

The Algebraic Structure of the KLT Relations for Gauge and Gravity Tree Amplitudes

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Abstract. We study the Kawai–Lewellen–Tye (KLT) relations for quantum field theory by reformulating it as an isomorphism between two Lie algebras. We also show how explicit formulas for KLT relations arise when studying rational functions on $\mathcal{M}_{0,n}$, and prove identities that allow for arbitrary rational functions to be expanded in any given basis. Via the Cachazo–He–Yuan formulas for, these identities also lead to new formulas for gauge and gravity tree amplitudes, including formulas for so-called Bern–Carrasco–Johansson numerators, in the case of non-linear sigma model and maximal-helicity-violating Yang–Mills amplitudes.

Key words: perturbative gauge theory; double copy; binary trees; Lie coalgebras; Lie polynomials

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1 Introduction

The *Kawai–Lewellen–Tye (KLT) relations* for field theory amplitudes express the tree amplitude of a gravity theory as a quadratic expression in the tree amplitudes of a gauge theory. The existence of such a relation is motivated by the *string KLT relation* between open and closed string tree amplitudes. These relations were proposed in [30], and explicit formulas for the components of the KLT matrix were derived more recently in [6]. The KLT relation for string amplitudes resembles the Riemann period relations, because the quadratic relation is obtained by taking the inverse of a matrix of the intersection numbers (in a certain homology theory) of $\mathcal{M}_{0,n}$, the moduli space of n points in \mathbb{P}^1 . The KLT relations are studied from this point of view in [10, 37]. There has been work done to understand the field theory KLT relation in similar terms [38, 39].

This article studies the field theory KLT relation directly, without reference to $\mathcal{M}_{0,n}$, to find a simple algebraic interpretation of the KLT matrix. The relationship to $\mathcal{M}_{0,n}$ is then exploited to find new formulas for some gauge theory tree amplitudes. The formulas obtained are in a special form, which manifests the numerators of Bern–Carrasco–Johansson (BCJ) [4]. The rest of this introduction gives a brief review of the field theory KLT relations, and then summarizes the results of the paper.

1.1 Partial amplitudes and the field theory KLT relations

The *colour factors* of a gauge theory amplitude organize the amplitude into *partial amplitudes* corresponding to surfaces with boundary marked points. For an $SU(N)$ gauge theory, with

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coupling constant g_{YM} , the full amplitude may be written as

$$A_n = (ig_{\text{YM}})^{n-2} \sum_{\ell=0}^{\infty} \lambda^\ell \sum_{h,p,g} \left(\frac{1}{N}\right)^{2g+h-1} A_{p,g,h}, \quad (1.1)$$

where $\lambda = g_{\text{YM}}^2 N$ is the 't Hooft coupling [46], and where $A_{p,g,h}$ is the sum of partial amplitudes corresponding to surfaces of genus g , with p punctures and h boundary components. The second sum in (1.1) is constrained by the relation $p + 2g + h = \ell + 1$.

Consider Yang–Mills (YM) gauge theory, with gluons in the matrix representation of $\mathfrak{su}(N)$. Fix some distinct $\lambda_1, \dots, \lambda_n \in \mathfrak{su}(N)$, corresponding to the external gluon colour states. A cubic Feynman tree diagram, α , gives a contribution to the tree amplitude which is proportional to

$$c_\alpha := \text{tr}(\alpha[\lambda_1, \dots, \lambda_{n-1}]\lambda_n), \quad (1.2)$$

where tr is the Killing form, and $\alpha[\lambda_1, \dots, \lambda_{n-1}]$ is the Lie bracketing of the λ_i according to the tree α , regarded as a rooted binary tree, with root n .¹ A Feynman tree diagram that contains quartic vertices gives a contribution which is a sum of terms, each proportional to c_α for some binary rooted tree α .² It is therefore possible to write the n -point tree amplitude as

$$A_{\text{YM}}^{\text{tree}} = \sum_{\substack{\text{trees} \\ \alpha}} A_\alpha c_\alpha,$$

where the sum is over all binary trees α , with external edges labelled by $1, \dots, n$. The coefficients A_α that appear in the sum depend only on the gluon momenta and polarizations. The colour factors c_α may be expanded as a sum over permutations:

$$c_\alpha = \sum_{a \in \mathfrak{S}_{n-1}} (a, \alpha) \text{tr}(\lambda_{a(1)} \dots \lambda_{a(n-1)} \lambda_n), \quad (1.3)$$

where \mathfrak{S}_{n-1} is the set of permutations. The bracketing (a, α) denotes the coefficient of the ordering a in the expansion of the monomial α , which can be either $+1$, -1 , or 0 . The partial amplitude expansion of $A_{\text{YM}}^{\text{tree}}$ can therefore also be written as a sum over permutations:

$$A_{\text{YM}}^{\text{tree}} = \sum_{a \in \mathfrak{S}_{n-1}} A(a, n) \text{tr}(\lambda_{a(1)} \dots \lambda_{a(n-1)} \lambda_n),$$

where $A(a, n)$ is the sum over binary trees,

$$A(a, n) = \sum_{\substack{\text{trees} \\ \alpha}} (a, \alpha) A_\alpha.$$

Given the gauge theory partial tree amplitudes, the *field theory KLT relation* expresses the n -point gravity tree amplitude M_n as a quadratic expression of the following form [6, 7]

$$M_n = \lim_{k_n^2 \rightarrow 0} \sum_{a,b \in \mathfrak{S}_{n-2}} \frac{A(1an)S(1a, 1b)A(\bar{b}1n)}{k_n^2}, \quad (1.4)$$

where the matrix entries $S(1a, 1b)$ depend only on the gluon momenta, k_i^μ . It is understood that, $S(1a, 1b)$ is defined ‘off-shell’, in the sense that it is valid for $k_n^2 \neq 0$. When $k_n^2 = 0$, $S(1a, 1b)$ is

¹A binary rooted tree with k leaves determines and is determined by a Lie bracketing of k variables, up to a sign.

²The cubic diagrams that appear in this sum can be found by considering all the possible ways to ‘expand’ the quartic vertices into a subdiagram with two cubic vertices.

a $(n-2)! \times (n-2)!$ matrix of full rank (as explained in Section 2). Its rank drops by 1 when $k_n^2 = 0$. The explicit formula for the entries of $S(1a, 1b)$ is given by [6]

$$S(1a, 1b) := \prod_{i=2}^{n-1} \left(\sum_{\substack{j <_{1a} i \\ j >_{1b} i}} s_{ij} \right), \quad (1.5)$$

where the variables

$$s_{ij} = 2k_i \cdot k_j$$

are the Mandelstam variables associated to the gluon momenta. For fixed i , the sum in this formula is over all j that both precede i in $1a$, and are preceded by i in $1b$. The notation \bar{b} denotes the reversal of the ordering b .

This formula, (1.4), was originally derived in [6] using an argument from the properties of the open string integral. [24] showed that the ‘off-shell’ KLT matrix, (1.5), is the matrix inverse of a matrix of ‘Berends–Giele’ currents for bi-adjoint scalar theory. Namely, define a $(n-2)! \times (n-2)!$ matrix, T :

$$T(1a, 1b) = \sum_{\substack{\text{trees} \\ \alpha}} \frac{(1a, \alpha)(1b, \alpha)}{k_n^2 s_\alpha},$$

where a and b are permutations of $2, \dots, n-1$, and the sum is over all binary trees. The brackets $(1a, \alpha)$ are defined as in (1.3), above. The denominator, s_α , is the *product of propagators*.³ Then, thinking of $S(1a, 1b)$ as an $(n-2)! \times (n-2)!$ matrix, S :

Proposition 1.1. *The matrices S and T are inverse: $ST = TS = \text{Id}$, i.e.,*

$$\sum_{b \in \mathfrak{S}_{n-2}} T(1a, 1b)S(1b, 1c) = \sum_{b \in \mathfrak{S}_{n-2}} S(1a, 1b)T(1b, 1c) = \delta_{b,c}, \quad (1.6)$$

where $\delta_{b,c}$ is the identity matrix.

