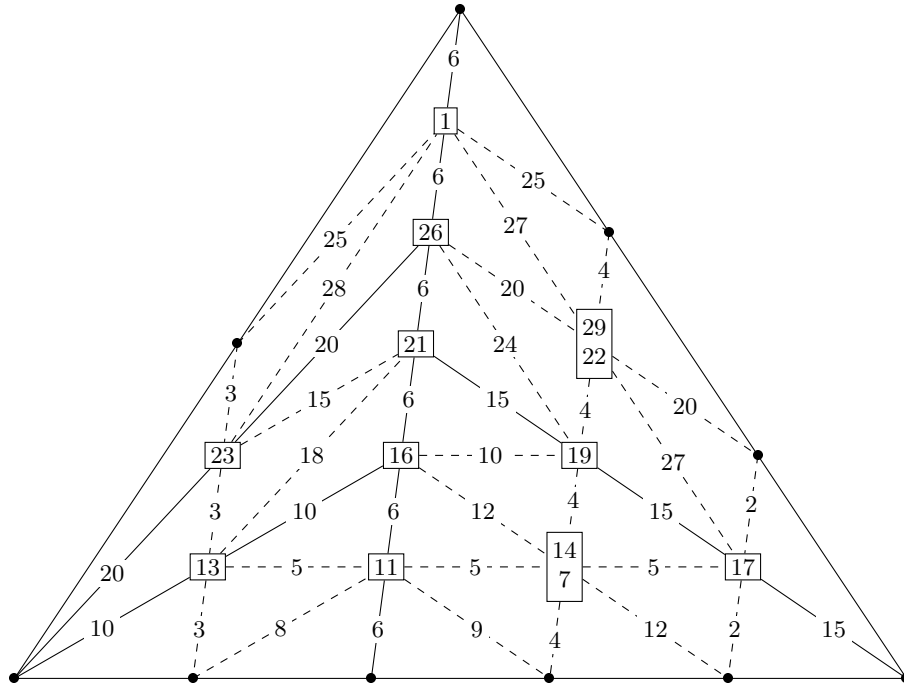
$$G\text{-ig}(C_2) = \{2, 17, 22, 27\}.$$

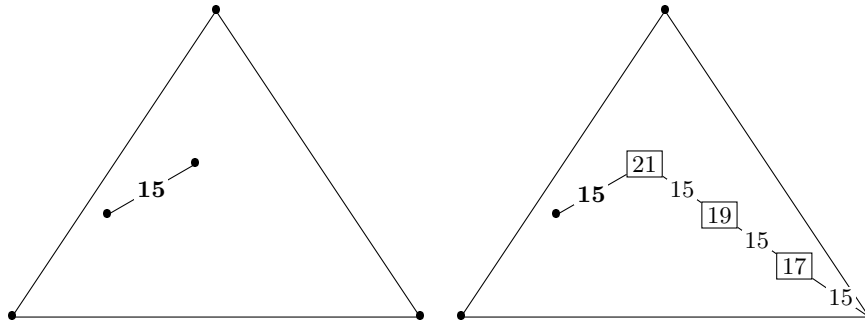**Figure 20.** Reid's recipe for $G = \frac{1}{30}(25, 2, 3)$.

Figure 21. Unlocking for a 15-curve.

Lastly, we will consider the boundary 15-curve C'_{15} shown in Fig. 24.

At the first step we include the 15-chain and the curves of Type Ix and Iy unlocked by it. These curves are marked with characters 10, 24, 18. The 18-curve and the 24-curve are of Type Iy and only contribute their own character to the G-igsaw piece. The 10-curve is of Type Ixb and so we add the Hirzebruch divisors along the 10-chain. As a result

$$G\text{-ig}(C'_{15}) = \{10, 13, 15, 16, 17, 18, 19, 21, 24\}.$$

3.8   Example: $G = \frac{1}{35}(1, 3, 31)$

We will use the example of $G = \frac{1}{35}(1, 3, 31)$ to illustrate a phenomenon implicit, but less clear in the long side picture. The triangulation for G-Hilb and Reid's recipe are found in Fig. 25.

Consider the 3-curve C_3 incident to e_1. The unlocking procedure for this curve is shown in Fig. 26 giving

$$G\text{-ig}(C_3) = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 24, 25, 26, 28, 29,$$
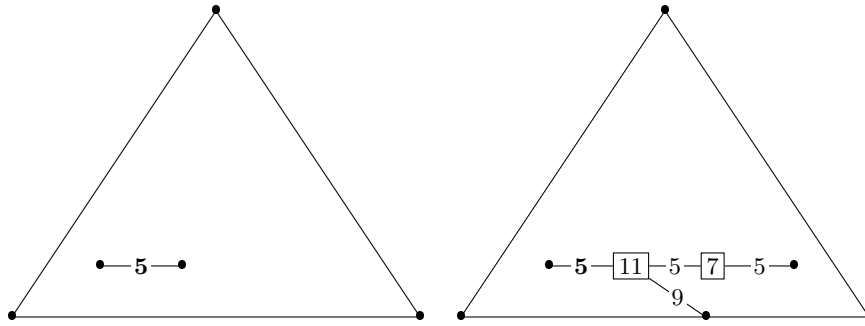$$30, 32, 33, 34\}.$$
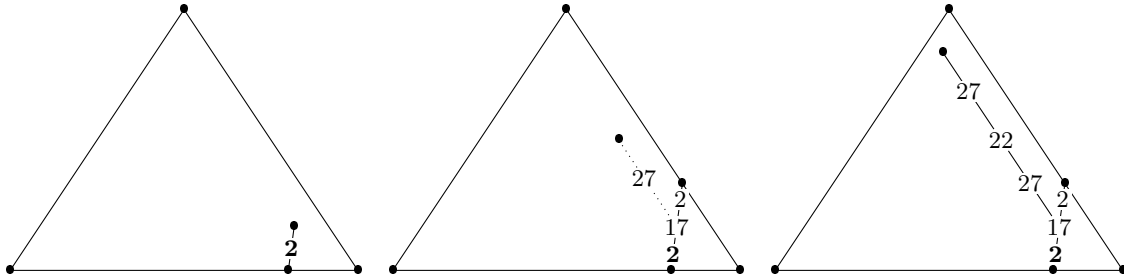
Figure 22. Unlocking for a 5-curve.

Figure 23. Unlocking for a 2-curve.

