Expansion for a Fundamental Solution of Laplace's Equation in Flat-Ring Cyclide Coordinates

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Abstract. We derive an expansion for the fundamental solution of Laplace's equation in flat-ring coordinates in three-dimensional Euclidean space. This expansion is a double series of products of functions that are harmonic in the interior and exterior of "flat rings". These internal and external flat-ring harmonic functions are expressed in terms of simply-periodic Lamé functions. In a limiting case we obtain the expansion of the fundamental solution in toroidal coordinates.

Key words: Laplace's equation; fundamental solution; separable curvilinear coordinate system; flat-ring cyclide coordinates; special functions; orthogonal polynomials

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1 Introduction

In 1875 Albert Wangerin [23] introduced a coordinate system s, t in the xy-plane by setting $x + \mathrm{i}y = \mathrm{dn}(s + \mathrm{i}t, k)$, where dn [18, formula (22.2.6)] is one of the Jacobian elliptic functions. By rotating the system about the y-axis he obtained coordinates s, t, ϕ in $\mathbb{R}^3$ that are called flat-ring coordinates by Moon and Spencer [16, p. 126]. Actually, Moon and Spencer use $x + \mathrm{i}y = \mathrm{sn}(s + \mathrm{i}t, k)$ [18, formula (22.2.4)] but this leads to the same coordinate system. Wangerin showed that the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.1}$$

can be solved by functions of the form

$$u(x, y, z) = \mathcal{R}(x, y, z) u_1(s) u_2(t) u_3(\phi), \tag{1.2}$$

where $\mathcal{R}$ is a known elementary function called the modulation factor (following Morse and Feshbach), and u_1, u_2, u_3 solve ordinary differential equations. In our case $\mathcal{R} = R^{-1/2}$, where R is the distance to the axis of rotation. This is referred to as $\mathcal{R}$-separation of variables and we express this by saying that the Laplace equation is $\mathcal{R}$-separable in flat-ring coordinates. Wangerin won

a prize for his original work but only for the first part of his paper. In the second part Wangerin tried to expand the fundamental solution of the Laplace equation (given as the inverse distance to a fixed point), in terms of the separated solutions (1.2). Such an expansion is needed in physical applications. However, there were some obvious mistakes in this part of the paper. It is the purpose of the present paper to derive the desired expansion of the fundamental solution in flat-ring coordinates. It is a little bit surprising, but, as far as we know, this work has not been carried out in all those years since 1875.

Flat-ring coordinates are a special instance of so-called cyclidic coordinate systems. In these systems the coordinate surfaces are of order 4 (they are zero sets of polynomials of degree 4 in the Cartesian coordinates x, y, z). This is in contrast to quadric coordinate systems in which the coordinate surfaces have order 2 (like spherical coordinates). Bôcher [3] listed quadric and cyclidic coordinate systems in which Laplace's equation can be solved using separation of variables in his book "Über die Reihenentwicklungen der Potentialtheorie" published in 1894. We can also find these quadric and cyclidic coordinates systems in Miller's book [15, Tables 14 on p. 164 and 17 on p. 210]. The quadric coordinates are Cartesian, cylindrical, parabolic cylindrical, elliptic cylindrical, spherical, prolate spheroidal, oblate spheroidal, parabolic, paraboloidal, spheroconal, and ellipsoidal coordinates. The cyclidic coordinates are flat-ring cyclide, flat-disk cyclide, bi-cyclide, cap-cyclide, and 5-cyclide coordinates. In the quadric systems we can choose $R = 1$ in (1.2) but this is not possible in the cyclidic systems. In several of these coordinate systems the expansion of the fundamental solution is known; see [2, 7, 11, 17].

Let us consider perhaps the simplest rotationally-invariant cyclidic coordinate system which $\mathcal{R}$-separates Laplace's equation in three-dimensions, the toroidal coordinate system. As we show in Section 7.1 flat-ring coordinates reduce to toroidal coordinates in a limiting case. Toroidal coordinates [13, Section 8.10] are given by

$$x = \frac{\sinh \tau \cos \phi}{\cosh \tau - \cos \psi}, \qquad y = \frac{\sinh \tau \sin \phi}{\cosh \tau - \cos \psi}, \qquad z = \frac{\sin \psi}{\cosh \tau - \cos \psi}, \tag{1.3}$$

where $\tau > 0$, $-\pi < \phi, \psi \leq \pi$. The coordinate surfaces $\tau = $ constant are tori, i.e.,