Section 2 gives a basis-independent statement of this result, and gives a streamlined version of the proof in [24].

1.2 Summary

This section summarizes the new results in Sections 3 and 4, which build on the approach taken in Section 2, and culminate in formulas for some gauge and gravity tree amplitudes. The proof of (1.6) involves a bracket operation (originally called the ‘S-map’ in [33, 34]) defined on orderings. Some examples are

$$\{1, 23\} = s_{12}123 - s_{13}132, \quad \{12, 34\} = s_{23}1234 - s_{13}2134 + s_{14}2143 - s_{24}1243. \quad (1.7)$$

The definition of $\{a, b\}$ is given in Section 2, where it is also shown that $\{, \}$ is a Lie bracket. As explained in Section 3, and exploited in Section 4, the bracket operation $\{a, b\}$ naturally arises in the context of rational functions on the configuration space, $\text{Conf}_{n-1}(\mathbb{C})$, of points in the complex plane. Write $\text{pt}(12\dots n)$ for the function

$$\text{pt}(12\dots n) = \prod_{i=1}^{n-1} \frac{1}{z_i - z_{i+1}}.$$

Then (proved in Section 3)

³ s_α is a polynomial expression in Mandelstam variables, determined by the tree α ; see (2.3).

Proposition 1.2. For an ordering a of $1, \dots, k$, and an ordering b of $k+1, \dots, n$:

$$\left(\sum_{i=1}^k \sum_{j=k+1}^n \frac{s_{ij}}{z_i - z_j} \right) \text{pt}(a)\text{pt}(b) = \text{pt}(\{a, b\}). \quad (1.8)$$

When combined with the definition of $S(1a, 1b)$, (1.8) can be used to derive a formula for the partial tree amplitudes of the non-linear sigma model (the gauge theory associated to maps from the Riemann sphere to the Lie group $\text{SU}(N)$). The result is

Proposition 1.3. The NLSM tree partial amplitudes are given by

$$A_{\text{NLSM}}(a, n) = \sum_{b \in \mathfrak{S}_{n-2}} S(1b, 1b) m(1b, n|a, n), \quad (1.9)$$

where the sum is over all permutations of $2, 3, \dots, n-1$, and $m(a, n|b, n) = k_n^2 T(1a, 1b)$ are the partial biadjoint scalar amplitudes.

This agrees with earlier results reported in [16, 32]; but the methods used to derive it are new. The idea that leads to (1.9) is to use the matrix tree theorem to expand the matrix determinant that appears in the integrand of the so-called CHY formula for A_{NLSM} . Identities proved in Section 3 are then used to re-arrange the integrand into a suitable form. The most useful identity is

Proposition 1.4. Let G be a tree with vertex set $1, \dots, n$. Orient the edges of G by fixing 1 to be a sink, and demanding that all vertices (except for 1) have only one outgoing edge. For $i \neq 1$, write $x(i)$ for the endpoint of the edge outgoing from i . Then

$$\prod_{\substack{\text{edges} \\ i \rightarrow j}} \frac{1}{z_{ij}} = \sum_{x(i) <_{1a} i} \text{pt}(1a). \quad (1.10)$$

The sum is over all orderings, a , such that $x(i)$ precedes i in $1a$, for all i .

The identity, (1.10), can also be used to obtain formulas for other tree amplitudes that have CHY formulas (or similar). In four dimensions, gravity tree amplitudes can be expressed in terms of determinants of matrices called Hodges' matrices. Applying the same idea that leads to (1.9) also leads to new formulas for 4D gravity (and then, by the KLT relations, for Yang–Mills) amplitudes. These formulas are not always easy to evaluate. However, in the MHV case (when only 2 gluons are + helicity, and the rest are – helicity), contact can be made with known results. The MHV result, expressed in the standard spinorial notation used, is

Proposition 1.5. The tree level gravity amplitude can be expanded as

$$M_{\text{GR}} = \sum_{\sigma \in \mathfrak{S}_{n-3}} A_{\text{YM}}(12b\sigma) N(12b\sigma), \quad (1.11)$$

where

$$N(12b\sigma) = \frac{\langle 12 \rangle [12]^2}{[b1][b2]} \prod_{\substack{j=3 \\ i \neq b}}^n \sum_{i <_{\sigma} j} \frac{[i2][ij]}{[j2]}.$$

This is closely related to the early result due to Berends–Giele–Kuijff [3], which gives a formula for M_{GR} , but in a different form. The methods used in the derivation lead directly to expressions of the form of (1.11), which is well suited to the KLT relations. Indeed, the field theory KLT relations imply that, given (1.11), the YM partial amplitudes can be written as

$$A_{\text{YM}}(\rho) = \sum_{\sigma \in \mathfrak{S}_{n-3}} m(12b\sigma|\rho)N(12b\sigma).$$

Finally, the discussion in Section 5 concludes by relating the approach taken in this paper to three important outstanding problems for our understanding of perturbative gauge theory amplitudes and the KLT relations.

2 The KLT kernel

The KLT kernel, (1.5), is the inverse of a certain map, as shown in [24]. This section revisits this result, emphasizing those aspects which are relevant for Sections 3 and 4.

To fix notation, write A for the set $\{1, \dots, n\}$. An *ordering* of A is a word that uses each letter $i \in A$ exactly once. Write \mathfrak{S}_A for the set of orderings of A , and W_A for the \mathbb{R} -vector space generated by \mathfrak{S}_A . There is a multilinear inner product, $(\ , \)$, on W_A , such that, for two distinct orderings a and b ,

$$(a, a) = 1, \quad \text{and} \quad (a, b) = 0.$$

Let $\mathcal{L}_A \subset W_A$ be the subspace of *multilinear Lie polynomials* in W_A .⁴ And let $\text{Sh}_A \subset W_A$ be the subspace of nontrivial shuffle products: Sh_A is linearly spanned by expressions of the form

$$a \sqcup b,$$

where a and b are two (non-empty) words whose concatenation, ab , is an ordering of A .⁵ By Ree’s theorem [41], \mathcal{L}_A is the orthogonal subspace

$$\mathcal{L}_A = \text{Sh}_A^\perp,$$

with respect to the given inner product. The dual vector space \mathcal{L}_A^\vee is

$$\mathcal{L}_A^\vee = W_A / \text{Sh}_A.$$

Remark 2.1. An alternative definition of Sh_A is as follows. It is the subspace in W_A spanned by the expressions

$$aib - (-1)^{|a|}i(\bar{a} \sqcup b), \tag{2.1}$$

where aib is an ordering in \mathfrak{S}_A , and $i \in A$ is a single letter, and $|a|$ denotes the length of the word a . See [44, Corollary 2.4] or [23, Lemma 3.6].

A Lie monomial $\alpha \in \mathcal{L}_A$ defines a *rooted binary tree* with leaves labelled by $1, \dots, n$. See Figure 1 for an example. If the Lie monomial α is written in its bracketed form, then the associated tree has one internal edge for each pair of brackets in α , and this edge can be labelled

⁴If $\text{Lie}(A)$ is the free Lie algebra on A , then \mathcal{L}_A is the intersection $\text{Lie}(A) \cap W_A$, using the inclusions of W_A and $\text{Lie}(A)$ into the free associative algebra on A .

