A Characterisation of Smooth Maps
into a Homogeneous Space

Anthony D. BLAOM

University of Auckland, New Zealand
E-mail: anthony.blaom@gmail.com
URL: https://ablaom.github.io

Abstract. We generalize Cartan's logarithmic derivative of a smooth map from a manifold into a Lie group G to smooth maps into a homogeneous space $M = G/H$, and determine the global monodromy obstruction to reconstructing such maps from infinitesimal data. The logarithmic derivative of the embedding of a submanifold $\Sigma \subset M$ becomes an invariant of Σ under symmetries of the "Klein geometry" M whose analysis is taken up in [*SIGMA* **14** (2018), 062, 36 pages, arXiv:1703.03851].

Key words: homogeneous space; subgeometry; Lie algebroids; Cartan geometry; Klein geometry; logarithmic derivative; Darboux derivative; differential invariants

2020 Mathematics Subject Classification: 53C99; 22A99; 53D17

1 Introduction

According to a theorem of Élie Cartan, a smooth map $f \colon \Sigma \to G$, from a connected manifold Σ into a Lie group G, is uniquely determined by its *logarithmic derivative*, up to right translations in G. This derivative, also known as the Darboux derivative of f, is a one-form δf on Σ taking values in the Lie algebra of $\mathfrak{g}$ of G. Here we formulate and prove a generalization of this result, Theorem 3.14, to smooth maps $f \colon \Sigma \to G/H$ into an arbitrary homogeneous space G/H. Our generalization describes explicitly the global obstruction to reconstructing such maps from infinitesimal data, data that generalizes logarithmic derivatives (generalized Maurer–Cartan forms).

In this introduction we generalize the notion of Maurer–Cartan forms and their monodromy, and state the main existence Theorem 1.5. The proof is straightforward, apart from a question about Lie algebroid integrability, which is addressed in Section 2, by applying [5]. Uniqueness, up to symmetry, is guaranteed under a mild topological condition on G/H, but we must take some care to qualify what is meant by "symmetry", a task postponed to Section 3.

Cartan's theorem is commonly associated with his method of moving frames for studying subgeometry. While the moving frames method can be reinterpreted within the present framework, it is possible to study subgeometry using the new theory without fixing frames or local coordinates. Both frame and frame-free illustrations are given in a sequel article [1]. It is instructive to review Cartan's approach here. For more detail we recommend [8].

Cartan's method of moving frames

To classify, with a unified approach, the submanifolds of Euclidean space, affine space, conformal spheres, projective space, and so on, the ambient space M is viewed as a homogeneous space G/H, i.e., as a "Klein geometry". Here G is the group of symmetries of the geometric structure on M, which acts transitively by assumption.

Using the group structure, one tries to replace the embedding $f\colon \Sigma \to G/H$ of a subma-nifold Σ with certain data defined just on Σ and amounting to an infinitesimalization of the map f. The infinitesimal data consists of *invariants* of f — that is, the data depends on f only up to symmetries of G/H (G-translations). However, these invariants ought to be *complete*, in the sense that they are sufficient for the reconstruction of f, up to symmetry.

Cartan's method for finding a complete set of invariants is in two steps. In the first step one attempts to lift the embedding $f\colon \Sigma \to M \cong G/H$ to a smooth map $\tilde{f}\colon \Sigma \to G$:

$$
\begin{array}{ccc}
 & & G \\
 & \overset{\tilde{f}}{\nearrow} & \downarrow \\
\Sigma & \overset{f}{\longrightarrow} & G/H.
\end{array}
$$

The lift, which is not unique, should be as canonical as possible, to make the identification of invariants easier later on. For example, given a curve in Euclidean three-space $f\colon [0,1] \to \mathbb{R}^3$, one obtains a lift $\tilde{f}\colon [0,1] \to G$ into the group of rigid motions by declaring $\tilde{f}(t)$ to be the rigid motion mapping the Frenet frame of the curve at $f(0)$ to the Frenet frame at $f(t)$ — the "moving frame".

Now the basic infinitesimal invariant of a Lie group G is the Maurer–Cartan form, a one-form on G taking values in its Lie algebra $\mathfrak{g}$. In the second step of Cartan's procedure, one pulls the Maurer–Cartan form back from G to a one-form $\delta\tilde{f}$ on Σ using the lifted map $\tilde{f}\colon \Sigma \to G$. By Cartan's theorem recalled below, one can reconstruct $\tilde{f}\colon \Sigma \to G$, and hence the map $f\colon \Sigma \to G/H$, from a knowledge of $\delta\tilde{f}$ alone, which accordingly encodes (indirectly) complete invariants for the embedding.

Smooth maps into a Lie group

Fix a Lie group G and let $\mathfrak{g}$ denote its Lie algebra. A *Maurer–Cartan form* on a smooth manifold Σ is a $\mathfrak{g}$-valued one-form ω satisfying the Maurer–Cartan equations,

$$
\mathrm{d}\omega_k + \sum_{i<j} c_k^{ij} \omega_i \wedge \omega_j = 0,
$$

where the ω_k are components of ω with respect to some basis of $\mathfrak{g}$, and c_k^{ij} the corresponding structure constants. We have written the Maurer–Cartan equations as they are most commonly recognized, although this is not best representation from the present point of view, as we shall see.

The Lie group G itself supports a unique right-invariant Maurer–Cartan form ω_G that is the identity on $T_e G = \mathfrak{g}$. Every smooth map $f\colon \Sigma \to G$ pulls ω_G back to a Maurer–Cartan form on Σ, here denoted δf. Since

$$
\delta f\left(\frac{\mathrm{d}}{\mathrm{d}t} x(t)\right) = \frac{\mathrm{d}}{\mathrm{d}\tau} f(x(\tau)) f(x(t))^{-1}\bigg|_{\tau=t},
$$

or $\delta f = \mathrm{d}\log(f)$ in the special case $G = (0,\infty)$, δf is called the *logarithmic derivative* of f.

Theorem 1.1 (Cartan). *Every Maurer–Cartan form ω on a simply-connected manifold Σ is the logarithmic derivative of some smooth map $f\colon \Sigma \to G$. If $f'\colon \Sigma \to G$ is a second map with logarithmic derivative ω, then there exists a unique $g \in G$ such that $f'(x) = f(x)g$.*

One says that f is a *primitive* of ω. If Σ is only *connected*, then the obstruction to the existence of a primitive is known as the *monodromy*. Anticipating our later generalization, we recall two forms of the monodromy here. For further details see, e.g., [10, Theorem 7.14, p. 124].