$$\left(1 + x^2 + y^2 + z^2\right)^2 = 4\left(x^2 + y^2\right) \coth^2 \tau.$$

The coordinate surfaces $\psi = $ constant are spherical bowls, i.e.,

$$(z - \cot \psi)^2 + x^2 + y^2 = \frac{1}{\sin^2 \psi}.$$

The coordinate surfaces $\phi = $ constant are half-planes, i.e.,

$$x \sin \phi = y \cos \phi.$$

Laplace's equation (1.1) has solutions of the form

$$u(x, y, z) = (\cosh \tau - \cos \psi)^{1/2} u_1(\tau) u_2(\psi) u_3(\phi),$$

where u_1, u_2, u_3 satisfy the following ordinary differential equations, respectively:

$$\frac{1}{\sinh \tau} \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\sinh \tau \frac{\mathrm{d}u_1}{\mathrm{d}\tau} \right) - \left(n^2 - \frac{1}{4} + \frac{m^2}{\sinh^2 \tau} \right) u_1 = 0, \tag{1.4}$$

$$\frac{\mathrm{d}^2 u_2}{\mathrm{d}\psi^2} + n^2 u_2 = 0, \tag{1.5}$$

$$\frac{\mathrm{d}^2 u_3}{\mathrm{d}\phi^2} + m^2 u_3 = 0, \tag{1.6}$$

where m, n are separation parameters. The general solution of (1.4) is

$$u_1(\tau) = c_1 P^m_{n-\frac{1}{2}}(\cosh\tau) + c_2 Q^m_{n-\frac{1}{2}}(\cosh\tau),$$

where c_1 and c_2 are constants, and $P^\mu_\nu \colon \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C}$ and $Q^\mu_\nu \colon \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C}$ are the associated Legendre functions of the first and second kind respectively defined in terms of the Gauss hypergeometric function by [18, functions (14.3.6) and (14.3.7)]

$$P^\mu_\nu(z) := \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} {}_2F_1\left(\begin{array}{c} -\nu, \nu+1 \\ 1-\mu \end{array}; \frac{1-z}{2}\right), \tag{1.7}$$

and

$$Q^\mu_\nu(z) := \frac{e^{i\mu\pi}\sqrt{\pi}\,\Gamma(\nu+\mu+1)(z^2-1)^{\frac{1}{2}\mu}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})z^{\nu+\mu+1}} {}_2F_1\left(\begin{array}{c} \frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2} \\ \nu+\frac{3}{2} \end{array}; \frac{1}{z^2}\right), \tag{1.8}$$

where $\nu + \mu \notin -\mathbb{N}$.

The internal toroidal harmonics [13, equation (8.11.8)]

$$G_{m,n}(x,y,z) := (\cosh\tau - \cos\psi)^{1/2} Q^m_{n-\frac{1}{2}}(\cosh\tau) e^{in\psi} e^{im\phi} \qquad \text{for} \quad m, n \in \mathbb{Z}$$

are harmonic in $\mathbb{R}^3$ except for the z-axis. The external toroidal harmonics [13, equation (8.11.9)]

$$H_{m,n}(x,y,z) := (\cosh\tau - \cos\psi)^{1/2} P^m_{n-\frac{1}{2}}(\cosh\tau) e^{in\psi} e^{im\phi} \qquad \text{for} \quad m, n \in \mathbb{Z}$$

are harmonic in $\mathbb{R}^3$ except for the unit circle $z = 0$, $x^2 + y^2 = 1$. The desired expansion of the fundamental solution of Laplace's equation in internal and external toroidal harmonics is

$$\frac{1}{\|\mathbf{r} - \mathbf{r}^*\|} = \frac{1}{\pi} \sum_{m,n\in\mathbb{Z}} (-1)^m \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})} G_{m,n}(\mathbf{r})\overline{H_{m,n}(\mathbf{r}^*)} \tag{1.9}$$

provided $\tau > \tau^*$, where τ, τ^* are toroidal coordinates of $\mathbf{r} = (x, y, z)$ and $\mathbf{r}^* = (x^*, y^*, z^*)$, respectively. Formula (1.9) follows by inserting [10, Theorem 8.795.2]

$$Q_\nu(\cosh\alpha) = \sum_{k=0}^{\infty} \epsilon_k (-1)^k P^{-k}_\nu(\cosh\tau) Q^k_\nu(\cosh\tau^*) \cos(k(\psi - \psi^*)),$$

where $\epsilon_k := 2 - \delta_{k,0}$ is the Neumann factor (commonly occurring in Fourier cosine series), $\cosh\alpha := \cosh\tau\cosh\tau^* - \sinh\tau\sinh\tau^*\cos(\psi - \psi^*)$, with $\nu = n - \frac{1}{2}$, into the single summation expression for the reciprocal distance between two points in toroidal coordinates given by Bateman [1, Section 10.3, equation (26)]

$$\frac{1}{\|\mathbf{r} - \mathbf{r}^*\|} = \frac{\sqrt{\cosh\tau - \cos\psi}\sqrt{\cosh\tau^* - \cos\psi^*}}{\sqrt{2}\,a\sqrt{\cosh\alpha - \cos(\psi - \psi^*)}}$$