⁵The shuffle product, $a \sqcup b$, is defined inductively by $(ia) \sqcup (jb) = i(a \sqcup jb) + j(ia \sqcup b)$, and $a \sqcup e = e \sqcup a = a$, where i, j are individual letters, e is the empty word, and a, b are nonempty words.

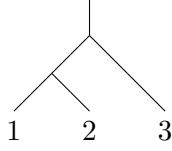


Figure 1. The labelled rooted binary tree corresponding to the Lie monomial $\pm[[1, 2], 3]$.

by the subset $I \subset A$ of letters that appear inside that pair of brackets. Write $P(\alpha)$ for the set of edges of α (including the root edge). For example,

$$P([[1, 2], 3]) = \{\{12\}, \{123\}\}.$$

Given null momenta k_i^μ for each $i = 1, \dots, n$ ($k_i^2 = 0$), form the associated Mandelstam variables

$$s_I = \left(\sum_{i \in I} k_i^\mu \right)^2,$$

or, equivalently,

$$s_I = \sum_{\{ij\} \subset I} s_{ij}, \quad (2.2)$$

where $s_{ij} = 2k_i \cdot k_j$. Given a tree α , associate to each edge, $I \in P(\alpha)$, a massless scalar propagator: $1/s_I$. For each rooted binary tree, α , introduce the following ‘product of propagators’ monomial

$$\tilde{s}_\alpha = \prod_{I \in P(\alpha)} s_I. \quad (2.3)$$

Finally, it is convenient to write M_A for the Laurent ring with variables s_I , $I \subset A$, subject to the linear relations above, (2.2).

For an ordering $a \in \mathfrak{S}(A)$, write

$$T(a) = \sum_{\alpha} \frac{(a, \alpha)\alpha}{\tilde{s}_\alpha} \in \mathcal{L}_A \otimes M_A,$$

where the sum is over all rooted binary trees with leaves labelled by A . $T(a)$ is a ‘prototype’ of a gauge theory partial tree amplitude. Kapranov [29] proposed to study the KLT relation by regarding $T(a)$ as defining a linear map

$$T: \mathcal{L}_A^\vee \otimes M_A \rightarrow \mathcal{L}_A \otimes M_A.$$

The map T is self-adjoint with respect to the pairing between \mathcal{L}_A^\vee and \mathcal{L}_A ,

$$(a, T(b)) = (b, T(a)). \quad (2.4)$$

Moreover, the functions $1/\tilde{s}_\alpha$ are linearly independent in M_A , and this implies that $\ker T$ is trivial. Indeed, for $P \in \mathcal{L}_A^\vee$, if $T(P) = 0$, then $(P, \alpha) = 0$ for all Lie monomials α . By dimension counting, T is onto, so it follows that T is an isomorphism.

Write $\tilde{\mathcal{L}}_A = \bigoplus_{B \subset A} \mathcal{L}_B$, and so on. T extends to an isomorphism

$$\tilde{T}: \tilde{\mathcal{L}}_A^\vee \otimes M_A \rightarrow \tilde{\mathcal{L}}_A \otimes M_A.$$

$\tilde{\mathcal{L}}_A \otimes M_A$ is a Lie algebra with the usual bracket.⁶ The isomorphism \tilde{T} then induces a Lie bracket on the dual vector space! Indeed, define a bracket operation, $\{ , \}$, on $\mathcal{L}_A^\vee \otimes M_A$ by

$$T(\{a, b\}) = [T(a), T(b)], \quad (2.5)$$

for $a, b \in \mathcal{L}_A^\vee$. Since \tilde{T} is an isomorphism, $\{ , \}$ is a Lie bracket. In fact, for disjoint orderings a and b , it can be shown that, explicitly

$$\{a, b\} = \sum_{\substack{a=a_1 a_2 \\ b=b_1 b_2}} (-1)^{|a_2|+|b_1|} s_{ij}(a_1 \sqcup \bar{a}_2) i j (\bar{b}_1 \sqcup b_2). \quad (2.6)$$

Examples of this were given above, in (1.7). See also [24, Section 4] or [23, Chapter 4].

For a Lie monomial α , written in bracketed form, let $S(\alpha) \in \mathcal{L}_A^\vee \otimes M_A$ be obtained from α by replacing every pair of brackets with a pair of braces. For example,

$$S([[1, 2], 3]) = \{\{1, 2\}, 3\}.$$

This extends to define a linear map

$$S: \mathcal{L}_A \otimes M_A \rightarrow \mathcal{L}_A^\vee \otimes M_A.$$

Repeated applications of (2.5) gives that

$$T(S(\alpha)) = \alpha.$$

This implies that S is self-adjoint:

$$(\beta, S(\alpha)) = (T(S(\beta)), S(\alpha)) = (S(\beta), T(S(\alpha))) = (S(\beta), \alpha),$$

using (2.4).

Proposition 2.2. *T and S are inverses.*

Proof. Let b_i, β_i be a pair of dual bases for \mathcal{L}_A and \mathcal{L}_A^\vee (with $i = 1, \dots, (n-1)!$). For an ordering $a \in W_A$,

$$S(T(a)) = \sum_i S(\beta_i)(b_i, T(a)) = \sum_{i,j} b_j(S(\beta_i), \beta_j)(b_i, T(a)).$$

Using that S is self-adjoint gives

$$rhs = \sum_{i,j} b_j(\beta_i, S(\beta_j))(b_i, T(a)) = \sum_j b_j(T(a), S(\beta_j)) = \sum_j b_j(a, \beta_j).$$

But b_i, β_i is a pair of dual bases, so $S(T(a)) = a$. ■

Fixing a pair of dual bases as above, the components of T and S are $(b_j, T(b_i)) = T_{ij}$ and $(\beta_j, S(\beta_i)) = S_{ij}$. These are $(|A|-1)! \times (|A|-1)!$ matrices, and the proposition says that S_{ij} is the matrix inverse of T_{ij} . This very simple definition of S was missed in the literature, possibly because of the limit that appears in (1.4). Moreover, notice that, trivially,

$$T_{il} = \sum_{j,k} T_{ij} S_{jk} T_{kl}. \quad (2.7)$$

⁶But note that the bracket of two Lie monomials must be zero in $\tilde{\mathcal{L}}_A$ if they share any letter in common.

This will be seen to imply the field theory KLT relations for the gauge and gravity-like theories studied in Section 4.

A possible choice of dual bases for \mathcal{L}_A and \mathcal{L}_A^\vee is to take the $(n-1)!$ words $1b$, for each b an ordering in $\mathfrak{S}(2, \dots, n)$, and dually the $(n-1)!$ Lie monomials

$$\ell(1b) = [[\dots [[1, b(1)], b(2)], \dots], b(n-1)] \in \mathcal{L}_A,$$

for each $b \in \mathfrak{S}(2, \dots, n)$. These are dual bases because

$$(1b, \ell(1b')) = \begin{cases} 1 & \text{if } b = b', \\ 0 & \text{otherwise.} \end{cases}$$

In these bases, the components of S are found to be [24]

$$S(1a, 1b) = (\ell(1b), S(\ell(1c))) = \prod_{i=2}^{n-1} \sum_{\substack{j <_{1b} i \\ j <_{1c} i}} s_{ij}. \quad (2.8)$$

This formula, discussed in the introduction, is the formula first found by [6] (albeit with the order of one of the words reversed). Many variations on this formula (including the original formula in [6]) can be obtained by choosing to compute the matrix elements of S using a different pair of bases.