Notice that every chain meeting the 3-chain in a vertex is broken there. Repeating for the next 3-curve along the chain produces the same unlocking sequence except that the topmost part including the 1-chain and the 12-chain are not included, capturing that the monomials in the corresponding character spaces are no longer divisible by r_3 there.

4   Walls of $\mathfrak{C}_0$

In this section we will compute explicit inequalities carving out $\mathfrak{C}_0$, and will determine which of these inequalities are necessary and hence define walls of $\mathfrak{C}_0$.

4.1   Type I walls

We know from [10, Theorem 9.12] that all flops in a single $(-1,-1)$-curve C are achieved by a wall-crossing from $\mathfrak{C}_0$. Moreover, we have $\deg(\mathcal{R}_\rho|_C) = 1$ for all $\rho \in G\text{-ig}(C)$ from [10, Corollary 6.3]. The unlocking procedure hence gives a combinatorial way of writing down the equations of these walls.

Proposition 4.1. *Suppose $C \subset G\text{-Hilb}\,\mathbb{A}^3$ is an exceptional $(-1,-1)$-curve marked with character χ by Reid's recipe. Then, the Type I wall corresponding to C is given by*

$$\theta(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)) = \sum_{\chi \in G\text{-ig}(C)} \theta(\chi) > 0,$$

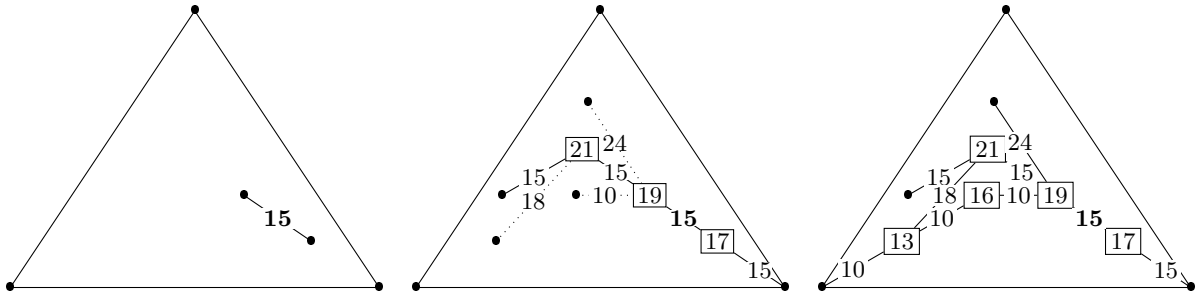*where $G\text{-ig}(C)$ is computed by the unlocking procedure.*
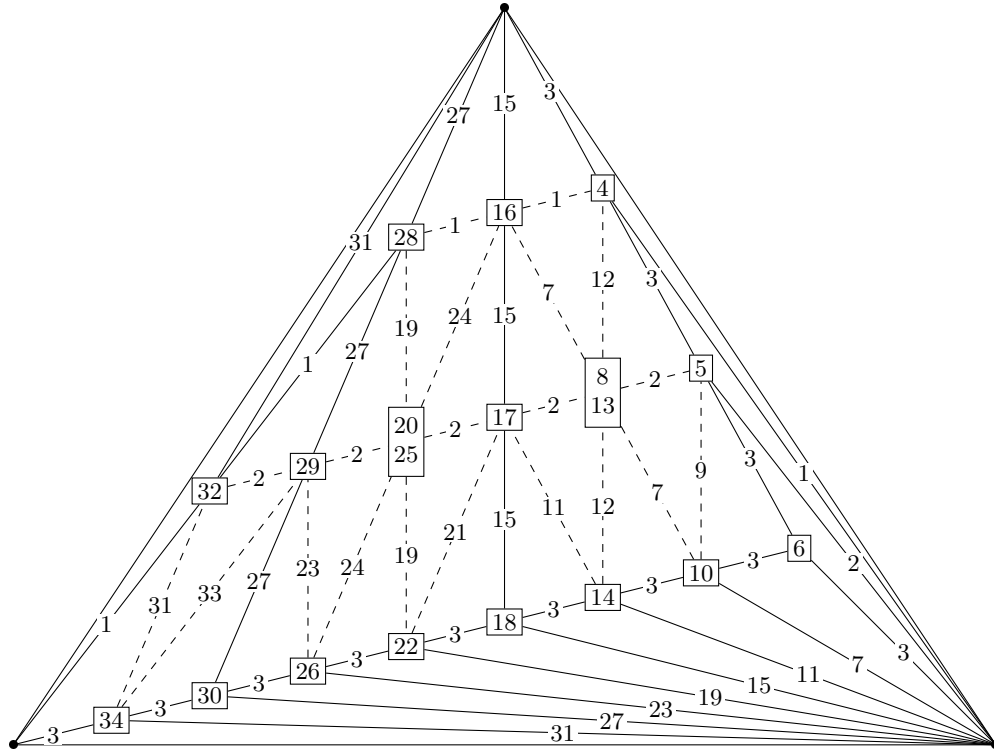
Figure 24. Unlocking for a boundary 15-curve.

Figure 25. Reid's recipe for $G = \frac{1}{35}(1, 3, 31)$.

4.2   No Type II walls

Proposition 4.2. *Suppose $C \subset G$-Hilb $\mathbb{A}^3$ is an exceptional $(1, -3)$-curve marked with charac-ter χ by Reid's recipe. Then, the inequality corresponding to C is given by*

$$\theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)\big) = 2 \cdot \theta\big(\chi^{\otimes 2}\big) + \sum_{\chi \in G\text{-ig}(C)\backslash\{\chi^{\otimes 2}\}} \theta(\chi) > 0,$$

where G-ig(C) is computed by the unlocking procedure.