$$= \frac{\sqrt{\cosh\tau - \cos\psi}\sqrt{\cosh\tau^* - \cos\psi^*}}{a\pi} \sum_{n=-\infty}^{\infty} Q_{n-\frac{1}{2}}(\cosh\alpha) e^{in(\psi - \psi^*)},$$

which is a direct consequence of Heine's reciprocal square root identity [5, formula (A5)]

$$\frac{1}{\sqrt{z - x}} = \frac{\sqrt{2}}{\pi} \sum_{n=-\infty}^{\infty} e^{in\theta} Q_{n-\frac{1}{2}}(z),$$

where $x = \cos\theta$ and $z > 1$.

In this work we carry out the corresponding analysis in flat-ring coordinates. "Flat-rings" now play the role of tori. In Section 2 we consider flat-ring coordinates in the form used by Poole [19, 20]. This form is the most convenient for our purposes. In Section 3 we solve the Laplace equation by $\mathcal{R}$-separation of variables. The differential equation for u_1 and u_2 is the Lamé equation. In Section 4 we collect properties of simply-periodic Lamé functions that we need to introduce flat-ring harmonics in Section 5. In Section 5 we prove the main results of this paper: we solve the Dirichlet problem on flat-rings (Theorem 5.3), we find an integral representation of external flat-ring harmonics in terms of internal flat-ring harmonics (Theorem 5.5), and we find the series expansion of the fundamental solution in terms of internal and external flat-ring harmonics (Theorem 5.6). In Sections 6 and 7 we connect flat-ring with toroidal coordinates and we compare the corresponding expansions of the fundamental solution.

2   Flat-ring coordinates

2.1   Flat-ring coordinates in algebraic from

Flat-ring coordinates μ, ρ, ϕ form an orthogonal coordinate system in $\mathbb{R}^3$ with rotational symmetry. We define these coordinates by

$$x = \frac{\cos\phi}{T}, \qquad y = \frac{\sin\phi}{T}, \qquad z = \frac{1}{T}\left(\frac{-\mu\rho}{a}\right)^{1/2}, \tag{2.1}$$

where

$$T = \left(\frac{(a-\mu)(a-\rho)}{a(a-1)}\right)^{1/2} + \left(\frac{(1-\mu)(1-\rho)}{a-1}\right)^{1/2}. \tag{2.2}$$

The parameter $a > 1$ is given. The coordinates μ, ρ vary according to

$$-\infty < \mu < 0 < \rho < 1.$$

The roots appearing in (2.1), (2.2) are understood as positive roots. Miller [15, system (15), p. 210] considers flat-ring coordinates μ', ρ', ϕ depending on a parameter a' in a slightly different form. The connection is given by

$$a' = \frac{a}{a-1}, \qquad \mu = a - (a-1)\mu', \qquad \rho = a - (a-1)\rho',$$

where

$$1 < \rho' < \frac{a}{a-1} < \mu' < \infty.$$

For $\phi = 0$ we obtain a planar coordinate system

$$x = \frac{1}{T}, \qquad z = \frac{1}{T}\left(\frac{-\mu\rho}{a}\right)^{1/2}. \tag{2.3}$$

This defines a function g by setting $(x, z) = g(\mu, \rho)$. We introduce the sets

$$A = (-\infty, 0) \times (0, 1), \qquad Q = \left\{(x, z) \colon x, z > 0, \ x^2 + z^2 < 1\right\}.$$

Proposition 2.1. *g maps A bijectively onto Q. If $(x, z) = g(\mu, \rho)$ then*

$$\frac{\left(x^2 + z^2 + 1\right)^2}{\tau - a} - \frac{\left(x^2 + z^2 - 1\right)^2}{\tau - 1} - \frac{4z^2}{\tau} = 0 \qquad for \quad \tau = \mu, \rho, \tag{2.4}$$

so the inverse map $(\mu, \rho) = g^{-1}(x, z)$ is given by

$$\mu = \frac{1}{8x^2}\left(-\omega - \left(\omega^2 + 64ax^2z^2\right)^{1/2}\right), \qquad \rho = \frac{1}{8x^2}\left(-\omega + \left(\omega^2 + 64ax^2z^2\right)^{1/2}\right), \tag{2.5}$$

where

$$\omega = 4(1+a)z^2 + a\left(x^2 + z^2 - 1\right)^2 - \left(x^2 + z^2 + 1\right)^2.$$

The map g and its inverse map are real-analytic.