3 Scattering equations identities

The ‘Cachazo–He–Yuan (CHY)’ formulas express the partial tree amplitudes of several gauge theories as a sum of residues of logarithmic forms on $\mathcal{M}_{0,n}$, as reviewed in Section 4.1. The logarithmic forms on $\mathcal{M}_{0,n}$ satisfy algebraic identities that imply, via the CHY formulas, identities among the associated gauge theory tree amplitudes. This section concludes by proving one such identity, Proposition 3.4, which is used in applications to gauge theory amplitudes in Section 4.

The open stratum of the moduli space $\mathcal{M}_{0,n}$ is defined as

$$\mathcal{M}_{0,n}(\mathbb{C}) = (\mathbb{C}\mathbb{P}^1)_*^{\oplus n} / \mathrm{PSL}_2\mathbb{C},$$

where $(\mathbb{C}\mathbb{P}^1)_*^{\oplus n}$ denotes n -tuples of pairwise distinct points in $\mathbb{C}\mathbb{P}^1$, and $\mathrm{PSL}_2\mathbb{C}$ acts by Möbius transformations. Write \mathbb{C}_{n-1}^* for the braid hyperplane arrangement, $\mathbb{C}_*^{n-1} := \mathbb{C}^{n-1} - \Delta$, where Δ is the big diagonal (i.e., the union of the hyperplanes $z_i - z_j = 0$). The open stratum of $\mathcal{M}_{0,n}$ is the quotient of this by the free action of $\mathbb{C}^* \times \mathbb{C}$, that acts as $(a, b): z \mapsto az + b$,

$$\mathcal{M}_{0,n}(\mathbb{C}) \simeq \mathbb{C}_*^{n-1} / \mathbb{C}^* \times \mathbb{C}.$$

This follows by setting $z_n = \infty$, and noticing that the stabilizer in $\mathrm{PSL}_2\mathbb{C}$ of a point in \mathbb{P}^1 is $\mathbb{C}^* \times \mathbb{C}$.

Write $z_{ij} := z_i - z_j$. In the ring of rational functions on \mathbb{C}_*^{n-1} , there is a natural submodule spanned by the *broken Parke–Taylor* functions, $\mathrm{pt}(a)$, defined for a given word $a = 123 \dots n-1$, as the product

$$\mathrm{pt}(123 \dots n-1) = \prod_{i=1}^{n-2} \frac{1}{z_{ii+1}}.$$

It can be shown (by an explicit induction, or by using general results from [9]) that these functions satisfy

$$\text{pt}(a \sqcup b) = 0, \quad (3.1)$$

for any two disjoint (non-empty) words a and b (i.e., two words with no letters in common). In view of (2.1), (3.1) further implies that

Lemma 3.1. *For an ordering $a1b \in \mathfrak{S}(1, \dots, n-1)$,*

$$\text{pt}(a1b) = (-1)^{|a|} \text{pt}(1(\bar{a} \sqcup b)).$$

This lemma implies that open string partial tree amplitudes satisfy the so-called Kleiss–Kuijff relations [31].

If a and b are two disjoint words, then for any $i \in a$ and $j \in b$, the Lemma implies that

$$\frac{1}{z_{ij}} \text{pt}(a) \text{pt}(b) = (-1)^{|a_2|+|b_1|} \text{pt}((a_1 \sqcup \bar{a}_2)ij(\bar{b}_1 \sqcup b_2)),$$

where $a = a_1ia_2$ and $b = b_1jb_2$. Recalling (2.6), this implies that

Lemma 3.2. *For a and b disjoint words as above,*

$$\text{pt}(\{a, b\}) = \text{pt}(a) \text{pt}(b) E_{a,b},$$

where

$$E_{a,b} = \sum_{i \in a, j \in b} \frac{s_{ij}}{z_{ij}}.$$

Note that $i \neq j$ for each term in the sum.

The equations $E_{a,b} = 0$ are known as the *scattering equations*.⁷ Fix $A = \{1, \dots, n-1\}$, and take the ordering $a = 123 \dots n-1$. Then repeated applications of Lemma 3.2 give that

$$\prod_{i=2}^{n-1} E_{i,123 \dots i-1} = \sum_{b \in \mathfrak{S}(2 \dots n-1)} S(12 \dots n-1, 1b) \text{pt}(1b).$$

In this way, the components of the KLT matrix S can be recovered from products of the functions $E_{a,b}$.

Remark 3.3. Lemma 3.2 can also be used [11, 33] to show that gauge theory partial tree amplitudes satisfy the *fundamental BCJ relations* [4]

$$A_{\text{YM}}(\{i, a\}, n) = \sum_{a=bjc} s_{ij} A_{\text{YM}}(ij(\bar{b} \sqcup c), n) = 0. \quad (3.2)$$

The open string partial tree amplitudes satisfy a more complicated relation of the form [7, 45]

$$(\alpha')^{(n-3)} A_{\text{string}}(\{i, a\}, n) = O(\alpha').$$

⁷Let ab be an ordering of $1, \dots, n$, then the functions $E_{a,b}$ arise as derivatives of the *Koba–Nielsen function*,

$$f_s(z) = \prod_{i=1, i < j}^n z_{ij}^{s_{ij}}.$$

Indeed,

$$E_{a,b} = \sum_{i \in a} \frac{\partial f_s(z)}{\partial z_i}.$$

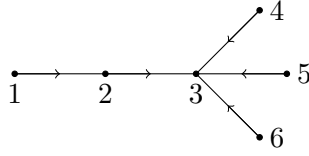


Figure 2. The tree associated to the rational function $1/z_{12}z_{23}z_{34}z_{35}z_{36}$; and the orientation induced by designating 3 a sink.

Let G be any spanning tree on the vertex set $\{1, 2, \dots, n-1\}$. Designate 1 to be the ‘sink’ of G . Then there is a unique assignment of directions to the edges of G , such that exactly *one* edge incident on any vertex i is outgoing, except for vertex 1, which has only incoming edges (see Figure 2). For a given vertex i in G , let $x(i)$ be the vertex connected to i by the one outgoing edge from i .

Proposition 3.4. *Let G be as above, and fix 1 to be the sink. Then the rational function associated to G is the following product over the edges of G :*

$$\prod_{\substack{\text{edges} \\ i \rightarrow j}} \frac{1}{z_{ij}} = \prod_{i=2}^{n-2} \frac{1}{z_{ix(i)}}, \quad (3.3)$$

and this can be expanded as a sum

$$\prod_{\substack{\text{edges} \\ i \rightarrow j}} \frac{1}{z_{ij}} = \sum_a \text{pt}(1a), \quad (3.4)$$

$x(i) <_{1a} i$

The sum is over all orderings, a , such that $x(i)$ precedes i in $1a$, for all i .

Proof. This follows by repeated applications of Lemma 3.1. The orientation of G induces a partial order on the vertices, with 1 the smallest. Let i be one of the largest vertices with valence greater than 1, and suppose that i has k incoming edges. All vertices greater than i have valence 1, so that the tree ‘greater than i ’ is comprised of some number of ‘branches’, as in Figure 3. By the lemma,

$$\text{pt}(ia)\text{pt}(ib) = (-1)^{|b|} \text{pt}(\bar{b}ia) = \text{pt}(i(a \sqcup b)),$$

and a product of k branches gives

$$\text{pt}(ia_1)\text{pt}(ia_2) \cdots \text{pt}(ia_k) = \text{pt}(i(a_1 \sqcup a_2 \sqcup \cdots \sqcup a_k)).$$

Moving ‘down the tree’ gives the identity. ■

Remark 3.5. Formulas related to (3.4) appear in the discussion of hyperplane arrangements in [43]. The functions, (3.3), associated to a spanning tree G are studied in [27], with interesting applications to CHY formulas.