The *global form* of the monodromy is a groupoid morphism

$$\Omega \colon \; \Pi(\Sigma) \to G, \tag{1.1}$$

where $\Pi(\Sigma)$ is the fundamental groupoid $\Pi(\Sigma)$ of Σ. By definition, an element of $[\gamma] \in \Pi(\Sigma)$ is the homotopy equivalence class of a path $\gamma \colon [0,1] \to \Sigma$ (endpoints fixed). Since the interval $[0,1]$ is simply-connected, the Maurer–Cartan form $\gamma^*\omega$ on $[0,1]$ admits, by Cartan's theorem, a unique primitive $\Gamma \colon [0,1] \to G$ satisfying $\Gamma(0) = 1_G$, known as the *development* of ω along the path γ. One shows that $\Gamma(1)$ depends only on the class $[\gamma]$ and one defines $\Omega([\gamma]) = \Gamma(1)$.

If ω is the logarithmic derivative of some map $f \colon \Sigma \to G$, then $\Omega([\gamma]) = f(\gamma(1))f(\gamma(0))^{-1}$. In particular, fixing some $x_0 \in \Sigma$,

$$f(x) = \Omega([\gamma])f(x_0), \tag{1.2}$$

where $\gamma \colon [0,1] \to \Sigma$ is any path from x_0 to x. If ω is an arbitrary Maurer–Cartan form, then we attempt to *define* a primitive $f \colon \Sigma \to G$ by (1.2). The group of all elements of $\Pi(\Sigma)$ beginning and ending at x_0 is the fundamental group $\pi_1(\Sigma, x_0)$ and f is well-defined if the restriction of Ω to a group homomorphism $\Omega_{x_0} \colon \pi_1(\Sigma, x_0) \to G$ — which we call the *pointed form* of the monodromy — is trivial, i.e., takes on the constant value 1_G. This condition is evidently independent of the choice of fixed point x_0.

Complete invariants without lifts

Global lifts as described above do not exist in general and Cartan's method has been largely limited to the *local* reconstruction of smooth maps into a homogeneous space, and the special case of curves ($\dim \Sigma = 1$). This is despite the fact that Theorem 1.1 and the monodromy obstruction are global results!

A generalization of Theorem 1.1 to smooth maps $f \colon \Sigma \to G/H$ obviates the need for lifts. Specifically, what we present here is a characterization of smooth maps $f \colon \Sigma \to M$, where M is an arbitrary space on which some Lie group G is acting transitively, a subtle but significant change in viewpoint, as we shall explain in Section 3. Our results are naturally formulated in the language of Lie algebroids, and the proof is an application of Cartan's fundamental theorems for Lie groups, known as Lie I, Lie II and Lie III, generalized to Lie groupoids, with which we will assume some familiarity (see, e.g., [5, 6]). Standard introductions to Lie groupoids and algebroids are [4, 6, 7, 9].

Logarithmic derivatives

In Lie algebroid language, a Maurer–Cartan form on Σ is nothing more than a morphism $\omega \colon T\Sigma \to \mathfrak{g}$ of Lie algebroids, and Theorem 1.1 a special case of Lie II, as is well-known. In the general setting, we replace the Maurer–Cartan form on G with the action algebroid $\mathfrak{g} \times M$ associated with the action of G on M, and use $f \colon \Sigma \to M$ to pull $\mathfrak{g} \times M$ back to a Lie algebroid $A(f)$ over Σ. Of course this pullback must be performed *in the category of Lie algebroids* rather than vector bundles (see, e.g., [9, Section 4.2]). The composite $\delta f \colon A(f) \to \mathfrak{g}$ of the natural maps $A(f) \to \mathfrak{g} \times M \to \mathfrak{g}$ is a Lie algebroid morphism, which becomes the *logarithmic derivative* of f. The results to be described here show that δf — or more precisely an appropriate equivalence class of δf, see Section 3 — is a complete invariant of f.

Example 1.2 (logarithmic derivative of an embedding)**.** Suppose $\Sigma \subset M$ is a submanifold and $f \colon \Sigma \to M$ the embedding. Then $A(f)$ is the subbundle of the trivial bundle $\mathfrak{g} \times \Sigma \to \Sigma$ consisting of all pairs (ξ, x) having the property that the integral curve on M through $x \in \Sigma$

of the infinitesimal generator $\xi^\dagger$ of $\xi \in \mathfrak{g}$ is tangent to $\Sigma \subset M$ at x. The anchor of $A(f)$ is $(\xi, x) \mapsto \xi^\dagger(x)$ and the bracket well-defined by

$$[X, Y] = \nabla_{\#X} Y - \nabla_{\#Y} X + \{X, Y\}.$$

Here ∇ is the canonical flat connection on $\mathfrak{g} \times \Sigma$ and, viewing sections of $\mathfrak{g} \times \Sigma$ as $\mathfrak{g}$-valued functions, $\{X, Y\}(x) := [X(x), Y(x)]_\mathfrak{g}$. The logarithmic derivative δf is the composite $A(f) \hookrightarrow \mathfrak{g} \times \Sigma \to \mathfrak{g}$.

Generalized Maurer–Cartan forms

Again let M be a smooth manifold on which some Lie group G is acting from the left transitively — what is hereafter referred to as a *homogeneous G-space*. With this data fixed, our next task is to introduce axioms for Lie algebroid morphisms $\omega \colon A \to \mathfrak{g}$, where A is a Lie algebroid over some manifold Σ, modeled on local properties of logarithmic derivatives δf of smooth maps $f \colon \Sigma \to M$.

To this end, observe that *logarithmic derivatives map Lie algebroid isotropy algebras isomorphically onto isotropy algebras of the action of G on M*. Specifically, if we denote the kernel of an anchor map $\# \colon A \to T\Sigma$ by A°, then, for any $x \in \Sigma$, $A(f)_x^\circ$ and $\mathfrak{g}_{f(x)}$ have the same dimension, and

$$\delta f(A(f)_x^\circ) = \mathfrak{g}_{f(x)}. \tag{1.3}$$

The following axioms, then, are no stronger than properties already satisfied by logarithmic derivatives:

M1 A is transitive.

M2 For some point $x_0 \in \Sigma$, the restriction $\omega \colon A_{x_0}^\circ \to \mathfrak{g}$ is injective.

M3 For some (possibly different) point $x_0 \in \Sigma$, there exists $m_0 \in M$ such that $\omega(A_{x_0}^\circ) \subset \mathfrak{g}_{m_0}$, which will be written $x_0 \xrightarrow{\omega} m_0$.

Using the shorthand defined in M3, (1.3) reads $x \xrightarrow{\delta f} f(x)$. A Lie algebroid morphism $\omega \colon A \to \mathfrak{g}$ is called a *generalized Maurer–Cartan form* if it satisfies M1–M3.