Proof. Let $(\mu, \rho) \in A$ and $(x, z) = g(\mu, \rho)$. Then

$$\frac{1}{T}\left(1 - \frac{\mu\rho}{a}\right) = \left(\frac{(a-\mu)(a-\rho)}{a(a-1)}\right)^{1/2} - \left(\frac{(1-\mu)(1-\rho)}{a-1}\right)^{1/2},$$

which leads to

$$\frac{x^2 + z^2 - 1}{2x} = \frac{1}{2}\left(\frac{1}{T} - \frac{1}{T}\frac{\mu\rho}{a} - T\right) = -\left(\frac{(1-\mu)(1-\rho)}{a-1}\right)^{1/2}. \tag{2.6}$$

By a similar calculation we obtain

$$\frac{x^2 + z^2 + 1}{2x} = \left(\frac{(a-\mu)(a-\rho)}{a(a-1)}\right)^{1/2}. \tag{2.7}$$

By (2.6) and (2.7),

$$\frac{\left(x^2 + z^2 + 1\right)^2}{\mu - a} - \frac{\left(x^2 + z^2 - 1\right)^2}{\mu - 1} - \frac{4z^2}{\mu} = 4x^2\left(\frac{\rho - a}{a(a-1)} - \frac{\rho - 1}{a-1} + \frac{\rho}{a}\right) = 0.$$

With the same calculation we show that (2.4) also holds for $\tau = \rho$.

If $(x, z) = g(\mu, \rho)$ with $(\mu, \rho) \in A$ then it is clear that $x, z > 0$, and (2.6) shows that $(x, z) \in Q$. Conversely, suppose that $(x, z) \in Q$. We consider the quadratic polynomial

$$F(\tau) = \tau(\tau - 1)(\tau - a)\left(\frac{\left(x^2 + z^2 + 1\right)^2}{\tau - a} - \frac{\left(x^2 + z^2 - 1\right)^2}{\tau - 1} - \frac{4z^2}{\tau}\right). \tag{2.8}$$

The factor of τ^2 is

$$\left(x^2 + z^2 + 1\right)^2 - \left(x^2 + z^2 - 1\right)^2 - 4z^2 = 4x^2 > 0.$$

Moreover,

$$F(0) = -4az^2 < 0,$$
$$F(1) = (a-1)\left(x^2 + z^2 - 1\right)^2 > 0.$$

It follows that there is a unique $(\mu, \rho) \in A$ such that

$$F(\tau) = 4x^2(\tau - \mu)(\tau - \rho) \qquad \text{for} \quad \tau \in \mathbb{R}. \tag{2.9}$$

By setting $\tau = 0, 1, a$ in (2.9) it follows that

$$4x^2\mu\rho = -4az^2,$$
$$4x^2(1-\mu)(1-\rho) = (a-1)\left(x^2 + z^2 - 1\right)^2,$$

$$4x^2(a-\mu)(a-\rho) = a(a-1)\big(x^2+z^2+1\big)^2.$$

Then we obtain

$$T = \frac{x^2+z^2+1}{2x} - \frac{x^2+z^2-1}{2x} = \frac{1}{x} \qquad \text{and} \qquad z^2 = x^2\frac{-\mu\rho}{a} = \frac{1}{T^2}\frac{-\mu\rho}{a}.$$

It follows that $g(\mu,\rho) = (x,z)$ and so we have shown that g is surjective.

Suppose that $g(\mu,\rho) = g(\tilde\mu,\tilde\rho) = (x,z)$. By (2.4), μ, ρ, $\tilde\mu$, $\tilde\rho$ are all roots of $F(\mu) = 0$ with F defined in (2.8). Since $(\mu,\rho),(\tilde\mu,\tilde\rho) \in A$, this implies $(\mu,\rho) = (\tilde\mu,\tilde\rho)$. Thus g is injective. It is clear that g and g^{-1} are analytic maps. ∎

Some coordinate lines are shown in Figure 1. Note that the boundary of the region Q is given by the quarter circle $\rho = 1$, the vertical segment $\mu = -\infty$, and two horizontal segments $[0,b]$, $[b,1]$ represented by $\rho = 0$ and $\mu = 0$ respectively, where

$$b = \frac{\sqrt{a}-1}{\sqrt{a-1}} = \frac{\sqrt{a}-1}{\sqrt{a}+1}.$$
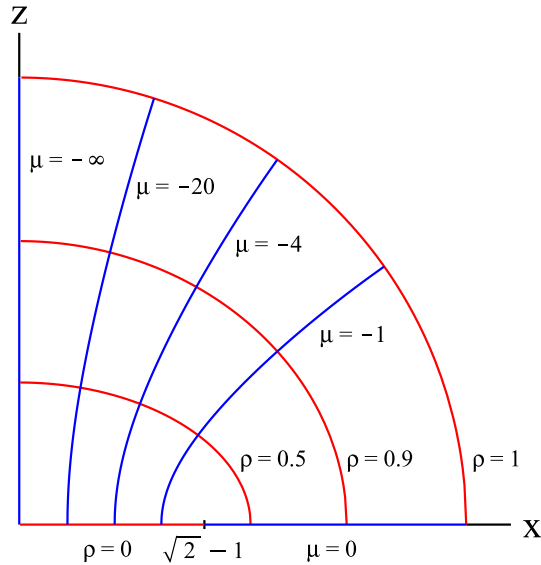

Figure 1. Coordinate lines of system (2.3) for $a = 2$.