4 Gauge and gravity tree amplitudes in BCJ form

This section uses the identities in Section 3 to obtain formulas for the tree partial amplitudes of the non-linear sigma model (NLSM), and also Yang–Mills in four dimensions. Moreover, the field theory KLT relation makes it possible to derive formulas also for the tree amplitudes of

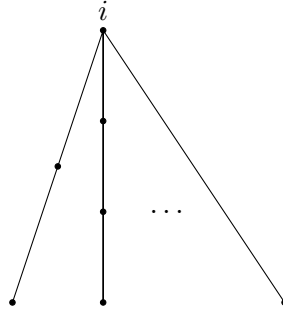


Figure 3. Outermost branches of a tree.

Einstein gravity, and Dirac-Born-Infeld theory, which are the gravity theories associated to YM and NLSM, respectively. There have been several previous studies that obtain formulas of this kind from CHY integrals, including [5, 21, 27]. The approach taken here is novel, and combines applications of the matrix tree theorem with the identities proved in Section 3.

4.1 CHY formulas

The identities in Section 3 are relevant to the problem of computing tree amplitudes. This is because of the CHY formulas for the partial tree amplitudes of NLSM and YM. In [12], these formulas are written as integrals of the form

$$A = \int d\mu(a) \left(z_{ij} z_{jk} z_{ki} \prod_{\substack{l=1 \\ l \neq i, j, k}}^n \delta(E_l) \right) I, \quad (4.1)$$

for functions I of appropriate weight under the action of $\mathrm{SL}_2\mathbb{C}$, and for any fixed choice of i, j, k . $\mathrm{SL}_2\mathbb{C}$ acts by Möbius transformations on the coordinates z_i . It is convenient to write

$$z_{ij} z_{jk} z_{ki} \prod_{\substack{l=1 \\ l \neq i, j, k}}^n \delta(E_l) = \prod_l' \delta(E_l),$$

for any choice of i, j, k . The functions

$$E_i = E_{i, 12 \dots \hat{i} \dots n} = \sum_{\substack{j=1 \\ j \neq i}}^n \frac{s_{ij}}{z_{ij}} \quad (4.2)$$

are the scattering equation functions introduced in Section 3. The natural logarithmic top forms on $(\mathbb{CP}^1)_*^n$ induce volume forms, $d\mu(a)$, on $\mathcal{M}_{0,n}$: for $a = 12 \dots n$, the associated $\mathcal{M}_{0,n}$ top form is⁸

$$d\mu(123 \dots n) = \frac{1}{\mathrm{Vol} \mathrm{PSL}_2\mathbb{C}} \bigwedge_{i=1}^n d \log z_{ii+1},$$

⁸For any choice of a top form, $\mathrm{Vol} \mathrm{PSL}_2\mathbb{C}$, on the fibres of the projection $(\mathbb{CP}^1)_*^n \rightarrow \mathcal{M}_{0,n}$, such as, for example,

$$z_{ij} z_{jk} z_{ki} dz_i dz_j dz_k,$$

for any distinct i, j, k .

so that $d\mu(123\dots n)$ is a $n-3$ top form on $\mathcal{M}_{0,n}$ (regarded as the quotient of $(\mathbb{CP}^1)_*$ by $\mathrm{PSL}_2(\mathbb{C})$). Or, choosing the gauge fixing $z_1 = 0, z_{n-1} = 1, z_n = \infty$,

$$d\mu(123\dots n) = \frac{dz_2 dz_3 \cdots dz_{n-2}}{\prod_{i=1}^{n-2} z_{ii+1}} \Big|_{z_1=0, z_{n-1}=1}.$$

The formula, (4.1), must be understood as a sum of residues at the solutions of $E_i = 0$. This can be written as

$$\int d\mu(a) \left(\prod_l' \delta(E_l) \right) I = \sum_{\substack{\text{solutions} \\ (z)}} \mathrm{Res}_{(z)} \left(\frac{d\mu(a)I}{\det(\partial_i E_j)} \right),$$

where the sum is over all solutions, (z_i^*) , to the equations $E_i = 0$, and $\mathrm{Res}_{(z)}$ denotes the Poincaré residue at some given solution, (z_i^*) . It is left implicit in these formulas that the residue at a solution is oriented by the form

$$+d \log E_1 \wedge \cdots \wedge d \log E_{n-3}.$$

An important example is the CHY formula for the *biadjoint scalar* partial tree amplitudes. In the notation of Section 2, for two orderings $a, b \in \mathfrak{S}(1, 2, \dots, n-1)$, these partial amplitudes are

$$m(a, n|b, n) = s_a(a, T(b)) = \sum_{\substack{\text{trees} \\ \alpha}} \frac{(a, \alpha)(b, \alpha)}{s_\alpha},$$

where the sum is over all rooted binary trees with $n-1$ labelled external edges (not including the root). The CHY formula for these amplitudes is,

$$m(a, n|b, n) = \int d\mu(a) \left(\prod_l' \delta(E_l) \right) \mathrm{PT}(b, n), \quad (4.3)$$

where the ‘Parke–Taylor function’ $\mathrm{PT}(a, n)$ is, for $a = 12\dots n-1$,

$$\mathrm{PT}(a, n) = \frac{1}{z_{12} z_{23} \cdots z_{n-1n} z_{n1}}.$$

Equation (4.3) is proved in [20].

4.2 Nonlinear sigma model

Define an $n \times n$ matrix, A , with off-diagonal entires ($i \neq j$)

$$A_{ij} = \frac{s_{ij}}{z_{ij}},$$

and diagonal entires

$$A_{ii} = - \sum_{\substack{j=1 \\ j \neq i}}^n A_{ij}.$$

The row sums of A clearly vanish. Write $A[i, j]$ for the matrix obtained by removing rows i, j and columns i, j from the matrix. The tree partial amplitudes for the non-linear sigma model (NLSM) are given by a CHY formula with integrand [13]

$$A_{\text{NLSM}}(a, n) = \int d\mu(a) \left(\prod'_l \delta(E_l) \right) \frac{\det A[i, j]}{z_{ij}^2}, \quad (4.4)$$

for some choice of i, j .