Contrary to the group case, a Lie algebroid is not necessarily integrable (the Lie algebroid of some Lie groupoid) and obstructions to integrability are subtle. See [6] for a fuller discussion and examples. Nevertheless, we have:

Proposition 1.3. *Assume* M1 *and* M2 *hold and that* Σ *is connected. Then:*

1. *A is an integrable Lie algebroid.*

2. M2 *holds with x_0 replaced by an arbitrary point $x \in \Sigma$.*

3. *If* M3 *holds, then it holds with x_0 replaced by an arbitrary point $x \in \Sigma$ (and suitable choice of replacement $m \in M$ for m_0).*

We shall see the first assertion readily implies the others. A simple proof of (1) is not known to us.[1] In Section 2, where the proposition is proven, we will easily deduce integrability from Crainic and Fernandes' generalization of Lie III [5].

---

[1]In an earlier version of this manuscript the main existence theorem was proven without assuming integrability, using substantially more complicated arguments, and integrability established post facto.

Principal primitives

In fact, the most naïve notion of a primitive is not unique "up to symmetry". However, the naïve notion will play a role and be given a name:

Definition 1.4. A smooth map $f\colon \Sigma \to M$ is a *principal primitive* of a generalized Maurer–Cartan form $\omega\colon A \to \mathfrak{g}$ if there exists a Lie algebroid morphism $L\colon A \to A(f)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \omega\ } & \mathfrak{g} \\
\downarrow{\scriptstyle L} & \nearrow{\scriptstyle \delta f} & \\
A(f). & &
\end{array}
\tag{1.4}
$$

Note that we do not assume L is an isomorphism, or even that A and $A(f)$ have the same rank.

Monodromy obstructions to the existence of primitives

We now offer this paper's main construction, and formulate the existence part of our results. Let M be a homogeneous G-space and $\omega\colon A \to \mathfrak{g}$ an associated generalized Maurer–Cartan form. We are going to explicitly describe the obstruction to the existence of a principal primitive $f\colon \Sigma \to M$ of ω, where Σ is the base of A. The most natural description is in terms of some abstract transitive Lie groupoid $\mathcal{G}$ integrating A, whose existence is guaranteed by Proposition 1.3, and which we may take to be source-simply connected, on account of Lie I. In the next subsection we will offer a more concrete interpretation using a generalization of Cartan's development along paths.

According to Lie II, ω is the derivative of a unique Lie groupoid morphism

$$
\Omega\colon\ \mathcal{G} \to G,
\tag{1.5}
$$

which we call the *global form* of the monodromy of ω, being the analogue of (1.1). Continuing the analogy, we choose $x_0 \in \Sigma$ and m_0 such that M2 and M3 hold, and attempt to define a principal primitive $f\colon \Sigma \to M$, mapping x_0 to m_0, by $f(x) = \Omega(p) \cdot m_0$, where $p \in \mathcal{G}$ is any arrow from x to x_0. In this ambition we are successful, so long as f is well-defined, i.e., provided

$$
\Omega(\mathcal{G}_{x_0}) \subset G_{m_0}.
\tag{1.6}
$$

Here $\mathcal{G}_{x_0} \subset \mathcal{G}$ denotes the group of all arrows $p \in \mathcal{G}$ beginning and ending at x_0, and G_{m_0} the isotropy at m_0 of the action of G on M.

Now the algebroid isotropy $A_{x_0}^{\circ}$ is the Lie algebra of $\mathcal{G}_{x_0}$ and, by our hypothesis M3, $\omega\big(A_{x_0}^{\circ}\big) \subset \mathfrak{g}_{m_0}$. Therefore,

$$
\Omega\big(\mathcal{G}_{x_0}^{\circ}\big) \subset G_{m_0},
\tag{1.7}
$$

where $\mathcal{G}_{x_0}^{\circ}$ is the connected component of $\mathcal{G}_{x_0}$. Moreover, as the transitive Lie groupoid $\mathcal{G}$ has simply-connected source-fibres, there is a natural exact sequence

$$
1 \to \mathcal{G}_{x_0}^{\circ} \to \mathcal{G}_{x_0} \xrightarrow{\ \rho\ } \pi_1(\Sigma, x_0) \to 1.
$$

From this and (1.7) we obtain a map $\Omega_{x_0}^{m_0}\colon \pi_1(\Sigma, x_0) \to M$ well-defined by

$$
\Omega_{x_0}^{m_0}(\rho(p)) = \Omega(p) \cdot m_0.
$$

We call this the *pointed form* of the monodromy. By construction, our requirement (1.6) is equivalent to $\Omega_{x_0}^{m_0}$ taking a constant value (which is necessarily m_0).

Theorem 1.5 (existence and uniqueness of principal primitives). *A Maurer–Cartan form ω: $A \to \mathfrak{g}$ over a connected manifold Σ admits a principal primitive $f \colon \Sigma \to M$ if and only if the pointed form of the monodromy $\Omega_{x_0}^{m_0} \colon \pi_1(\Sigma, x_0) \to M$ is constant for some (and consequently any) choice of x_0 and m_0 with $x_0 \xrightarrow{\omega} m_0$. In that case there is a unique principal primitive f of ω such that $f(x_0) = m_0$.*

Proof. The preceding arguments establish the existence of a primitive, given constant monodromy. Conversely, given the existence of a primitive f with $m_0 = f(x_0)$, one easily establishes constancy of the monodromy. For example, an elementary observation stated later as Proposition 3.12 shows that

$$f(x) = \Omega(x_0) \cdot m_0 \tag{1.8}$$

for any arrow $p \in \mathcal{G}$ from x_0 to x and so, in particular, $m_0 = \Omega(p) \cdot m_0$ for any $p \in \mathcal{G}_{x_0}$. Since (1.8) applies to any principle primitive with $m_0 = f(x_0)$, the last statement of the theorem also holds. ∎

Monodromy as development along A-paths

Let A be a Lie algebroid over a connected manifold Σ. Then a piece-wise smooth map $a \colon [0, 1] \to A$, covering an ordinary path $\gamma \colon [0, 1] \to \Sigma$, is called an *$A$-path* if $\#a(t) = \dot{\gamma}(t)$, for all $t \in [0, 1]$. Here $\# \colon A \to T\Sigma$ denotes the anchor of A.