Proof. Notice that such a curve C lies inside the exceptional $\mathbb{P}^2$ in the meeting of champions case when the meeting of champions triangle has side length 0. Thus the $\mathbb{P}^2$ is marked with $\chi^{\otimes 2}$ and lies in the socle of any torus-invariant G-cluster. From Theorem 2.4 $r_{\chi^{\otimes 2}} = r_\chi^2$ and so r_χ^2 is the furthest character from r_χ in the G-igsaw piece in some direction. Note that

$$\deg(\mathcal{R}_\rho|_C) = \min\big\{k \colon r_\chi^k \,|\, r_\rho\big\}$$
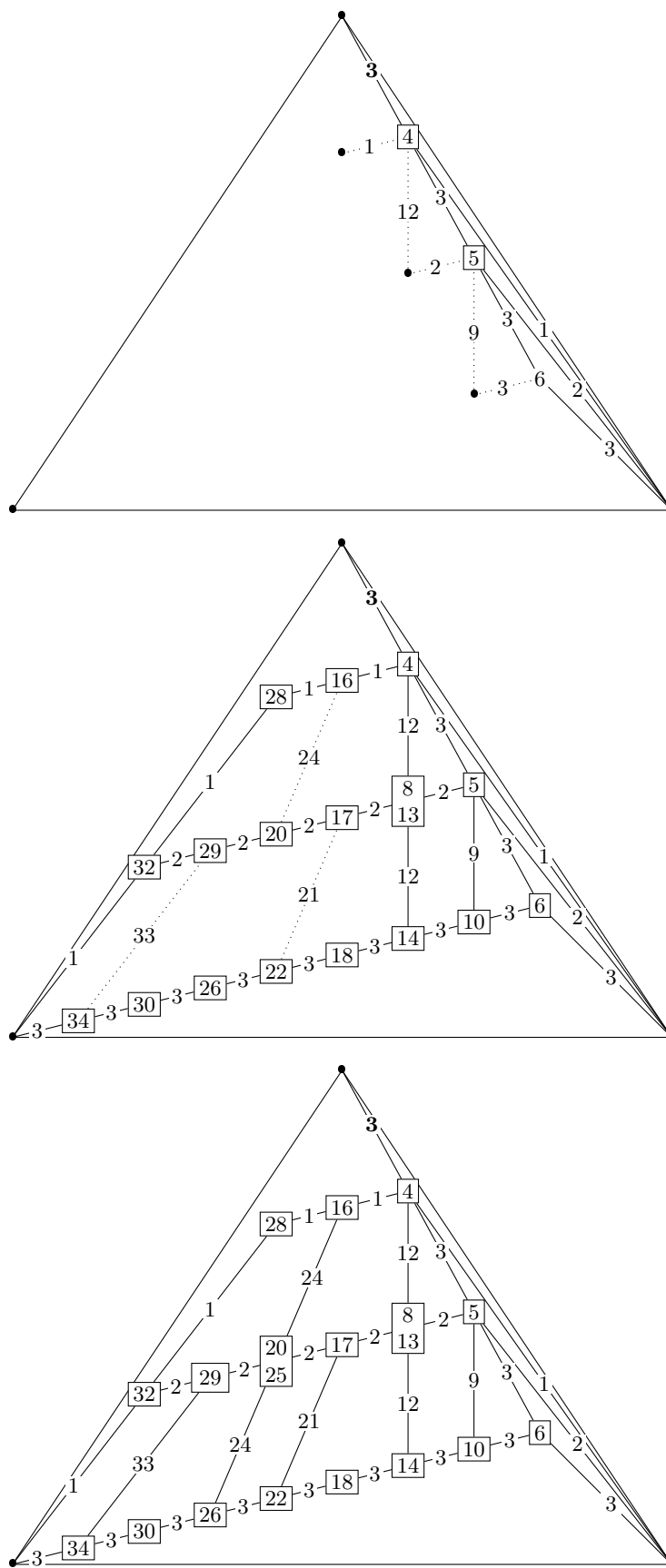
Figure 26. Unlocking for a 3-curve.

and so all the characters in G-ig(C) appear with multiplicity 1 except for r_χ^2, which appears with multiplicity 2. This gives the required formula. ∎

As a result we can immediately deduce the conclusion of [10, Proposition 3.8] for $\mathfrak{C}_0$.

Corollary 4.3. $\mathfrak{C}_0$ *has no Type II walls.*

Proof. Suppose C is an exceptional $(1,-3)$-curve marked with χ. From the unlocking procedure a total G-igsaw piece for C consists of χ, χ^2, and the characters marking the (Hirzebruch) divisors along the χ-chain. Let D' be the exceptional $\mathbb{P}^2$ containing C. Consider the inequality for rigid quotients parameterised by D': from Proposition 2.6 the characters appearing in this inequality are exactly the characters in the G-igsaw pieces of all three χ-curves converging at D'. These are

$$\left\{\chi, \chi^{\otimes 2}\right\} \cup \operatorname{Hirz}(\chi),$$

which are exactly the characters appearing in the inequality for C. However, the inequality for rigid quotients parameterised by D' has multiplicities all equal to 1. When combined with the inequality $\theta\left(\chi^{\otimes 2}\right) > 0$ coming from rigid subsheaves parameterised by D' – note that we can use both inequalities from subsheaves and from quotients since D' is irreducible – this implies that the inequality $\varphi_{\mathfrak{C}_0}(\mathcal{O}_C) > 0$ is redundant. ∎

4.3 All flops in $(-1,-1)$-curves

Using Proposition 4.1 and the unlocking procedure one can show directly that every $(-1,-1)$-curve produces a necessary inequality, recovering [10, Theorem 9.12] by purely combinatorial means.

In order to test redundancy of inequalities we say that an inequality $\sum_i \alpha_i \theta(\chi_i) > 0$ with nonnegative coefficients is a *summand* of another inequality $\sum_j \beta_j \theta(\rho_j) > 0$ with nonnegative coefficients if the difference $\sum_i \alpha_i \theta(\chi_i) - \sum_j \beta_j \theta(\rho_j)$ also has nonnegative coefficients in the basis $\operatorname{Irr} G$. If an inequality coming from curves or divisors decomposes into other inequalities as summands, then it is redundant and does not define a wall of $\mathfrak{C}_0$.

Proposition 4.4. *Suppose C is an exceptional $(-1,-1)$ curve inside G-Hilb $\mathbb{A}^3$. Then the inequality $\theta(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)) > 0$ is necessary and so defines a wall of $\mathfrak{C}_0$.*

Proof. Suppose C is marked with χ. From the unlocking procedure we can write the inequality corresponding to C in the form

$$\theta(\chi) + \sum_i \theta(\psi_i) + \theta(\rho_1) + \sum_i \theta\left(\psi_i^1\right) + \cdots + \theta(\rho_m) + \sum_i \theta\left(\psi_i^m\right) > 0, \tag{4.1}$$

where ρ_j are the characters marking curves C_j unlocked by C and ψ_i^j are the characters in the G-igsaw piece for C_j. Note that curves unlocked by C cannot continue on both sides of the χ-chain, since they meet the χ-chain at a Hirzebruch divisor found at the intersection of the χ-chain and an edge of a regular triangle, where only two chains can continue. The inequality for the $(-1,-1)$-curve C_j is

$$\theta\left(\varphi_{\mathfrak{C}_0}(\mathcal{O}_{C_j})\right) = \theta(\rho_j) + \sum_i \theta\left(\psi_i^j\right) > 0.$$