To evaluate (4.4), it is convenient to choose $i = 1, j = n$, and to gauge fix, say, $z_1 = 0, z_{n-1} = 1, z_n = \infty$. With these choices, the integrand simplifies to

$$A_{\text{NLSM}}(a, n) = \int d^{n-3}z \text{pt}(a) \left(\prod'_l \delta(E_l) \right) \det A[1, n],$$

where now the diagonal entries of $A[1, n]$ are

$$A_{ii} = - \sum_{\substack{j=1 \\ j \neq i}}^{n-1} A_{ij}.$$

To evaluate $\det A[1, n]$ in this limit, Kirchoff's matrix tree theorem gives [17]

$$\det A[1, n] = \sum_{\substack{\text{spanning} \\ \text{trees}, \\ G}} \prod_{i \rightarrow j} \frac{s_{ij}}{z_{ij}}, \quad (4.5)$$

where the orientations of the edges of G are determined by designating vertex 1 the sink, as in the paragraph before Proposition 3.4. Given this, Proposition 3.4 implies that

$$\det A[1, n] = \sum_{\substack{\text{spanning} \\ \text{trees}, \\ G}} \prod_{i=2}^n s_{ix_i} \sum_{\substack{a \\ x_i <_{1a} i}} \text{pt}(1a), \quad (4.6)$$

where x_i is the vertex in G reached from vertex i along an outgoing edge, and the second sum is over all permutations $a \in \mathfrak{S}(2, \dots, n-1)$ such that x_i precedes i in $1a$ for all i . After reversing the two summations, (4.6) becomes

$$\det A[1] = \sum_{a \in \mathfrak{S}_{n-2}} \text{pt}(1a) \prod_{i=2}^n \sum_{j <_{1a} i} s_{ij}.$$

The sum in the product is over all letters appearing before i in the ordering $1a$. The product appearing in this sum can be identified with components of the KLT map, using equation (2.8):

$$\prod_{i=2}^n \sum_{j <_{1a} i} s_{ij} = S(1a, 1a).$$

So the NLSM partial amplitudes, (4.4), are

$$A_{\text{NLSM}}(a, n) = \sum_{b \in \mathfrak{S}_{n-2}} S(1b, 1b) m(1b, n|a, n). \quad (4.7)$$

As in [13], this result can be substituted into the KLT relation (equation (1.4)) to obtain the tree amplitudes of the special Galileon theory, which is

$$M_{\text{SG}} = \lim_{s_{1a} \rightarrow 0} \frac{1}{s_{1a}} \sum_{a \in \mathfrak{S}_{n-2}} A_{\text{NLSM}}(1a, n) S(1a, 1b) A_{\text{NLSM}}(1b, n).$$

Using Proposition 2.2 (or equation (2.7)), together with (4.7), the formula for M_{SG} can be written compactly as

$$M_{\text{SG}} = \sum_{a \in \mathfrak{S}_{n-2}} S(1a, 1a) A_{\text{NLSM}}(1a, n). \quad (4.8)$$

Proposition 4.1. *The NLSM and special Galileon tree amplitudes can be expressed as the following sums over binary trees:*

$$A_{\text{NLSM}}(a, n) = \sum_{\substack{\text{trees,} \\ \alpha}} \frac{(a, \alpha) n_\alpha}{s_\alpha} \quad \text{and} \quad M_{\text{SG}} = \sum_{\alpha} \frac{n_\alpha n_\alpha}{s_\alpha},$$

where

$$n_\alpha = \sum_{b \in \mathfrak{S}_{n-2}} (1b, \alpha) S(1b, 1b). \quad (4.9)$$

The numerators n_α have no poles in the s_I variables, and the replacement $\alpha \mapsto n_\alpha$ defines a homomorphism out of \mathcal{L}_A . Numerators

The formula (4.9) was found also in [16, 32]; the derivation here is new. One interest of this derivation is that the methods used here can also be easily adapted to study other CHY integrals. The next section discusses the case of Yang–Mills gauge theory in four dimensions.

4.3 Yang–Mills

Some of the methods used in the previous section to study NLSM amplitudes also lead to results about Yang–Mills tree amplitudes. This is because there exist formulas for Yang–Mills tree amplitudes (in four dimensions) that involve determinants similar to those computed above. This section first recalls these formulas, and then manipulates them using the identities from Section 3.

In four dimensions, gluons have two helicity states, and it is conventional to further refine the partial amplitude decomposition by specifying the helicities of the gluons. Using spinors, the null momenta k_i may be written as $k_i^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}$, unique up to a complex rescaling $\lambda_i \rightarrow \alpha \lambda_i$, $\tilde{\lambda}_i \rightarrow \alpha^{-1} \tilde{\lambda}_i$ (with $\alpha \neq 0$). The two helicities correspond two polarizations $\epsilon_+^{\alpha\dot{\alpha}} \propto \lambda^\alpha \tilde{\xi}^{\dot{\alpha}}$ and $\epsilon_-^{\alpha\dot{\alpha}} \propto \xi^\alpha \tilde{\lambda}^{\dot{\alpha}}$, for some reference spinors ξ and $\tilde{\xi}$. In practice, the partial amplitudes themselves do not depend on the choice of reference spinors, and are functions of the invariants

$$\langle ij \rangle = \lambda_i^\alpha \lambda_j^\beta \epsilon_{\alpha\beta}, \quad [ij] = \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_j^{\dot{\beta}} \epsilon_{\dot{\alpha}\dot{\beta}},$$

where $\epsilon_{12} = -\epsilon_{21} = 1$. Fix k gluons ‘1, …, k ’ with + helicity, and $n - k$ gluons ‘ $k + 1, \dots, n$ ’ with – helicity.

The CHY-like formulas for 4D amplitudes that we will consider are sums over solutions to so-called ‘polarized scattering equations’, which have fewer solutions than the scattering equations. To write these equations, it is helpful to define the following two spinor-valued functions on \mathbb{CP}^1 :

$$\lambda^\alpha(z) = \sum_{j=k+1}^n \frac{t_j \lambda_j^\alpha}{z - z_j} \quad \text{and} \quad \tilde{\lambda}^{\dot{\alpha}}(z) = \sum_{i=1}^k \frac{t_i \tilde{\lambda}_i^{\dot{\alpha}}}{z - z_i},$$

where the t_i are non-zero complex scalars. Then the polarized scattering equations are

$$\begin{aligned}\lambda_i^\alpha - t_i \lambda^\alpha(z_i) &= 0, & \text{for } i = 1, \dots, k, \\ \tilde{\lambda}_j^{\dot{\alpha}} - t_j \tilde{\lambda}^{\dot{\alpha}}(z_j) &= 0, & \text{for } j = k+1, \dots, n.\end{aligned}\tag{4.10}$$

Solutions to these equations are also solutions to the ‘original scattering equations’, (4.2), because (4.10) ensure that the following residues vanish:

$$\text{Res}_{z=z_i} k_i^{\alpha\dot{\alpha}} \lambda_\alpha(z) \tilde{\lambda}_{\dot{\alpha}}(z).$$

The equations (4.10) also imply that the spinor data satisfies momentum conservation,

$$\sum_{i=1}^n \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} = 0,$$

which can be checked by breaking the sum into two parts: $i = 1, \dots, k$ and $i = k+1, \dots, n$.

In order to present the formula for 4D gravity amplitudes, first define the $k \times k$ Hodges’ matrix, H : the off-diagonal entries are [15, 28]

$$H_{ij} = \frac{t_i t_j \langle ij \rangle}{z_{ij}} \quad \text{for } i \neq j,$$

and the diagonal entries are

$$H_{ii} = - \sum_{\substack{i=1 \\ j \neq i}}^k H_{ij}.$$

Likewise, define the $(n-k) \times (n-k)$ matrix \tilde{H} to have entries

$$\tilde{H}_{ij} = \frac{\tilde{t}_i \tilde{t}_j [ij]}{z_{ij}} \quad \text{for } i \neq j,$$

and

$$\tilde{H}_{ii} = - \sum_{\substack{j=n-k \\ j \neq i}}^n \tilde{H}_{ij}.$$

Fix some a in $1, \dots, k$, and fix some b in $k+1, \dots, n$. Write $H[a]$ for the matrix formed by removing the a^{th} row and column from H . Likewise for $\tilde{H}[b]$. Given these Hodges’ matrices, the 4D gravity tree amplitude can be written as

$$M_{\text{GR}} = \int d\mu \frac{\det H[a] \det \tilde{H}[b]}{\prod_{i=1}^n t_i^2} \prod_{\substack{i=1 \\ i \neq a}}^k \delta^2(\lambda_i - t_i \lambda(z_i)) \prod_{\substack{j=k+1 \\ j \neq b}}^n \delta^2(\tilde{\lambda}_i - \tilde{t}_i \tilde{\lambda}(z_i)).\tag{4.11}$$