If we change the signs of the square roots in (2.1) and (2.2) we obtain coordinates for other regions in $\mathbb{R}^2$ according to Table 1.

2.2   Flat-ring coordinates in transcendental form

We write $a = k^{-2}$ with $k \in (0,1)$, and set

$$\rho = \operatorname{sn}^2(s,k), \qquad \mu = \operatorname{sn}^2(it,k), \tag{2.10}$$

where $\operatorname{sn}(u,k)$ denotes the Jacobian elliptic function of modulus k (see [24, Chapter XXII], [18, Chapter 22]). We will also use the Jacobian elliptic functions $\operatorname{cn}(z,k)$, $\operatorname{dn}(z,k)$, the complementary modulus $k' = \sqrt{1-k^2}$, and the complete elliptic integrals of the first kind

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad K' = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}}.$$

Table 1. Signs of roots in regions of (x, z)-plane.

	$\left(\dfrac{(a-\mu)(a-\rho)}{a(a-1)}\right)^{1/2}$	$\left(\dfrac{(1-\mu)(1-\rho)}{a-1}\right)^{1/2}$	$\left(\dfrac{-\mu\rho}{a}\right)^{1/2}$
$x > 0,\ z > 0,\ x^2 + z^2 < 1$	$+$	$+$	$+$
$x > 0,\ z > 0,\ x^2 + z^2 > 1$	$+$	$-$	$+$
$x > 0,\ z < 0,\ x^2 + z^2 > 1$	$+$	$-$	$-$
$x > 0,\ z < 0,\ x^2 + z^2 < 1$	$+$	$+$	$-$

By (2.10), $s \in (0, K)$ is mapped bijectively onto $\rho \in (0, 1)$, and $t \in (0, K')$ is mapped bijectively onto $\mu \in (-\infty, 0)$. Substituting (2.10) into (2.1) and (2.2), we obtain

$$x = \frac{\cos\phi}{T}, \qquad y = \frac{\sin\phi}{T}, \qquad z = \frac{-ik\,\mathrm{sn}(s,k)\,\mathrm{sn}(it,k)}{T}, \tag{2.11}$$

where

$$T = \frac{1}{k'}\,\mathrm{dn}(s,k)\,\mathrm{dn}(it,k) + \frac{k}{k'}\,\mathrm{cn}(s,k)\,\mathrm{cn}(it,k).$$

By Proposition 2.1, the coordinate surface $s = s_0$ is given by

$$\frac{k^2\left(x^2 + y^2 + z^2 + 1\right)^2}{\mathrm{dn}^2(s_0,k)} - \frac{\left(x^2 + y^2 + z^2 - 1\right)^2}{\mathrm{cn}^2(s_0,k)} + \frac{4z^2}{\mathrm{sn}^2(s_0,k)} = 0,$$

and the coordinate surface $t = t_0$ is given by

$$\frac{k^2\left(x^2 + y^2 + z^2 + 1\right)^2}{\mathrm{dn}^2(it_0,k)} - \frac{\left(x^2 + y^2 + z^2 - 1\right)^2}{\mathrm{cn}^2(it_0,k)} + \frac{4z^2}{\mathrm{sn}^2(it_0,k)} = 0. \tag{2.12}$$

When $\phi = 0$ we have the planar coordinate system

$$x = \frac{1}{T}, \qquad z = \frac{-ik\,\mathrm{sn}(s,k)\,\mathrm{sn}(it,k)}{T}. \tag{2.13}$$

We note that the coordinates s, t can also be written in complex form [20, equation (8)] as

$$x + iz = f(s + it), \qquad f(w) = \frac{1}{k'}(\mathrm{dn}(w,k) - k\,\mathrm{cn}(w,k)).$$

The corresponding metric coefficients

$$h_s = \left(\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2\right)^{1/2}, \qquad h_t = \left(\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2\right)^{1/2},$$

are [20, equation (9)]

$$h_s = h_t = \frac{k}{T}\left(\mathrm{sn}^2(s,k) - \mathrm{sn}^2(it,k)\right)^{1/2}. \tag{2.14}$$

If we take $s \in (0, 2K)$, $t \in (0, K')$ then we obtain a coordinate system for the first quadrant $x, z > 0$. If the point (x, z) is given by coordinates s, t then the point

$$(\tilde{x}, \tilde{z}) = \left(\frac{x}{x^2 + z^2}, \frac{z}{x^2 + z^2}\right),$$

obtained from (x, z) by inversion at the unit circle, has coordinates $\tilde{s} = 2K - s$, $\tilde{t} = t$.