In order to express (4.1) in terms of other inequalities, we must have an inequality featuring the character χ. These can only arise from other χ-curves or divisors parameterising rigid quotients not featuring χ. Other χ-curves will feature at least one different character in their G-igsaw piece compared to G-ig(C): indeed, other curves in the same regular triangle will feature a different

collection of del Pezzo divisors, curves in other regular triangles will either feature different del Pezzo divisors or unlock different curves, and χ-curves along a boundary edge will have different unlocking behaviour. In particular, the inequalities from these curves will not be summands of the inequality (4.1). Inequalities from rigid quotients not containing χ will also not be summands of (4.1) since the unlocking procedure implies that there are no divisors D_ρ along the χ-chain for which all characters marking curves incident to D_ρ are represented in G-ig(C). It follows that (4.1) is necessary. ∎

Proposition 4.4 has an analog for Type III walls in Lemma 4.15 where we classify the $(0, -2)$-curves producing those walls in terms of explicit combinatorics.

4.4   Irredundant inequalities – examples

The aim of these final sections is to precisely describe all of the walls of $\mathfrak{C}_0$, which primarily means identifying which curves produce redundant inequalities and subsequently classifying the walls of Type III. We start with an example.

Example 4.5. Consider $G = \frac{1}{6}(1, 2, 3)$. G-Hilb and Reid's recipe are shown in Fig. 27.
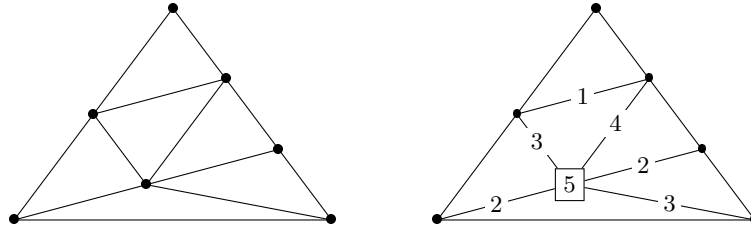

Figure 27. G-Hilb and Reid's recipe for $\frac{1}{6}(1, 2, 3)$.

We compute the inequalities coming from curves and divisors that define $\mathfrak{C}_0$ via the unlocking procedure:

$$\theta(\chi_1) > 0, \tag{A_1}$$
$$\theta(\chi_2) + \theta(\chi_5) > 0 \tag{A_2}$$
$$\theta(\chi_2) + \theta(\chi_3) + 2\theta(\chi_4) + 2\theta(\chi_5) > 0, \tag{B_2}$$
$$\theta(\chi_3) + \theta(\chi_5) > 0, \tag{A_3}$$
$$\theta(\chi_3) + \theta(\chi_4) + \theta(\chi_5) > 0, \tag{B_3}$$
$$\theta(\chi_4) > 0, \tag{A_4}$$
$$\theta(\chi_5) > 0, \tag{A_5}$$
$$\theta(\chi_2) + \theta(\chi_3) + \theta(\chi_4) + \theta(\chi_5) > 0. \tag{B_5}$$

(A_1) is from the curve marked with the essential character 1. Similarly for (A_4). We then have two inequalities (A_2) and (B_2) coming from the two 2-curves, and two (A_3) and (B_3) from the two 3-curves. The 5-divisor gives two inequalities (A_5) and (B_5) for rigid subsheaves and quotients it parameterises.

We can see that (B_2) is redundant by expressing it as a combination of (A_2), (A_3) and (A_4). Similarly, (B_3) can be expressed in terms of (A_3) and (A_4). No further reductions are possible, and so the walls of $\mathfrak{C}_0$ (with their types) in this example are

$$\theta(\chi_1) = 0, \tag{I}$$
$$\theta(\chi_2) + \theta(\chi_5) = 0, \tag{III}$$

$$\theta(\chi_3) + \theta(\chi_5) = 0, \tag{I}$$
$$\theta(\chi_4) = 0, \tag{I}$$
$$\theta(\chi_5) = 0, \tag{O}$$
$$\theta(\chi_2) + \theta(\chi_3) + \theta(\chi_4) + \theta(\chi_5) = 0. \tag{O}$$

Example 4.6. We continue with a more detailed example for $G = \frac{1}{30}(25, 2, 3)$. Continuing the calculations in Section 3.7, we find that the inequalities from curves in G-Hilb are