This formula as given can be found in [1, 26], and is proved in [25]. It is equivalent to and closely related to the Cachazo–Skinner formula [14, 15], and also to the RSVW formula [42]. The 4D Yang–Mills tree partial amplitude can likewise be written as

$$A_{\text{YM}}(an) = \int d\mu \text{PT}(an) \prod_{\substack{i=1 \\ i \neq a}}^k \delta^2(\lambda_i - t_i \lambda(z_i)) \prod_{\substack{j=k+1 \\ j \neq b}}^n \delta^2(\tilde{\lambda}_i - \tilde{t}_i \tilde{\lambda}(z_i)).\tag{4.12}$$

$\mathrm{GL}_2\mathbb{C}$ acts on the integrands in (4.11) and (4.12) by matrix multiplication on the pair $(z_i/t_i, 1/t_i)$. In both formulas, the measure $d\mu$ is given by

$$d\mu = \frac{1}{\mathrm{Vol}\mathrm{GL}_2\mathbb{C}} \prod_{i=1}^n \frac{dz_i dt_i}{t_i},$$

so that $d\mu$ is a top dimensional $2n - 4$ form. More explicitly, adopting the gauge fixing $z_1 = 0$, $z_2 = 1$, $z_n = \infty$, $t_n = 1$,

$$d\mu = \prod_{i=3}^{n-1} \frac{dz_i dt_i}{t_i} \Big|_{z_1=0, z_2=1, z_n=\infty, t_n=1}.$$

Remark 4.2. By defining homogeneous coordinates $\sigma_i = (z_i/t_i, 1/t_i)$, it is possible to write the integrands of (4.11) and (4.12) in $\mathrm{GL}_2\mathbb{C}$ -covariant form, using the pairing

$$(\sigma_i, \sigma_j) = \frac{z_i - z_j}{t_i t_j}.$$

These leads to the formulas in the form presented in [1, 26].

The rest of this section uses the tools from Section 3 to expand the integrand in (4.11), in order to write M_{GR} as a sum

$$M_{\mathrm{GR}} = \sum_{a \in \mathfrak{S}_{n-2}} N_{\mathrm{YM}}(1an) A_{\mathrm{YM}}(1an). \quad (4.13)$$

Computing the coefficients of this expansion, $N_{\mathrm{YM}}(1an)$, also suffices to compute A_{YM} itself: this follows from the KLT relation, (1.4), as seen in the case of NLSM amplitudes, in equations (4.7) to (4.8).

To compute $N_{\mathrm{YM}}(1an)$, the first step is to expand the determinants $\det H[a]$ and $\det \tilde{H}[b]$. The can be done using Kirkchoff's tree theorem, as in (4.5), above, with the difference that H is a *symmetric* matrix, whereas A is not. Fix some a from $1, \dots, k$. Then the determinant of $H[a]$ is

$$\det H[a] = \sum_G \prod_{\substack{\text{edges} \\ i-j}} \frac{t_i t_j \langle ij \rangle}{z_{ij}}, \quad (4.14)$$

where the sum is over all spanning trees, G , of the vertex set $1, \dots, k$. The Hodges matrix H is symmetric, so it is not necessary to orient the edges in order for (4.14) to be well defined. Also note that the result, (4.14), is independent of the choice of a . A single summand in (4.14), for a tree G , can also be written as

$$\left(\prod_{i=1}^k t_i^{d_i} \right) \prod_{\substack{\text{edges} \\ i-j}} \frac{\langle ij \rangle}{z_{ij}},$$

where d_i is the degree of the vertex i in G . The determinant $\det' \tilde{H}$ is a sum of similar such terms. Fixing some b from $k+1, \dots, n$,

$$\det \tilde{H}[b] = \sum_G \left(\prod_{j=k+1}^n t_j^{d_j} \right) \prod_{\substack{\text{edges} \\ i-j \\ \text{in } G}} \frac{[ij]}{z_{ij}},$$

where d_j is the degree of the vertex j in the spanning tree G .

The full amplitude, M_{GR} , can therefore be expanded as a sum over pairs of trees (G, G') , with G spanning vertex set $1, \dots, k$ and G' spanning vertex set $k+1, \dots, n$. Explicitly,

$$M_{\text{GR}} = \sum_{\substack{\text{trees} \\ G, G'}} \int d\mu I_{G, G'} \prod_{\substack{i=1 \\ i \neq a}}^k \delta^2(\lambda_i - t_i \lambda(z_i)) \prod_{\substack{j=k+1 \\ j \neq b}}^n \delta^2(\tilde{\lambda}_i - \tilde{t}_i \tilde{\lambda}(z_i)), \quad (4.15)$$

where

$$I_{G, G'} = \left(\prod_{i=1}^n t_i^{d_i-2} \right) \left(\prod_{\substack{i-j \\ \text{in } G}} \frac{\langle ij \rangle}{z_{ij}} \right) \left(\prod_{\substack{i-j \\ \text{in } G'}} \frac{[ij]}{z_{ij}} \right).$$

The coefficients $n_{\text{YM}}(1a)$ in (4.13) can in principle be computed from (4.15) in two steps. First, it is necessary to express the t_i in terms of the z_i by solving for them using the polarized scattering equations. Second, the integrands $I_{G, G'}$ should be expanded in Parke–Taylor factors using the identities in Section 3. The resulting terms can then be regrouped to give a sum of the form (4.13). This is carried out for the maximal-helicity-violating (MHV) case in the following subsection.

4.4 Maximal-helicity-violating Yang–Mills amplitude

The Maximal-helicity-violating (MHV) case is $k = 2$, when only two gluons are + helicity, and $n - 2$ are – helicity. In this case, the matrix tree expansion, (4.15), simplifies to the following sum over spanning trees on $3, \dots, n$:

$$M_{\text{GR}} = \sum_{\substack{\text{trees} \\ G}} \int d\mu I_G \prod_{\substack{i=1 \\ i \neq a}}^k \delta^2(\lambda_i - t_i \lambda(z_i)) \prod_{\substack{j=k+1 \\ j \neq b}}^n \delta^2(\tilde{\lambda}_i - \tilde{t}_i \tilde{\lambda}(z_i)),$$

where

$$I_G = \frac{\langle 12 \rangle}{z_{12}} \left(\frac{1}{t_1 t_2} \prod_{i=3}^n t_i^{d_i-2} \right) \left(\prod_{\substack{i-j \\ \text{in } G}} \frac{[ij]}{z_{ij}} \right).$$

A tree on $n - 2$ vertices has $n - 3$ edges, so that

$$\sum_{i=3}^n (d_i - 2) = 2(n - 3) - 2(n - 2) = -2.$$

It follows that I_G may also be written as

$$I_G = \frac{\langle 12 \rangle}{z_{12}} \left(\frac{t_1}{t_2} \prod_{i=3}^n (t_1 t_i)^{d_i-2} \right) \left(\prod_{\substack{i-j \\ \text{in } G}} \frac{[ij]}{z_{ij}} \right).$$

Now fix some b from $3, \dots, n$. Choosing b to be a source vertex induces an orientation of each spanning tree G , such that every vertex (apart from b) has exactly 1 incoming edge. Given that G is oriented this way, I_G may be further re-written as

$$I_G = \frac{\langle 12 \rangle}{z_{12}} \left(\frac{t_1}{t_2} (t_1 t_b)^{-2} \prod_{\substack{i=3 \\ i \neq b}}^n (t_1 t_i)^{-1} \right) \left(\prod_{\substack{i \rightarrow j \\ \text{in } G}} \frac{(t_1 t_i)[ij]}{z_{ij}} \right). \quad (4.16)$$