There are three different ways to extend the coordinates (2.13) to the half-plane $x > 0$.

1) We take $s \in (-2K, 2K)$ and $t \in (0, K')$. The reflection $(x, z) \mapsto (x, -z)$ is given by $s \mapsto -s$. Then (2.13) establishes a bijective map between (s, t) and the region,

$$Q_1 = \{(x, z) \colon x > 0\} \setminus \{(x, 0) \colon x \geq b\}, \qquad b = \frac{1 - k}{k'}, \tag{2.15}$$

which together with its inverse is real-analytic. Q_1 is the right-hand half-plane with a cut from $x = b$ to $x = +\infty$ along the x-axis. In order to express s, t in terms of x, z form μ, ρ according to (2.5) and set

$$s_1 = \operatorname{arcsn}\left(\sqrt{\rho}, k\right), \qquad t_1 = \operatorname{arcsc}\left(\sqrt{-\mu}, k'\right),$$

using the inverse Jacobian elliptic functions from [18, Section 22.15]. Then the flat-ring coordinates s, t of x, z are given by $t = t_1$ and

$$s = \begin{cases} s_1 & \text{if} \quad R^2 + z^2 \leq 1 \quad \text{and} \quad z \geq 0, \\ 2K - s_1 & \text{if} \quad R^2 + z^2 > 1 \quad \text{and} \quad z \geq 0, \\ -s_1 & \text{if} \quad R^2 + z^2 \leq 1 \quad \text{and} \quad z < 0, \\ s_1 - 2K & \text{if} \quad R^2 + z^2 > 1 \quad \text{and} \quad z < 0. \end{cases}$$

Some coordinate lines of this coordinate system are depicted in Figure 2.
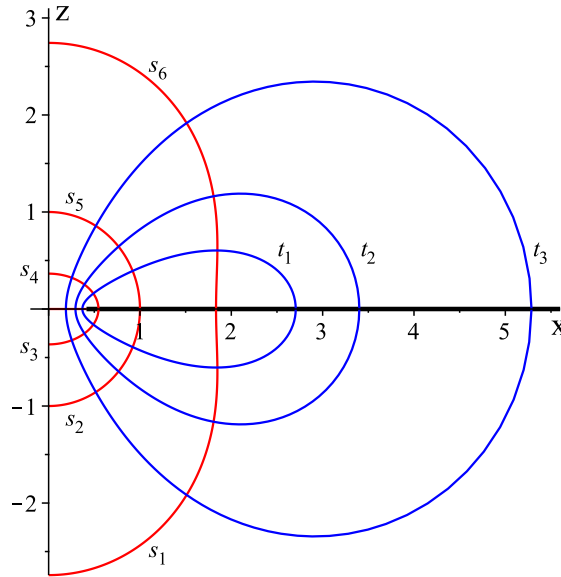
Figure 2. Coordinate lines $s = s_i$, $s_1 = -1.5K$, $s_2 = -K$, $s_3 = -0.5K$, $s_4 = 0.5K$, $s_5 = K$, $s_6 = 1.5K$ and $t = t_i$, $t_1 = 0.3K'$, $t_2 = 0.5K'$, $t_3 = 0.7K'$ of system (2.13) for $a = 2$.

A second variant is as follows.

2) We take $s \in (0, 2K)$ and $t \in (-K', K')$. The reflection $(x, z) \mapsto (x, -z)$ is given by $t \mapsto -t$. Then (2.13) establishes a bijective map between (s, t) and the region,

$$Q_2 = \{(x, z) \colon x > 0\} \setminus \{(x, 0) \colon 0 < x \leq b \text{ or } x \geq b^{-1}\}.$$

Q_2 is the right-hand half-plane with cuts along the x-axis from $x = 0$ to $x = b$ and from b^{-1} to ∞.

3) We take $s \in (0, 4K)$ and $t \in (0, K')$. The reflection $(x, z) \mapsto (x, -z)$ is given by $s \mapsto 4K - s$. Then (2.13) establishes a bijective map between (s, t) and the region,

$$Q_3 = \{(x, z)\colon x > 0\} \setminus \{(x, 0)\colon 0 < x \le b^{-1}\}.$$

3 $\mathcal{R}$-separation of the Laplace equation

Let $u(x, y, z)$ be a function defined in the region

$$\tilde{Q}_1 = \{(x \cos \phi, x \sin \phi, z)\colon (x, z) \in Q_1, \ \phi \in (-\pi, \pi)\} \tag{3.1}$$

with Q_1 from (2.15). Note that $\tilde{Q}_1$ consists of all of $\mathbb{R}^3$ with the exception of the exterior of a disk in the xy-plane centered at the origin with radius b, and the half-plane $y = 0$, $x \le 0$. In this region, we have the first variant of transcendental flat-ring coordinates with $s \in (-2K, 2K)$, $t \in (0, K')$, $\phi \in (-\pi, \pi)$.