$$\theta_2 + \theta_{27} + \theta_{22} + \theta_{17} > 0, \tag{A_2}$$
$$\theta_2 + \theta_5 + \theta_8 + \theta_{11} + \theta_{14} > 0, \tag{B_2}$$
$$\theta_3 + \theta_{13} + \theta_{18} + \theta_{23} + \theta_{28} > 0, \tag{A_3}$$
$$\theta_3 + \theta_5 + \theta_7 + \theta_9 + \theta_{11} + \theta_{13} + \theta_{23} + \theta_{28} > 0, \tag{B_3}$$
$$\theta_3 + \theta_5 + \theta_7 + \theta_9 + \theta_{11} + \theta_{13} + \theta_{15} + \theta_{17} + \theta_{19} + \theta_{21} > 0, \tag{C_3}$$
$$\theta_4 + \theta_{29} + \theta_{24} + \theta_{19} + \theta_{14} > 0, \tag{A_4}$$
$$\theta_4 + \theta_7 + \theta_{29} + \theta_{24} + \theta_{19} > 0, \tag{B_4}$$
$$\theta_4 + \theta_7 + \theta_{10} + \theta_{13} + \theta_{16} + \theta_{19} + \theta_{29} > 0, \tag{C_4}$$
$$\theta_4 + \theta_7 + \theta_{10} + \theta_{13} + \theta_{16} + \theta_{19} + \theta_{22} > 0, \tag{D_4}$$
$$\theta_5 + \theta_7 + \theta_9 + \theta_{11} > 0, \tag{A_5}$$
$$\theta_5 + \theta_7 + \theta_8 + \theta_{11} > 0, \tag{B_5}$$
$$\theta_5 + \theta_8 + \theta_{11} + \theta_{14} > 0, \tag{C_5}$$
$$\theta_6 + \theta_8 + \theta_9 + \theta_{10} + \theta_{11} + \theta_{13} + 2\theta_{12} + 2\theta_{14} + 2\theta_{16} + 2\theta_{15} + 2\theta_{17} + 2\theta_{19} + 3\theta_{18}$$
$$+ 3\theta_{20} + 3\theta_{22} + 3\theta_{21} + 3\theta_{23} + 3\theta_{25} + 4\theta_{24} + 4\theta_{26} + 4\theta_{28} + 4\theta_{27} + 4\theta_{29} + 4\theta_1 > 0, \tag{$\mathbf{A_6}$}$$
$$\theta_6 + \theta_8 + \theta_{10} + 2\theta_{12} + 2\theta_{14} + 2\theta_{16} + 3\theta_{18} + \theta_9 + \theta_{11} + \theta_{13} + 2\theta_{15} + 2\theta_{17}$$
$$+ 2\theta_{19} + 3\theta_{21} + \theta_1 + 4\theta_{26} + 2\theta_{16} + 3\theta_{23} + 3\theta_{20} + 3\theta_{22} + 4\theta_{24} + 4\theta_{26} > 0, \tag{$\mathbf{B_6}$}$$
$$\theta_6 + \theta_8 + \theta_{10} + 2\theta_{12} + 2\theta_{14} + 2\theta_{16} + 3\theta_{18} + \theta_9 + \theta_{11} + \theta_{13}$$
$$+ 2\theta_{15} + 2\theta_{17} + 2\theta_{19} + 3\theta_{21} + \theta_1 + \theta_{26} + \theta_{21} + \theta_{16} > 0, \tag{$\mathbf{C_6}$}$$
$$\theta_6 + \theta_8 + \theta_{10} + 2\theta_{12} + 2\theta_{14} + 2\theta_{16} + \theta_1 + \theta_{26} + \theta_{21} + \theta_{16} + \theta_9 + \theta_{11} + \theta_{13} > 0, \tag{$\mathbf{D_6}$}$$
$$\theta_6 + \theta_1 + \theta_{26} + \theta_{21} + \theta_{16} + \theta_{11} + \theta_8 + \theta_9 > 0, \tag{$\mathbf{E_6}$}$$
$$\theta_6 + \theta_1 + \theta_{26} + \theta_{21} + \theta_{16} + \theta_{11} > 0, \tag{$\mathbf{F_6}$}$$
$$\theta_8 > 0, \tag{A_8}$$
$$\theta_9 > 0, \tag{A_9}$$
$$\theta_{10} + \theta_{13} + \theta_{16} > 0, \tag{A_{10}}$$
$$\theta_{10} + \theta_{12} + \theta_{14} + \theta_{16} + \theta_{18} + \theta_5 + \theta_7 + \theta_9 + \theta_{11} + \theta_{13} > 0, \tag{$\mathbf{B_{10}}$}$$
$$\theta_{10} + \theta_{13} + \theta_{12} + \theta_{14} + \theta_{16} > 0, \tag{$\mathbf{C_{10}}$}$$
$$\theta_{12} + \theta_7 > 0, \tag{A_{12}}$$
$$\theta_{12} + \theta_{14} > 0, \tag{B_{12}}$$
$$\theta_{15} + \theta_{17} + \theta_{19} + \theta_{21} > 0, \tag{A_{15}}$$
$$\theta_{15} + \theta_{17} + \theta_{19} + \theta_{18} + \theta_{21} > 0, \tag{$\mathbf{B_{15}}$}$$
$$\theta_{15} + \theta_{17} + \theta_{18} + \theta_{21} + \theta_{24} + \theta_{10} + \theta_{13} + \theta_{16} + \theta_{19} > 0, \tag{$\mathbf{C_{15}}$}$$
$$\theta_{15} + \theta_{18} + \theta_{21} + \theta_{24} + \theta_{27} + \theta_{10} + \theta_{13} + \theta_{16} + \theta_{19} + \theta_{22} + \theta_5 + \theta_8$$
$$+ \theta_{11} + \theta_{14} + \theta_{17} > 0, \tag{$\mathbf{D_{15}}$}$$
$$\theta_{18} > 0, \tag{A_{18}}$$
$$\theta_{20} + \theta_{23} + \theta_{26} + \theta_{29} > 0, \tag{A_{20}}$$

$$\theta_{20} + \theta_{22} + \theta_{23} + \theta_{26} > 0, \tag{B_{20}}$$

$$\theta_{20} + \theta_{23} + \theta_{22} + \theta_{24} + \theta_{26} > 0, \tag{$\mathbf{C_{20}}$}$$

$$\theta_{20} + \theta_{15} + \theta_{17} + \theta_{19} + \theta_{21} + \theta_{22} + \theta_{24} + \theta_{26} + \theta_{28} > 0, \tag{$\mathbf{D_{20}}$}$$

$$\theta_{24} > 0, \tag{A_{24}}$$

$$\theta_{25} + \theta_{27} + \theta_{29} + \theta_1 > 0, \tag{A_{25}}$$

$$\theta_{25} + \theta_{28} + \theta_1 > 0, \tag{B_{25}}$$

$$\theta_{27} + \theta_{22} > 0, \tag{A_{27}}$$

$$\theta_{27} + \theta_{29} > 0, \tag{B_{27}}$$

$$\theta_{28} > 0. \tag{A_{28}}$$

The bolded inequalities correspond to curves C with $\mathcal{N}_C$ not of type $(-1, -1)$. We know by [10, Theorem 9.12] that the other inequalities are necessary and define Type $\mathtt{I}$ walls of $\mathfrak{C}_0$. The inequalities from divisors parameterising rigid subsheaves are

$$\theta_1 > 0, \tag{A_1}$$

$$\theta_7 > 0, \tag{A_7}$$

$$\theta_{11} > 0, \tag{A_{11}}$$

$$\theta_{13} > 0, \tag{A_{13}}$$

$$\theta_{14} > 0, \tag{A_{14}}$$

$$\theta_{16} > 0, \tag{A_{16}}$$

$$\theta_{17} > 0, \tag{A_{17}}$$

$$\theta_{19} > 0, \tag{A_{19}}$$

$$\theta_{21} > 0, \tag{A_{21}}$$

$$\theta_{22} > 0, \tag{A_{22}}$$

$$\theta_{23} > 0, \tag{A_{23}}$$

$$\theta_{26} > 0, \tag{A_{26}}$$

$$\theta_{29} > 0. \tag{A_{29}}$$

We record the redundancies for the bold (or potentially redundant) inequalities:

$$
\begin{aligned}
(\mathbf{F_6}) + (A_8) + (A_9) + (A_{10}) + (B_{12}) + (A_{15}) + (A_{18}) + (A_{20}) + (A_{24}) & \\
+ (A_{25}) + (B_{27}) + (A_{28}) & \implies (\mathbf{A_6}), \\
(\mathbf{F_6}) + (A_8) + (A_9) + (A_{10}) + (B_{12}) + (A_{15}) + (A_{18}) + (A_{20}) + (A_{24}) & \implies (\mathbf{B_6}), \\
(\mathbf{F_6}) + (A_8) + (A_9) + (A_{10}) + (B_{12}) + (A_{15}) + (A_{18}) & \implies (\mathbf{C_6}), \\
(\mathbf{F_6}) + (A_8) + (A_9) + (A_{10}) + (B_{12}) & \implies (\mathbf{D_6}), \\
(\mathbf{F_6}) + (A_8) + (A_9) & \implies (\mathbf{E_6}), \\
(A_5) + (B_{12}) + (A_{18}) & \implies (\mathbf{B_{10}}), \\
(A_{10}) + (B_{12}) & \implies (\mathbf{C_{10}}), \\
(A_{15}) + (A_{18}) & \implies (\mathbf{B_{15}}), \\
(A_{15}) + (A_{18}) + (A_{10}) + (A_{24}) & \implies (\mathbf{C_{15}}), \\
(A_{15}) + (A_{18}) + (A_{10}) + (A_{24}) + (A_{27}) + (C_5) & \implies (\mathbf{D_{15}}), \\
(B_{20}) + (A_{24}) & \implies (\mathbf{C_{20}}), \\
(A_{15}) + (B_{20}) + (A_{24}) & \implies (\mathbf{D_{20}}).
\end{aligned}
$$

We have killed off the inequalities from all curves except for the $(-1, -1)$-curves and one curve $(\mathbf{F_6})$ from the long side.

4.5 Redundant inequalities from curves

Observe that the vast majority of inequalities in Examples 4.5–4.6 define walls of Type I. We should be unsurprised by the cancellation of all except one bolded inequality in Example 4.6 due to the following result from [10].

Lemma 4.7 ([10, Corollaries 6.3 and 6.5]). *Suppose $w = (\sum \alpha_i \theta_i = 0)$ is a Type I or III wall of $\mathfrak{C}_0$. Then all $\alpha_i \in \{0, 1\}$.*

Chambers other than $\mathfrak{C}_0$ can have coefficients $\alpha_i = -1$, however since the trivial representation does not appear in G-ig(C) for any curve C we can exclude this possibility.

Corollary 4.8. *Suppose G-Hilb $\mathbb{A}^3$ has a meeting of champions of side length 0. Then the inequality for any curve along one of the three champions is redundant.*

Proof. Suppose χ is the character marking each of the champions. Then, by Theorem 2.4, $r_\chi^2 = r_{\chi^2}$ globally on G-Hilb and so $\deg(\mathcal{R}_{\chi^2}|_C) = 2$ for all χ-curves C. It follows from Lemma 4.7 that none of these inequalities can be strict. ∎

We can also show this directly via unlocking. This reproves Corollary 4.3.

Lemma 4.9. *Suppose C is a χ-curve. If the unlocking procedure for C doesn't unlock a curve or divisor marked with χ^2 then all the coefficients in the inequality $\theta(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)) > 0$ are equal to 0 or 1.*

Proof. This is because if some ρ has $\deg(\mathcal{R}_\rho|_C) \geq 2$ then $r_\chi^2 \mid r_\rho$ and so r_χ^2 must feature in the G-igsaw piece for C and is hence equal to r_{χ^2} near C. ∎

Lemma 4.10. *Suppose a curve C_0 unlocks a curve C_1 of character ρ. Let $\psi \in G$-ig(C_1). If C is a curve that unlocks C_0, then $\deg(\mathcal{R}_\psi|_C) \geq \deg(\mathcal{R}_\rho|_C)$.*

Proof. As used previously, $\deg(\mathcal{R}_\rho|_C) = \max\{k \in \mathbb{Z}_{\geq 0}: r_\chi^k \mid r_\rho\}$. From this formulation, clearly if $r_\rho \mid r_\psi$ then $\deg(\mathcal{R}_\psi|_C) \geq \deg(\mathcal{R}_\rho|_C)$, but this is the case by definition of G-igsaw piece. ∎

Lemma 4.11. *Suppose C is a curve on the boundary of a regular triangle marked with a character χ. Suppose the χ-chain contains a $(-1, -1)$-curve. Then the inequality $\theta(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)) > 0$ is redundant.*

Proof. Suppose C is marked with character χ. Let C_0 be the first $(-1, -1)$-curve in the χ-chain moving inwards from C. Then the G-igsaw piece for C consists of exactly the characters in the G-igsaw piece for C_0 along with the characters in the G-igsaw pieces for any curves $C_1, \ldots, C_n$ unlocked by C at Hirzebruch divisors before C_0. Let the character marking C_i be χ_i. The inequality for C decomposes as

$$\theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)\big) = \sum_{\rho \in G\text{-ig}(C_0)} \alpha_\rho \theta(\rho) + \sum_{i=1}^{n} \sum_{\rho \in G\text{-ig}(C_i)} \beta_\rho^i \theta(\rho), \tag{4.2}$$

where α_ρ and β_ρ^i are nonnegative multiplicities given by the appropriate calculation of $\deg(\mathcal{R}_\rho|_?)$, possibly computing the degree of $\mathcal{R}_\rho$ on multiple curves. Note that $\alpha_\chi = 1$. One can thus write

$$\theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)\big) = \theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_{C_0})\big)$$
$$+ \sum_{\rho \in G\text{-ig}(C_0)} (\alpha_\rho - 1)\theta(\rho) + \sum_{i=1}^{m} \left(\beta_{\chi_i}^i \theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_{C_i})\big) + \sum_{\rho \in G\text{-ig}(C_i)} (\beta_\rho^i - \beta_{\chi_i}^i)\theta(\rho) \right).$$