Every A-path a can be understood as Lie algebroid morphism $\hat{a} \colon TI \to A$ defined by $\hat{a}(\partial/\partial t) = a$ and all such morphisms arise from A-paths. In particular, given any Lie groupoid $\mathcal{G}$ integrating A, we may apply Lie II, obtaining a Lie groupoid morphism $I \times I \to \mathcal{G}$. The image of $(0, t)$ under this morphism, denoted

$$\int_0^t a \in \mathcal{G},$$

is an arrow from $\gamma(0)$ to $\gamma(t)$. If A is a Lie algebra, and $\mathcal{G}$ a Lie group with Lie algebra A, then a is simply a piece-wise smooth path in the Lie algebra, and the integral above the usual integral to an element in the group. This familiar case is the one applying in the proposition below:

Proposition 1.6. *Consider a Maurer–Cartan form $\omega \colon A \to \mathfrak{g}$ as in the preceding theorem, and suppose $x_0 \xrightarrow{\omega} m_0$, for some $x_0 \in \Sigma$ and $m_0 \in M$. Let $[\gamma] \in \pi_1(\Sigma, x_0)$ be given and let $a \colon [0, 1] \to A$ be any A-path covering γ. Then the monodromy is given by*

$$\Omega_{x_0}^{m_0}([\gamma]) = \left(\int_0^1 \omega \circ a \right) \cdot m_0.$$

Proof. The proposition follows immediately from the definition of $\Omega_{x_0}^{m_0}$ and the following elementary property of A-paths: Every Lie algebroid morphism $\omega \colon A_1 \to A_2$ maps A_1-paths to A_2-paths, and if ω is the derivative of a Lie groupoid morphism $\Omega \colon \mathcal{G}^1 \to \mathcal{G}^2$, then, for any A_1-path a,

$$\Omega \left(\int_0^t a \right) = \int_0^t \omega \circ a, \qquad t \in I. \qquad \blacksquare$$

Invariants for subgeometry and Bonnet-type theorems

As far as we know, Cartan's method of moving frames is the only general technique for obtaining invariants of a submanifold Σ of a Klein geometry $M \cong G/H$, and for proving theorems which

reconstruct the submanifold from its invariants (up to symmetry). The fundamental theorem of surfaces (Bonnet theorem) is a prototype for results of this kind. For the special class of parabolic geometries (G/H a flag manifold) an approach based on tractor bundles is outlined in [2] and successfully applied to conformal geometry (see also [3]). These authors do not describe the monodromy, however, restricting their attention to the case of simply-connected submanifolds.

While the logarithmic derivative δf introduced here delivers a complete invariant of an embedding $f \colon \Sigma \hookrightarrow G/H$, it is usually too abstract to be immediately useful. In [1] we take up the problem of "deconstructing" this invariant, and offer illustrations to concrete geometries.

Bracket convention

Throughout this article, brackets on Lie algebras and Lie algebroids are defined using *right-invariant* vector fields.

2   Integrability

In this section we prove Proposition 1.3 by applying Crainic and Fernandes' generalization of Lie III [5].

Let A be a Lie algebroid over Σ. In [5, 6] the kernel of the anchor of $A \to T\Sigma$ is denoted by $\mathfrak{g}$. However, as this conflicts with our use as the Lie algebra of G, we continue to denote the kernel by A°. We otherwise follow the notation and terminology of [6].

In particular, the Weinstein groupoid of A is denoted $\mathcal{G}(A)$. An element of $\mathcal{G}(A)$ is a certain equivalence class of A-paths. The obstruction to the existence of a bona fide Lie groupoid integrating A (that is, to the topological groupoid $\mathcal{G}(A)$ being a Lie groupoid) is measured by the *monodromy groups* $\tilde{\mathcal{N}}_{x_0}(A)$, $x_0 \in \Sigma$. By definition, $\tilde{\mathcal{N}}_{x_0}(A)$ is the kernel of the natural homomorphism

$$\mathcal{G}\left(A^\circ_{x_0}\right) \to \mathcal{G}(A)^\circ_{x_0}. \tag{2.1}$$

On the right-hand side $^\circ$ denotes connected component. At the level of Lie algebroid paths, this homomorphism is just inclusion. The object on the left is a Lie group, while that on the right may only be a topological group. Specializing [5] to the transitive case, we have

Theorem 2.1. *Assuming its base manifold Σ is connected, the Lie algebroid A is integrable if and only if $\tilde{\mathcal{N}}_{x_0}(A) \subset \mathcal{G}\left(A^\circ_{x_0}\right)$ is discrete, for some $x_0 \in \Sigma$.*

Now assume M1 and M2 hold. Then the restriction $\omega \colon A^\circ_{x_0} \to \mathfrak{g}$ is an injection, integrating to a homomorphism $\Omega \colon \mathcal{G}\left(A^\circ_{x_0}\right) \to G$ *of Lie groups*, whose kernel K_0 is accordingly discrete. On the other hand, we may include this homomorphism in the following commutative diagram, whose vertical arrow is (2.1):

$$\mathcal{G}\left(A^\circ_{x_0}\right) \xrightarrow{\ \Omega\ } G$$
$$\downarrow \qquad \nearrow$$
$$\mathcal{G}(A)^\circ_{x_0}.$$

Here the diagonal map is the restriction of the natural topological groupoid morphism $\mathcal{G}(A) \to \mathcal{G}(\mathfrak{g}) = G$, i.e., the map sending the equivalence class of an A-path a to the equivalence class of the $\mathfrak{g}$-path $\omega \circ a$. Commutativity of the diagram implies the kernel of the vertical map must lie in K_0, but this kernel is, by definition, $\tilde{\mathcal{N}}_{x_0}(A)$. This shows $\tilde{\mathcal{N}}_{x_0}(A)$ is a discrete subset of $\mathcal{G}\left(A^\circ_{x_0}\right)$ and the proof of Proposition 1.3(1) now follows from the theorem above.

Now let $x \in \Sigma$ be arbitrary and let $\mathcal{G}$ be a Lie groupoid integrating A, which is necessarily transitive. Then, assuming Σ is connected, there exists an arrow $p \in \mathcal{G}$ from x_0 to x. Conjugation $C_p \colon q \mapsto pqp^{-1}$ maps the isotropy group $\mathcal{G}_{x_0}$ isomorphically onto $\mathcal{G}_x$. Differentiating, we get a Lie algebra isomorphism $\mathrm{d}C_p \colon A^\circ_{x_0} \to A^\circ_x$ and a commutative diagram

$$
\begin{array}{ccc}
A^\circ_{x_0} & \xrightarrow{\ \mathrm{d}C_p\ } & A^\circ_x \\[2pt]
{\scriptstyle \omega}\big\downarrow & & \big\downarrow{\scriptstyle \omega} \\[2pt]
\mathfrak{g} & \xrightarrow{\ \mathrm{Ad}_{\Omega(p)}\ } & \mathfrak{g}.
\end{array}
$$

From this observation parts (2) and (3) of Proposition 1.3 immediately follow.

3   The uniqueness of primitives up to symmetry

This section establishes the uniqueness of primitives, appropriately defined, up to symmetry. The central result is Theorem 3.9. Combining our uniqueness result with Theorem 1.5, we obtain the existence and uniqueness Theorem 3.14.