Using cylindrical coordinates

$$x = r \cos \phi, \qquad y = r \sin \phi, \qquad z = z,$$

we obtain

$$r^{1/2} \Delta u = \left[\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2} \left(\frac{\partial^2}{\partial \phi^2} + \frac{1}{4} \right) \right] (r^{1/2} u).$$

Transforming to coordinates s, t we find [20, equation (10)]

$$r^{5/2} M \Delta u = \left[\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} + M \left(\frac{\partial^2}{\partial \phi^2} + \frac{1}{4} \right) \right] (r^{1/2} u), \tag{3.2}$$

where

$$M = k^2 \big(\operatorname{sn}^2(s, k) - \operatorname{sn}^2(\mathrm{i}t, k) \big).$$

Substituting

$$r^{1/2} u = u_1(s) u_2(\mathrm{i}t) u_3(\phi)$$

we obtain the following theorem.

Theorem 3.1. *Suppose that u has the form*

$$u(x, y, z) = \big(x^2 + y^2 \big)^{-1/4} u_1(s) u_2(\mathrm{i}t) u_3(\phi), \tag{3.3}$$

where $s \in (-2K, 2K)$, $t \in (0, K')$, and $\phi \in (-\pi, \pi)$. If u_1 and u_2 satisfy the Lamé equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\zeta^2} + \left(h - \left(m^2 - \frac{1}{4} \right) k^2 \operatorname{sn}^2(\zeta, k) \right) w = 0, \tag{3.4}$$

and u_3 satisfies

$$\frac{\mathrm{d}^2 u_3}{\mathrm{d}\phi^2} + m^2 u_3 = 0 \tag{3.5}$$

for some constants m, h, then $u(x, y, z)$ satisfies $\Delta u = 0$.

Note that u_1 and u_2 satisfy the same differential equation but on different intervals $\zeta \in (-2K, 2K)$ and $\zeta \in (0, \mathrm{i}K')$, respectively. Since the function $\operatorname{sn}^2(\zeta, k)$ is analytic on the strip $|\operatorname{Im} \zeta| < K'$, u_1, u_2 extend analytically to this strip.

4 Lamé functions of the first and second kind

In this section we consider particular solutions of the Lamé equation that are needed to introduce flat-ring harmonics in Section 5. The Lamé equation is

$$\frac{d^2 \mathsf{E}}{ds^2} + \left(h - \nu(\nu+1)k^2 \operatorname{sn}^2(s,k)\right)\mathsf{E} = 0, \tag{4.1}$$

where $\nu \geq -\frac{1}{2}$ and the modulus k of the Jacobian elliptic function sn satisfies $0 < k < 1$. The simply-periodic Lamé functions are those functions which are solutions of the Lamé equation (4.1) with period $4K$. We refer to these as Lamé functions of the first kind. In the application to flat-ring coordinates k is fixed (k is determined by the coordinate system) while $\nu = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$. The constant h is the eigenvalue parameter. The function $\operatorname{sn}^2(s,k)$ is even and has period $2K$ so that (4.1) is a special case of an even Hill equation [18, Section 28.29(i)]. The eigenvalue problem asks for values of h such that (4.1) has a nontrivial solution $\mathsf{E}(s)$ with period $4K$. This eigenvalue problem is treated in [9, equation (15.5.1)] and [14, Theorem 1.2]. The eigenvalue problem splits into four regular Sturm–Liouville problems with separated boundary conditions [22, Chapter VI] according to Table 2.

Table 2. Notation of simply-periodic Lamé functions.

boundary conditions	eigenvalues	eigenfunctions	period
$\mathsf{E}(0) = \mathsf{E}(K) = 0$	$b_\nu^{2n+2}(k)$	$\operatorname{Es}_\nu^{2n+2}(s,k)$	$\mathsf{E}(s+2K) = \mathsf{E}(s)$
$\mathsf{E}'(0) = \mathsf{E}(K) = 0$	$b_\nu^{2n+1}(k)$	$\operatorname{Es}_\nu^{2n+1}(s,k)$	$\mathsf{E}(s+2K) = -\mathsf{E}(s)$
$\mathsf{E}(0) = \mathsf{E}'(K) = 0$	$a_\nu^{2n+1}(k)$	$\operatorname{Ec}_\nu^{2n+1}(s,k)$	$\mathsf{E}(s+2K) = -\mathsf{E}(s)$
$\mathsf{E}'(0) = \mathsf{E}'(K) = 0$	$a_\nu^{2n}(k)$	$\operatorname{Ec}_\nu^{2n}(s,k)$	$\mathsf{E}(s+2K) = \mathsf{E}(s)$