Having used the matrix tree theorem to evaluate I_G , the next step is to use the polarized scattering equations to solve for the t_i appearing in (4.16). In the MHV case, the polarized scattering equations include,

$$\tilde{\lambda}_j = t_j \left(\frac{t_1 \tilde{\lambda}_1}{z_{i1}} + \frac{t_2 \tilde{\lambda}_2}{z_{i2}} \right),$$

for j in $3, \dots, n$. These equations imply that

$$t_1 t_j = \frac{[j2]}{[12]} z_{j1},$$

and that, for any given j in $3, \dots, n$,

$$\frac{t_1}{t_2} = - \frac{[j2] z_{j1}}{[j1] z_{j2}}.$$

Making these substitutions, it follows that, on the support of the polarized scattering equations,

$$I_G = - \frac{\langle 12 \rangle}{z_{12}} \left(\frac{[b2] z_{b1}}{[b1] z_{b2}} \right) \left(\frac{[12] 1}{[b2] z_{b1}} \right)^2 \left(\prod_{\substack{j=3 \\ j \neq b}}^n \frac{[12] 1}{[j2] z_{j1}} \right) \left(\prod_{\substack{i \rightarrow j \\ \text{in } G}} \frac{[i2][ij] z_{i1}}{[12] z_{ij}} \right).$$

Both of the products appearing in this equation have $n - 3$ terms, and the factors of $[12]$ in them cancel out. Combining the remaining factors gives

$$I_G = \frac{\langle 12 \rangle [12]^2}{[b1][b2]} \frac{1}{z_{12} z_{2b} z_{b1}} \prod_{\substack{i \rightarrow j \\ \text{in } G}} \left(\frac{[i2][ij] z_{i1}}{[j2] z_{j1} z_{ij}} \right). \quad (4.17)$$

Proposition 3.4 can be used to reexpress the product in (4.17) in terms of Parke–Taylor functions. A variation on the argument in Proposition 3.4 gives⁹

$$\prod_{\substack{i \rightarrow j \\ \text{in } G}} \frac{z_{i1}}{z_{j1} z_{ij}} = \sum_{\substack{\sigma \\ x_i < i}} \text{pt}(b\sigma) \frac{z_{b1}}{z_{\sigma^* 1}},$$

where σ^* is the last entry of σ . The resulting expression for I_G is

$$I_G = \frac{\langle 12 \rangle [12]^2}{[b1][b2]} \left(\prod_{\substack{i \rightarrow j \\ \text{in } G}} \frac{[i2][ij]}{[j2]} \right) \sum_{\substack{\sigma \\ x_i < i}} \text{PT}(12b\sigma).$$

Or, reordering the summations, it follows that, on the support of the polarized scattering equations,

$$\sum_G I_G = \sum_{\sigma \in \mathfrak{S}_{n-3}} \text{PT}(12b\sigma) N(12b\sigma),$$

where

$$N(12b\sigma) = \frac{\langle 12 \rangle [12]^2}{[b1][b2]} \prod_{j=3}^n \sum_{\substack{i < \sigma j \\ i \neq b}} \frac{[i2][ij]}{[j2]}. \quad (4.18)$$

⁹This follows by telescoping the factors of z_{i1}/z_{j1} .

This gives an expansion of the gravity tree amplitude into a sum of YM partial amplitudes,

$$M_{\text{GR}} = \sum_{\sigma \in \mathfrak{S}_{n-3}} A_{\text{YM}}(12b\sigma)N(12b\sigma).$$

For example, when combined with the Parke–Taylor formula for the MHV Yang–Mills amplitude, $A_{\text{YM}}(12b\sigma)$, this yields, via (4.12) and (4.11), the MHV gravity amplitude

$$M_{\text{GR}}(1234) = \frac{\langle 12 \rangle [12]^8}{[34]} \left(\prod_{\substack{i < j \\ i=1}}^4 [ij] \right)^{-1}.$$

Remark 4.3. Formulas for the MHV gravity amplitude were first obtained by Berends–Giele–Kuijf (BGK), and were inspired by the KLT relations [3]. Variations on the BGK formula have appeared in several places, including [2, 22, 36]. However, the formula above appears to be a new variant, and it has been derived here using new methods motivated by Section 3. The studies cited above mostly use inductive arguments based on recursion relations.

5 Discussion

The formulas for partial NLSM tree amplitudes in Section 4 can be expressed in the form

$$A(a, n) = \sum_{\text{trees } \alpha} \frac{(a, \alpha)n_\alpha}{s_\alpha}, \quad (5.1)$$

for numerators n_α that satisfy two key properties. First, the replacement $\alpha \mapsto n_\alpha$ extends to define a homomorphism out of the space of Lie polynomials, \mathcal{L}_A . Second, the n_α are *polynomial* in the Mandelstam variables s_I . The numerators n_α , satisfying these two properties, are called ‘BCJ numerators’, after [4]. The results for MHV gravity also lead to formulas of the form (5.1) for YM MHV amplitudes, but the n_α have spurious poles coming from the denominator factors in equation (4.18). Beyond giving formulas for these numerators, there are two important unresolved questions about the numerators for further research.

First, the BCJ numerators in Section 4 were of the form

$$n_\alpha = \sum_a (1a, \alpha)n(1a),$$

for some functions $n(1a)$. A number of authors have asked whether there exists a ‘kinematic’ Lie algebra such that the n_α can be expressed instead as a Lie bracketing of $n - 1$ Lie algebra elements, by analogy with the definition of c_α , (1.2), [8, 18, 19, 40]. The formulas obtained using the methods in this paper may suggest further clues for identifying such ‘kinematic algebras’ for gauge theories like Yang–Mills and NLSM.

Second, as has been widely observed, if the partial amplitudes of a gauge theory can be written in the form (5.1), for BCJ numerators n_α , then that gauge theory can participate in a KLT relation to produce a gravity amplitude (for some gravity-like theory). It would therefore be desirable to characterise or classify all gauge theories whose tree amplitudes can be obtained using BCJ numerators. A full answer to this question should consider the space of all possible perturbative gauge theories, which is beyond the immediate scope of the methods used in this paper.

These questions are all concerned with the KLT relations satisfied by *tree level* amplitudes. An important further aim is to discover whether the KLT relation, (1.4), can be extended to a statement about higher order terms in the perturbation series. There have been some attempts

where the surfaces Σ have boundary marked points labelled by $1, \dots, n$. Likewise, there is a partial amplitude series for biadjoint scalar theory,

$$A_{\phi^3}(1, \dots, n) = \sum_D A_D = \sum_{\Sigma, \Sigma'} c_{\Sigma} \tilde{c}_{\Sigma'} A(\Sigma, \Sigma').$$

The double partial amplitudes $A(\Sigma, \Sigma')$ are the analog, at higher orders in perturbation theory, of the matrix $m(1a, n|1b, n) = s_{1a}(1a, T(1b))$ whose inverse (away from $s_{1a} = 0$) produced the field theory KLT kernel in Section 2. It is reasonable to expect that there exists a KLT relation for the partial amplitudes $A(\Sigma)$ at fixed loop order, involving genus g surfaces, with h boundaries and p punctures, subject to the Euler characteristic constraint,

$$p + 2g + h = \ell + 1.$$

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