Symmetries of a homogeneous space

For the purposes of constructing a theory with the proper invariance, we have been regarding our fixed data as a homogeneous G-space. While a choice of point $m_0 \in M$ gives us an identification $M \cong G/G_{m_0}$, formulations depending on a choice of base point are to be eschewed.[2] This decision has a somewhat unexpected consequence, anticipated by reconsidering the simplest case.

According to Cartan's theorem, a smooth map $f \colon \Sigma \to G$ is uniquely determined by its logarithmic derivative, up to symmetries of G. Here a "symmetry" is a right group translation. However, to obtain an invariant version of Cartan's result we must broaden both the notion of symmetry and what it means to be a primitive. To see why, consider a smooth map $f \colon \Sigma \to M$, where M is a smooth manifold on which G is acting transitively and *freely*, so that $M \cong G$, up to choice of base-point. In order to drop the right-invariant Maurer–Cartan form ω_G on G to a one-form on M, we must suppose here that G is acting on M from the *left*. For then, fixing $m_0 \in M$ and defining a $\Phi(g) = g \cdot m_0$, the diffeomorphism $\Phi \colon G \to M$ pushes ω_G forward to a one-form ω_M on M that is independent of the choice of m_0. But then ω_M is *not* invariant with respect to the action of G — rather it is *equivariant*, if we regard G as acting on $\mathfrak{g}$ by adjoint action. In particular, two smooth maps $f_1, f_2 \colon \Sigma \to M$ with $f_2(x) = g \cdot f_1(x)$, $x \in \Sigma$, have, in general, *different* logarithmic derivatives: $f_2^* \omega_M = \langle \mathrm{Ad}_g, f_1^* \omega_M \rangle$.

To proceed one defines $f \colon \Sigma \to M$ to be a *primitive* of a one-form ω on Σ if $f^* \omega_M$ and ω agree "up to adjoint action". The price one pays for this relaxed definition is that the logarithmic derivative $f^* \omega_M$ only determines f up to a larger class of symmetries of M. Under an identification $M \cong G$, these symmetries consist of the diffeomorphisms generated by all right *and* left translations.

Symmetries in the general case are formalised as follows:

Definition 3.1. Let M be a homogeneous G-space. Then a *symmetry* of M is any diffeomorphism $\phi \colon M \to M$ for which there exists some $l \in G$ such that $\phi(g \cdot m) = lgl^{-1} \cdot \phi(m)$ for all $g \in G$, $m \in M$.

---

[2]Geometries in the real world do not come with a preferred choice of base-point. Base-points are an artifact of Klein's abstraction of geometry, not an intrinsic feature.

The symmetries of M form a Lie group henceforth denoted $\mathrm{Aut}(M)$. Evidently, $\mathrm{Aut}(M)$ contains every left translation $\phi(m) := k \cdot m$, $k \in G$ (take $l = k$). The following characterization of symmetries is readily verified:

Proposition 3.2. *Fix a point $m_0 \in M$ and identify M with the left coset space G/H, where H denotes the isotropy at m_0. Let $l \in G$ be arbitrary and suppose $r \in G$ is in the normaliser of H, so that there exists a map $\phi \colon G/H \to G/H$ making the following diagram commute:*

$$
\begin{array}{ccc}
G & \xrightarrow{\;g \,\mapsto\, lgr^{-1}\;} & G \\[2pt]
\downarrow & & \downarrow \\[4pt]
G/H & \xrightarrow{\quad\phi\quad} & G/H.
\end{array}
$$

Then ϕ is a symmetry of M and all symmetries of M arise in this way. In other words, the Lie group $W := N_G(H)/H$ acts on the left of G/H according to

$$rH \cdot gH = gr^{-1}H,$$

(an action commuting with the left action of G) and $\mathrm{Aut}(M)$ is the Lie group generated by both left translations and those transformations of $M \cong G/H$ defined by the action of W. Here $N_G(H)$ denotes the normaliser of H in G.

In contrast to the special case in which G acts freely, $\mathrm{Aut}(M)$ is frequently not much larger than the group G of left translations, in applications of interest to geometers:

Examples 3.3.

1. Take $M = \mathbb{R}^n$, let $H \subset \mathrm{GL}(n, \mathbb{R})$ be any linear Lie group whose fixed point set is the origin, and let $G \cong H \ltimes \mathbb{R}^n$ be the group of transformations of $\mathbb{R}^n$ generated by translations and elements of H. Then $N_G(H) = H$ and accordingly $\mathrm{Aut}(M) = G$.

2. (*Affine geometry*) As special cases of item (1), we may take $G = \mathrm{GL}(n, \mathbb{R})$ or $G = \mathrm{SL}(n, \mathbb{R})$ and obtain the affine and equi-affine geometries, with $\mathrm{Aut}(M) = G$.

3. (*Euclidean, elliptic and hyperbolic geometry*) Take M to be one of Riemannian space forms $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$, and let G be the full group of isometries. Then in every case it is possible to show that each element of $N_G(H)/H$ has a representative $r \in N_G(H)$ lying in the centre of G, and it follows easily that $\mathrm{Aut}(M) = G$.

4. (*Special elliptic geometry*) Take $M = \mathbb{S}^n$ but let G be the group of *orientation-preserving* isometries, $\mathrm{SO}(n + 1)$. In this case a little more work reveals that

$$
\mathrm{Aut}(M) = \begin{cases} \mathrm{SO}(n + 1) & \text{if } n \text{ is odd,} \\ \mathrm{O}(n + 1) & \text{if } n \text{ is even.} \end{cases}
$$

   That is, for even-dimensional spheres, we must add to G the orientation-reversing isometries to obtain the full symmetry group.

5. Suppose M is a homogeneous G-space where G is compact and connected and has trivial centre, and suppose that the isotropy subgroup H at some point of M is a maximal torus. Then $N_G(H)/H$ is the Weyl group, well-known to be *finite*.

6. (*Parabolic geometries*) For a flag manifold M, such as a conformal sphere or projective space, G is a connected semi-simple Lie group and the isotropy group H is a parabolic subgroup of G. In this case also $N_G(H)/H$ is known to be finite.

Morphisms between Maurer–Cartan forms

Henceforth we drop the qualification "generalized": All Maurer–Cartan forms and logarithmic derivatives will be understood in the generalized sense.