From Lemma 4.10, $\alpha_\rho - 1$ and $\beta_\rho - \beta_{\chi'}$ are both nonnegative. If all the remaining ρ in these sums with nonzero coefficients after this reduction are characters marking divisors then one can express each term $\gamma_\rho \theta(\rho) = \gamma_\rho \theta\big(\varphi_{\mathfrak{C}_0}\big(\mathcal{R}_\psi^{-1}|_D\big)\big)$ for some divisor D, thus evidencing that (4.2) is redundant. Suppose instead that some $\rho = \rho_1$ marks a curve unlocked by C_0 or some C_i. We assume the latter; the former is treated identically. Denote this new curve by $C_{i,1}$. Then

$$\sum_{\rho \in G\text{-ig}(C_i)} \big(\beta_\rho^i - \beta_{\chi_i}^i\big)\theta(\rho) = \big(\beta_{\rho_1}^i - \beta_{\chi_i}^i\big)\theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_{C_{i,1}})\big)$$

$$+ \sum_{\rho \in G\text{-ig}(C_{i,1})} \big(\beta_\rho^i - \beta_{\rho_1}^i\big)\theta(\rho) + \sum_{\rho \notin G\text{-ig}(C_{i,1})} \big(\beta_\rho^i - \beta_{\chi_i}^i\big)\theta(\rho),$$

where again each coefficient is nonnegative by Lemma 4.10 applied to $C_{i,1}$. Observe that there are strictly fewer nonzero coefficients in this expression than before, since at the least we removed the term for ρ_1. Continuing in this way for each character appearing that marks a curve, we can reduce to the situation where the only characters with nonzero coefficients in the error term are those that mark divisors. At that point we have already seen how to express the error term in terms of inequalities coming from divisors, and so we have shown that (4.2) is redundant. ∎

4.6   Classifying Type III walls

We provide a combinatorial classification of the Type III walls for $\mathfrak{C}_0$. We start with the following definition.

Definition 4.12. Let χ be a character marking a curve in G-Hilb. We say that the χ-chain is a *generalised long side* if it starts and ends on the boundary of the junior simplex, and all the edges along the χ-chain are boundary edges of regular triangles. We exclude the lines meeting at a trivalent vertex if there is a meeting of champions of side length 0 from this definition.

For example, any long side is a generalised long side. The 15-chain for $\frac{1}{35}(1, 3, 31)$ is a generalised long side as can be seen in Fig. 25.

Example 4.13. We compute the inequalities for curves along the 15-chain in G-Hilb for $G = \frac{1}{35}(1, 3, 31)$. From the unlocking procedure or computing G-igsaw pieces directly, the inequalities for the 15-curves starting from e_1 and moving downwards are

$$\theta_{15} + \theta_{18} + \theta_{21} + \theta_{24} + \theta_7 + \theta_{10} + \theta_{13} + \theta_{16} + \theta_{11} + \theta_{14} + \theta_{17} + \theta_{20} > 0, \tag{A_{15}}$$
$$\theta_{15} + \theta_{18} + \theta_{21} + \theta_{16} + \theta_{11} + \theta_{14} + \theta_{17} > 0, \tag{B_{15}}$$
$$\theta_{15} + \theta_{16} + \theta_{17} + \theta_{18} > 0, \tag{C_{15}}$$
$$\theta_{15} + \theta_{16} + \theta_{17} + \theta_{18} > 0. \tag{D_{15}}$$

Clearly (C_{15}) and (D_{15}) depend on each other; the inequality is the same since they are fibres of the $\mathbb{P}^1$-bundle structure on the Hirzebruch surface marked with 18, and so contracting one must contract the other. We consider some of the additional inequalities coming from $(-1, -1)$-curves:

$$\theta_7 + \theta_{10} + \theta_{13} > 0, \tag{A_7}$$
$$\theta_{11} + \theta_{14} > 0, \tag{A_{11}}$$
$$\theta_{21} > 0, \tag{A_{21}}$$
$$\theta_{24} + \theta_{20} > 0. \tag{A_{24}}$$

We can deduce

$$(C_{15}) + (A_7) + (A_{11}) + (A_{21}) + (A_{24}) \Longrightarrow (A_{15}),$$
$$(C_{15}) + (A_{11}) + (A_{21}) \Longrightarrow (B_{15}),$$

so that (A_{15}) and (B_{15}) are redundant.

Definition 4.14. Consider a generalised long side marked with character χ. Recall that each χ-chain consists of potentially several straight line segments. We call a curve in the χ-chain *final* if it is the furthest curve along the χ-chain away from a vertex along such a line segment.

For example, for $G = \frac{1}{35}(1,3,31)$, the bolded curves in Fig. 28 are final.
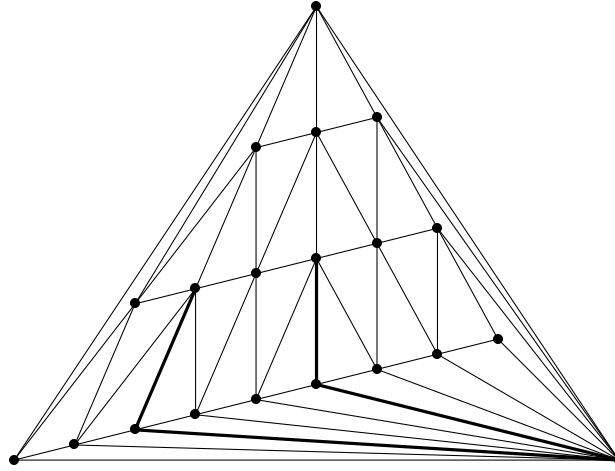

Figure 28. Final curves for $G = \frac{1}{35}(1,3,31)$.

Final curves not along a long side are also those contained in an exceptional Hirzebruch surface (with no blowups) or, equivalently, those corresponding to edges incident to a 4-valent vertex. There can be at most two final curves for each generalised long side, with exactly one when the generalised long side is actually a long side.

Lemma 4.15. *Suppose χ is a character marking a curve and that the χ-chain is a generalised long side. Then, the inequality for each non-final curve C in the χ-chain is redundant. The final curves all produce the same inequality:*

$$\theta(\chi) + \sum_{\psi \in \mathrm{Hirz}(\chi)} \theta(\psi) > 0,$$

which is a necessary inequality defining a Type III wall of $\mathfrak{C}_0$.

Proof. First, the inequality for a final χ-curve C features only the Hirzebruch divisors along the χ-chain by the unlocking procedure. It has all nonzero coefficients equal to 1 for the following reason. χ^2 cannot mark a Hirzebruch divisor along the χ-chain because to do so one would require another chain, say with character ρ, to cross the χ-chain and have $\chi \otimes \rho = \chi^2$. Of course, this would mean that $\rho = \chi$, but chains do not self-intersect. Hence, χ^2 does not appear in the G-igsaw piece for C and so all multiplicities must be equal to 1 by Lemma 4.9. This is clearly a necessary inequality, as χ is the only character in the inequality coming from a curve and there is no divisor that contains only χ-curves – in contrast to the case of a trivalent vertex.