With G, M and Σ fixed as in the Introduction (under "Generalized Maurer–Cartan forms") we collect all associated Maurer–Cartan forms into the objects of a category. In this category a morphism $\omega_1 \to \omega_2$ between objects $\omega_1 \colon A_1 \to \mathfrak{g}$ and $\omega_2 \colon A_2 \to \mathfrak{g}$ consists of a Lie algebroid morphism $\lambda \colon A_1 \to A_2$ covering the identity on Σ and an element $l \in G$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\ \omega_1\ } & \mathfrak{g} \\
{\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\mathrm{Ad}_l} \\
A_2 & \xrightarrow{\ \omega_2\ } & \mathfrak{g}.
\end{array}
$$

If λ is injective, we will say that $\omega_1 \to \omega_2$ is *monic*. The preceding abstractions are justified by the following observation (strengthened in special cases in Theorem 3.9 below):

Proposition 3.4. *Let $f_1 \colon \Sigma \to M$ be a smooth map into a homogeneous G-space M and define a second smooth map $f_2 \colon \Sigma \to M$ by $f_2 = \phi \circ f_1$, for some $\phi \in \mathrm{Aut}(M)$. Then δf_1 and δf_2 are isomorphic in the category of Maurer–Cartan forms.*

That is, smooth maps $f_1, f_2 \colon \Sigma \to M$ agreeing up to a symmetry of M have isomorphic logarithmic derivatives.

Proof. Supposing $f_2 = \phi \circ f_1$, $\phi \in \mathrm{Aut}(M)$, define $l \in G$ as in Definition 3.1. Then the map $\mathrm{Ad}_l \times \phi$, defined by

$$
\begin{aligned}
(\xi, x) &\mapsto \big(\mathrm{Ad}_l\, \xi, \phi(x)\big), \\
\mathfrak{g} \times M &\to \mathfrak{g} \times M
\end{aligned}
$$

is a Lie algebroid automorphism of the action algebroid $\mathfrak{g} \times M$ covering $\phi \colon M \to M$. In particular, the composite $A(f_1) \to \mathfrak{g} \times M \xrightarrow{\mathrm{Ad}_l \times \phi} \mathfrak{g} \times M$ is a Lie algebroid morphism J sitting in a commutative diagram

$$
\begin{array}{ccc}
A(f_1) & \xrightarrow{\ J\ } & \mathfrak{g} \times M \\
\downarrow & & \downarrow \\
T\Sigma & \xrightarrow[\ Tf_2\ ]{} & TM.
\end{array}
$$

The vertical arrows indicate anchor maps. Explicitly, we have

$$
J(X) = \big(\mathrm{Ad}_l\, \delta f_1(X),\, f_2(\lrcorner X)\big),
$$

where $\lrcorner X \in \Sigma$ denotes the base point of X.

As $A(f_2)$ is the pullback of $\mathfrak{g} \times M$ under f_2, we obtain, from the universal property of pullbacks, a unique Lie algebroid morphism $\lambda \colon A(f_1) \to A(f_2)$ such that J is the composite

$$
A(f_1) \xrightarrow{\lambda} A(f_2) \xrightarrow{\delta f_2} \mathfrak{g} \times M.
$$

This immediately implies commutativity of the diagram

$$
\begin{array}{ccc}
A(f_1) & \xrightarrow{\ \delta f_1\ } & \mathfrak{g} \\
{\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\mathrm{Ad}_l} \\
A(f_2) & \xrightarrow{\ \delta f_2\ } & \mathfrak{g}.
\end{array}
$$

One argues that λ is an isomorphism by replacing ϕ with ϕ^{-1} and reversing the roles of f_1 and f_2. ∎

Primitives

A smooth map $f\colon \Sigma \to M$ will be called a *primitive* of the Maurer–Cartan form $\omega\colon A \to \mathfrak{g}$ if there exists a morphism $\omega \to \delta f$. Evidently, every principal primitive is a primitive.

Maximal Maurer–Cartan forms

Note that Axioms M1 and M2, together with Proposition 1.3, imply the following restrictions on the necessarily constant rank of A, whenever $\omega\colon A \to \mathfrak{g}$ is a Maurer–Cartan form:

$$\dim \mathfrak{g} - \dim M \leqslant \operatorname{rank} A \leqslant \dim \mathfrak{g} - \dim M + \dim \Sigma.$$

We say ω is *maximal* if A has maximal rank, i.e., if

$$\dim \mathfrak{g} - \operatorname{rank} A = \dim M - \dim \Sigma.$$

In this case it follows from M3, Proposition 1.3, and a dimension count that

M3′. For any point $x \in \Sigma$ there exists $m \in M$ such that $\omega(A_x^\circ) = \mathfrak{g}_m$.

Logarithmic derivatives and ordinary Maurer–Cartan forms are always maximal.

Lemma 3.5. *Every morphism $\omega \to \delta f$ is monic. In particular, if ω is maximal, then $\omega \to \delta f$ is an isomorphism.*

Proof. A morphism $\omega \to \delta f$ consists of a Lie algebroid morphism $\lambda\colon A \to A(f)$ covering the identity on Σ, and $l \in G$, such that

$$\delta f(\lambda(a)) = \operatorname{Ad}_l \omega(a), \qquad a \in A. \tag{3.1}$$

Suppose $\lambda(a) = 0$, a an element of A with base-point $x \in \Sigma$. Since λ is a Lie algebroid morphism covering the identity, we have $\#a = 0$, i.e., $a \in A_x^\circ$. Since $\omega(a) = 0$, by (3.1), Axiom M2 and Proposition 1.3 imply $a = 0$. ∎

The existence Theorem 1.5 has the following corollary (of which we make no further use):

Corollary 3.6. *Every Maurer–Cartan form $\omega\colon A \to \mathfrak{g}$ with constant monodromy has an extension to a maximal Maurer–Cartan form $\omega\colon A' \to \mathfrak{g}$, for some Lie algebroid $A' \supset A$.*

Proof. By the existence theorem, ω admits a principal primitive $f\colon \Sigma \to M$. That is, there exists a morphism $\lambda\colon A \to A(f)$, injective by the lemma, whose logarithmic derivative $\delta f\colon A(f) \to \mathfrak{g}$ fits into the commutative diagram (1.4). The logarithmic derivative of f is then a maximal Maurer–Cartan form extending ω. ∎

Uniqueness of primitives

As usual, suppose G acts transitively on M, and let $G_{m_0}^\circ$ denote the connected component of the isotropy G_{m_0} at some $m_0 \in M$. Then since $G_{m_0}^\circ$ is *path*-connected, $N_G(G_{m_0}) \subset N_G(G_{m_0}^\circ)$.

Definition 3.7. We say the isotropy groups of the G action are *weakly connected* if for some (and hence any) $m_0 \in M$, we have $N_G(G_{m_0}) = N_G(G_{m_0}^\circ)$.

Example 3.8. If M is one of the Riemannian space forms $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$, and G is the full group of isometries, then although the isotropy groups of the action of G on M are not connected, they *are* weakly connected.

A proof of the following central result appears below.

Theorem 3.9. *Suppose the action of G on M has weakly connected isotropy groups. Let $f_1, f_2 \colon \Sigma \to M$ be smooth maps. Then there exists an isomorphism $\delta f_1 \cong \delta f_2$ in the category of Maurer–Cartan forms if and only if there exists $\phi \in \operatorname{Aut}(M)$ such that $f_2 = \phi \circ f_1$.*

In contrast to the classical setting (Theorem 1.1) there may exist more than one choice of $\phi \in \operatorname{Aut}(M)$ for which $f_2 = \phi \circ f_1$, even if G acts faithfully on M. For example, consider two *constant* maps f_1, f_2.

Combining the theorem with the lemma above, we obtain:

Corollary 3.10 (uniqueness of primitives)**.** *If the action of G on M has weakly connected isotropy groups then primitives $f \colon \Sigma \to M$ of a maximal Maurer–Cartan form are unique, up to symmetries of M.*

A non-maximal Maurer–Cartan form may have distinct primitives not related by a symmetry:

Example 3.11. Let G be the group of isometries of the plane $M = \mathbb{R}^2$ with Lie algebra $\mathfrak{g}$ identified with the Killing fields. Let $x, y \colon \mathbb{R}^2 \to \mathbb{R}$ denote the standard coordinate functions and let $\omega \colon T\mathbb{R} \to \mathfrak{g}$ be the generalized Maurer–Cartan form[3] defined by

$$\omega\left(\frac{\partial}{\partial t}\right) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \qquad \text{(a constant element of } \mathfrak{g}\text{)}.$$

Then for any $r \geqslant 0$ the map $f(t) = (r\cos t, r\sin t)$ is a primitive of ω.

For the proof of the theorem we need one additional observation:

Proposition 3.12. *Suppose $f \colon \Sigma \to M$ is a principal primitive of a Maurer–Cartan form $\omega \colon A \to \mathfrak{g}$. Let $\Omega \colon \mathcal{G} \to G$ be the global form of the monodromy of ω, as defined in (1.5). Then, for any $x \in \Sigma$, one has $x \xrightarrow{\omega} f(x)$, and for any $x_0 \in \Sigma$,*

$$f(x) = \Omega(p) \cdot f(x_0),$$

where $p \in \mathcal{G}$ is any arrow from x_0 to x.

Proof. For some Lie algebroid morphism $\lambda \colon A \to A(f)$, we have a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ \omega\ } & \mathfrak{g} \\
{\scriptstyle \lambda}\big\downarrow & & \big\uparrow \\
A(f) & \longrightarrow & \mathfrak{g} \times M.
\end{array}
$$

Since λ covers the identity, the claim $x \xrightarrow{\omega} f(x)$ follows easily from commutativity and the definition of the bottom map. Let $\mathcal{G}(f)$ denote the pullback of the action groupoid $G \times M$ by f. Since $\mathcal{G}$ is source-simply-connected, λ is the derivative of a Lie groupoid morphism $\Lambda \colon \mathcal{G} \to \mathcal{G}(f)$

---

[3]Actually ω is an ordinary Maurer–Cartan form in this case but we are understanding primitives as maps into $\mathbb{R}^2$, not maps into G!

and the following diagram commutes (because the composites being compared have a source-connected domain and identical derivatives, by the commutativity of the preceding diagram):

$$
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\ \Omega\ } & G \\
{\scriptstyle \Lambda}\big\downarrow & & \big\uparrow \\
\mathcal{G}(f) & \longrightarrow & G \times M.
\end{array}
$$

In particular, if we define F to be the composite Lie groupoid morphism $\mathcal{G} \xrightarrow{\Lambda} \mathcal{G}(f) \to G \times M$, then F covers $f \colon \Sigma \to M$ and, by the commutativity,

$$
F(p) = (\Omega(p), \alpha(p)), \tag{3.2}
$$

where α denotes source projection. But as $F \colon \mathcal{G} \to G \times M$ must respect the target projections, denoted β, we also have $f(\beta(p)) = \beta(F(p))$. Now (3.2) gives

$$
f(\beta(p)) = \Omega(p) \cdot \alpha(p),
$$

which proves the proposition. $\blacksquare$

Proof of theorem (for Σ simply-connected). That δf_1 and δf_2 must be isomorphic when $f_2 = \phi \circ f_1$, $\phi \in \mathrm{Aut}(M)$, is Proposition 3.4. Suppose $\delta f_1 \cong \delta f_2$ and assume initially that Σ is simply-connected (needed in the proof of the lemma below). By definition, there exists $l \in G$ and a Lie algebroid isomorphism $\lambda \colon A(f_2) \to A(f_1)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A(f_1) & \xrightarrow{\ \delta f_1\ } & \mathfrak{g} \\
{\scriptstyle \lambda}\big\uparrow & & \big\downarrow{\scriptstyle \mathrm{Ad}_l} \\
A(f_2) & \xrightarrow{\ \delta f_2\ } & \mathfrak{g}.
\end{array} \tag{3.3}
$$

Arbitrarily fixing a point $x_0 \in \Sigma$, (1.3) gives

$$
\delta f_1\big(A(f_1)_{x_0}\big) = \mathfrak{g}_{f_1(x_0)}, \qquad \delta f_2\big(A(f_2)_{x_0}\big) = \mathfrak{g}_{f_2(x_0)}. \tag{3.4}
$$

For $i = 1$ or 2, let $\Omega_i \colon \mathcal{G}^i \to G$ denote the global form of the monodromy of δf_i, as defined at (1.5). The Lie algebroid of $\mathcal{G}^i$ is $A(f_i)$ and, by Lie II for Lie groupoids, there is a unique Lie groupoid isomorphism $\Lambda \colon \mathcal{G}^2 \to \mathcal{G}^1$ whose derivative is λ. Taking $\omega = \delta f_i$ in the preceding proposition, we obtain

$$
f_1(x) = \Omega_1(p_1) \cdot f_1(x_0), \qquad f_2(x) = \Omega_2(p_2) \cdot f_2(x_0), \tag{3.5}
$$

whenever $p_i \in \mathcal{G}^i$ is an arrow from x_0 to x. By the commutativity of (3.3), the Lie groupoid morphisms $\Omega_2 \colon \mathcal{G}^2 \to G$ and $p_2 \mapsto l\Omega_1(\Lambda(p_2))l^{-1}$ have the same derivative, namely δf_2, so they must coincide, because $\mathcal{G}^2$ is source-connected:

$$
\Omega_2(p_2) = l\Omega_1(\Lambda(p_2))l^{-1}, \qquad p_2 \in \mathcal{G}^2. \tag{3.6}
$$

Since λ, and hence Λ, covers the identity on Σ, $p_1 \in \mathcal{G}^1$ is an arrow from x_0 to x if and only if $p_2 := \Lambda(p_1) \in \mathcal{G}^2$ is an arrow from x_0 to x. This fact and (3.6) allow us to rewrite the second equation in (3.5) as $f_2(x) = l\Omega_1(p_1)l^{-1} \cdot f_2(x_0)$. Or, choosing $r \in G$ such that

$$
f_2(x_0) = lr^{-1} \cdot f_1(x_0), \tag{3.7}
$$

we have

$$
f_1(x) = \Omega_1(p_1) \cdot f_1(x_0), \qquad f_2(x) = l\Omega_1(p_1)r^{-1} \cdot f_1(x_0), \tag{3.8}
$$

whenever $p_1 \in \mathcal{G}^1$ is an arrow from x_0 to x.