To see that the other inequalities coming from curves along a generalised long side are redundant, we will decompose these inequalities similarly to before. Let C be such a curve and write

$$\theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_C)\big) = \theta(\chi) + \sum_{\psi \in \mathrm{Hirz}(C)} \alpha_\psi \theta(\psi) + \sum_{i=1}^{n} \sum_{\rho \in G\text{-ig}(C_i)} \beta_\rho^i \theta(\rho),$$

where $C_1, \ldots, C_n$ are the curves unlocked by C. By exactly the same methods as in the proof of Lemma 4.11, one can express the final term as a sum of inequalities from curves and divisors.

The first two terms are equal to

$$\theta(\chi) + \sum_{\psi \in \mathrm{Hirz}(C)} \alpha_\psi \theta(\psi) = \theta\big(\varphi_{\mathfrak{C}_0}(\mathcal{O}_{C'})\big) + \sum_{\psi \in \mathrm{Hirz}(\chi)} (\alpha_\psi - 1)\theta\big(\varphi_{\mathfrak{C}_0}\big(\mathcal{R}_\psi^{-1}|_{D_\psi}\big)\big),$$

where C' is a final χ-curve and D_ψ is the divisor marked with ψ. Of course $\alpha_\psi \geqslant 1$ and so we have shown that the inequality from C is redundant. ∎

We consider the example $G = \frac{1}{25}(1, 3, 21)$, which has a meeting of champions of side length 2.

Example 4.16. We show the triangulation for G-Hilb and Reid's recipe for $G = \frac{1}{25}(1, 3, 21)$ in Fig. 29. Observe that of the three champions, the 3-chain and 9-chain are generalised long sides whilst the 1-chain contains a $(-1, -1)$-curve. We hence obtain two Type III walls from the champions and another for the 21-chain that is also a generalised long side, with inequalities

$$\theta_3 + \theta_4 + \theta_8 + \theta_{12} + \theta_{16} + \theta_{20} + \theta_{24} > 0, \tag{F_3}$$

$$\theta_9 + \theta_{10} + \theta_{11} + \theta_{12} > 0, \tag{C_9}$$

$$\theta_{21} + \theta_{22} + \theta_{23} + \theta_{24} > 0. \tag{C_{21}}$$
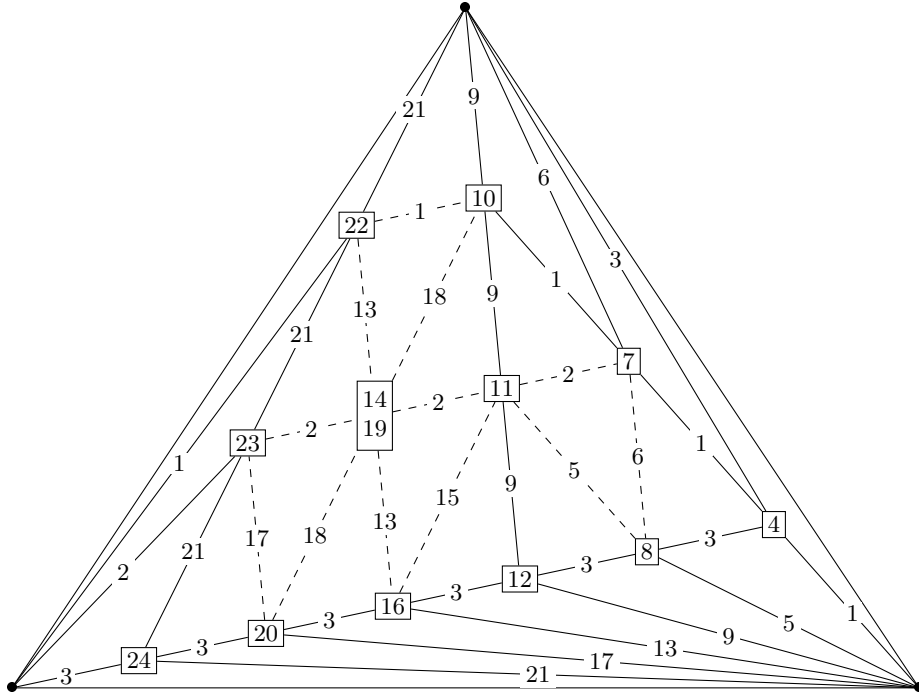

Figure 29. Reid's recipe for $G = \frac{1}{25}(1, 3, 21)$.

4.7 Summary

We compile the main results – Corollary 4.3, Proposition 4.4, Lemmas 4.11 and 4.15 – of this section.

Theorem 4.17. *Suppose $G \subset \mathrm{SL}_3(\mathbb{C})$ is a finite abelian subgroup. The walls of the chamber $\mathfrak{C}_0$ for G-Hilb $\mathbb{A}^3$ and their types are as follows:*

- *a Type I wall for each exceptional $(-1, -1)$-curve,*

- *a Type III wall for each generalised long side,*
- *a Type 0 wall for each irreducible exceptional divisor,*
- *each remaining wall is of Type 0 and comes from a divisor parameterising a rigid quotient.*

Moreover, for every contraction of Type I or III for G-Hilb $\mathbb{A}^3$ there is a wall of the corresponding type that induces the contraction by VGIT.

Proposition 2.6 describes how to recover the unstable locus or the corresponding reducible divisor D' for each wall of Type 0 from a rigid quotient. Let $\mathfrak{w}$ be a wall of $\mathfrak{C}_0$. Denote by $E(\mathfrak{w})$ the set of edges in the Craw–Reid triangulation corresponding to curves C for which all characters in G-ig(C) appear in the equation of the wall. The desired divisor D' inducing $\mathfrak{w}$ is then the union of the divisors corresponding to vertices for which all incident edges are in $E(\mathfrak{w})$. We observe that the unlocking procedure allows the check of which walls from rigid quotients are necessary to be performed combinatorially.

5 Future directions

There are several natural avenues of further study opened up by the results of this paper, three of which are

- attuning the results here with the derived interpretation of Reid's recipe [6],
- exploring any relations between analogs to Reid's recipe in other settings and walls in stability (for instance, dimer models [3, 15]),
- reverse-engineering a partial Reid's recipe for other resolutions $\mathcal{M}_{\mathfrak{C}}$ from an explicit description of the walls of $\mathfrak{C}$, and examining whether this has any categorical content.

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