Lemma 3.13. *$r \in G$ lies in the normaliser of $G_{f_1(x_0)}$.*

Assuming the lemma holds, there exists, by the characterization of symmetries in Proposition 3.2, an element $\phi \in \mathrm{Aut}(M)$ well-defined by $\phi(g \cdot f_1(x_0)) = lgr^{-1} \cdot f_1(x_0)$. Then (3.8) gives us $f_2(x) = \phi(f_1(x))$, as required. ∎

Proof of lemma. Since we assume the isotropy groups of the action of G on M are weakly connected, it suffices to show $r \in N_G\big(G^\circ_{f_1(x_0)}\big)$. We claim

$$\Omega_1\big(\mathcal{G}^1_{x_0}\big) = G^\circ_{f_1(x_0)}, \tag{3.9}$$

$$\Omega_2\big(\mathcal{G}^2_{x_0}\big) = G^\circ_{f_2(x_0)}. \tag{3.10}$$

Since $\mathcal{G}^i$ is transitive and source-connected ($i \in \{1,2\}$) the restriction of the target projection of $\mathcal{G}^i$ to the source-fibre over x_0 is a principal $\mathcal{G}^i_{x_0}$-bundle over Σ. Since we assume Σ is simply-connected, $\mathcal{G}^i_{x_0}$ is connected, by the long exact homotopy sequence for this principal bundle. It follows that (3.9) and (3.10) are consequences of their infinitesimal analogues, which already appear in (3.4) above.

Because $\Lambda\colon \mathcal{G}^2 \to \mathcal{G}^1$ is a Lie groupoid isomorphism covering the identity, we have

$$\Lambda\big(\mathcal{G}^2_{x_0}\big) = \mathcal{G}^1_{x_0}. \tag{3.11}$$

We now compute

$$rG^\circ_{f_1(x_0)}r^{-1} = l^{-1}G^\circ_{lr \cdot f_1(x_0)}l = l^{-1}G^\circ_{f_2(x_0)}l = l^{-1}\Omega_2\big(\mathcal{G}^2_{x_0}\big)l = \Omega_1\big(\Lambda\big(\mathcal{G}^2_{x_0}\big)\big)$$
$$= \Omega_1\big(\mathcal{G}^1_{x_0}\big) = G^\circ_{f_1(x_0)}.$$

The second and subsequent equalities in this computation follow from equations (3.7), (3.10), (3.6), (3.11) and (3.9) respectively. ∎

Proof of theorem (general case). If $\delta f_1 \cong \delta f_2$ but Σ is not simply-connected, then $\delta(f_1 \circ \pi) \cong \delta(f_2 \circ \pi)$, where $\pi\colon \tilde\Sigma \to \Sigma$ denotes the universal covering map, as it is not difficult to see. By the result just proven in the simply-connected case, there exists $\phi \in \mathrm{Aut}(M)$ such that $f_1 \circ \pi = \phi \circ f_2 \circ \pi$. But as π is surjective, this immediately implies $f_1 = \phi \circ f_2$. ∎

Summary of results

Suppose f is a primitive of a Maurer–Cartan form ω, so that $\delta f(\lambda(X)) = \mathrm{Ad}_l\, \omega(X)$, for some Lie algebroid morphism λ and element $l \in G$. Then it is not hard to show that $f'(x) = l^{-1} \cdot f(x)$ defines a *principal* primitive of ω. That is, the existence of primitives already implies the existence of principal primitives. We may therefore summarise the results cited in the Introduction and our uniqueness result, Corollary 3.10, as follows:

Theorem 3.14 (main theorem)**.** *Let M be a homogeneous G-space and $\omega\colon A \to \mathfrak{g}$ an associated generalized Maurer–Cartan form, where A is a Lie algebroid over some manifold Σ. Then A is integrable. Furthermore, ω admits a primitive $f\colon \Sigma \to M$ if and only if it has constant monodromy $\Omega^{m_0}_{x_0}\colon \pi_1(\Sigma, x_0) \to M$, for some choice of $x_0 \in \Sigma$ and $m_0 \in M$ with $x_0 \xrightarrow{\omega} m_0$. Assuming ω is maximal, and the isotropy groups of the action of G on M are weakly connected, the primitive f is unique up to symmetry.*

We reiterate that "symmetry" is to be understood in the sense Definition 3.1.

Acknowledgements

References

[1] Blaom A.D., Lie algebroid invariants for subgeometry, *SIGMA* **14** (2018), 062, 36 pages, arXiv:1703.03851.

[2] Burstall F.E., Calderbank D.M.J., Submanifold geometry in generalized flag manifolds, *Rend. Circ. Mat. Palermo (2) Suppl.* **72** (2004), 13–41.

[3] Burstall F.E., Calderbank D.M.J., Conformal submanifold geometry I–III, arXiv:1006.5700.

[4] Cannas da Silva A., Weinstein A., Geometric models for noncommutative algebras, *Berkeley Mathematics Lecture Notes*, Vol. 10, Amer. Math. Soc., Providence, RI, Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1999.

[5] Crainic M., Fernandes R.L., Integrability of Lie brackets, *Ann. of Math.* **157** (2003), 575–620, arXiv:math.DG/0105033.

[6] Crainic M., Fernandes R.L., Lectures on integrability of Lie brackets, in Lectures on Poisson Geometry, *Geom. Topol. Monogr.*, Vol. 17, Geom. Topol. Publ., Coventry, 2011, 1–107, arXiv:math.DG/0611259.

[7] Dufour J.P., Zung N.T., Poisson structures and their normal forms, *Progress in Mathematics*, Vol. 242, Birkhäuser Verlag, Basel, 2005.

[8] Ivey T.A., Landsberg J.M., Cartan for beginners: differential geometry via moving frames and exterior differential systems, *Graduate Studies in Mathematics*, Vol. 61, Amer. Math. Soc., Providence, RI, 2003.

[9] Mackenzie K.C.H., General theory of Lie groupoids and Lie algebroids, *London Mathematical Society Lecture Note Series*, Vol. 213, Cambridge University Press, Cambridge, 2005.

[10] Sharpe R.W., Differential geometry. Cartan's generalization of Klein's Erlangen program, *Graduate Texts in Mathematics*, Vol. 166, Springer-Verlag, New York, 1997.