In this table, in each case $n \in \mathbb{N}_0$ denotes the number of zeros of the eigenfunction in the interval $(0,K)$. All eigenfunctions are normalized so that

$$\int_0^K \mathsf{E}(s)^2 \, ds = 1.$$

Theorem 4.1. *Each of the systems*

$$\left\{\operatorname{Es}_\nu^{2n+2}\right\}_{n=0}^\infty, \qquad \left\{\operatorname{Es}_\nu^{2n+1}\right\}_{n=0}^\infty, \qquad \left\{\operatorname{Ec}_\nu^{2n+1}\right\}_{n=0}^\infty, \qquad \left\{\operatorname{Ec}_\nu^{2n}\right\}_{n=0}^\infty,$$

forms an orthonormal basis in the Hilbert space $L^2(0,K)$. All four systems combined form an orthonormal basis of $L^2(-2K, 2K)$ provided each function is multiplied by $\frac{1}{2}$.

In our application of simply-periodic Lamé functions to flat-ring coordinates we also need some estimates for these functions that we provide in the following.

Lemma 4.2. *We have*

$$\frac{\pi^2 n^2}{4K^2} \leq a_\nu^n(k) \leq \nu(\nu+1)k^2 + \frac{\pi^2 n^2}{4K^2} \qquad if \quad \nu \geq 0,$$

$$\nu(\nu+1)k^2 + \frac{\pi^2 n^2}{4K^2} \leq a_\nu^n(k) \leq \frac{\pi^2 n^2}{4K^2} \qquad if \quad -\frac{1}{2} \leq \nu < 0.$$

The same estimates hold with b_ν^{n+1} in place of a_ν^n.

Proof. The stated inequalities follow from

$$0 \leq \nu(\nu+1)k^2 \operatorname{sn}^2(s,k) \leq \nu(\nu+1)k^2 \qquad \text{if} \quad \nu \geq 0,$$

$$\nu(\nu+1)k^2 \leq \nu(\nu+1)k^2 \operatorname{sn}^2(s,k) \leq 0 \qquad \text{if} \quad -\frac{1}{2} \leq \nu < 0,$$

using a comparison theorem for eigenvalues [22, Section 27.IX]. ∎

The next lemma is a refinement of a result in [8, p. 288].

Lemma 4.3. *Let $q\colon [a,b] \to \mathbb{R}$ be a continuous function, λ a real number, and $y\colon [a,b] \to \mathbb{R}$ a solution of the differential equation*

$$-y'' + q(s)y = \lambda y \tag{4.2}$$

such that $y(a)y'(a) = y(b)y'(b) = 0$, and $\int_a^b y(s)^2\, \mathrm{d}s = 1$. Then

$$y(s)^2 \leq 2(b-a)^{-1} + 2(R(q))^{1/2} \qquad \text{for} \quad s \in [a,b],$$

where

$$R(q) := \max\{q(s)\colon a \leq s \leq b\} - \min\{q(s)\colon a \leq s \leq b\}.$$

Proof. We set

$$u = \min\{q(s)\colon a \leq s \leq b\}, \qquad v = R(q).$$

Obviously, the lemma is true when $v = 0$ so we assume that $v > 0$. Since we may add the same constant to q and λ, we also assume that $u > 0$. Multiplying (4.2) by y and integrating from a to b we obtain

$$\int_a^b y'(s)^2\, \mathrm{d}s + \int_a^b q(s)y(s)^2\, \mathrm{d}s = \lambda.$$

It follows that

$$\int_a^b y'(s)^2\, \mathrm{d}s \leq \lambda - u \leq \lambda. \tag{4.3}$$

Multiplying (4.2) by y' and integrating from s to t, we obtain

$$y'(t)^2 + \lambda y(t)^2 = 2f(s,t) + y'(s)^2 + \lambda y(s)^2, \tag{4.4}$$

where

$$f(s,t) := \int_s^t q(r)y(r)y'(r)\, \mathrm{d}r.$$

Integrating (4.4) from $s = a$ to $s = b$, we find

$$y'(t)^2 + \lambda y(t)^2 = (b-a)^{-1}\left(2\int_a^b f(s,t)\, \mathrm{d}s + \int_a^b y'(s)^2\, \mathrm{d}t + \lambda\right). \tag{4.5}$$

Using (4.3) we have

$$|f(s,t)| \leq (u+v)\left(\int_a^b y'(s)^2\, \mathrm{d}t\right)^{1/2} \leq (u+v)(\lambda-u)^{1/2}.$$

Therefore, (4.3) and (4.5) give

$$\lambda y(t)^2 \leq y'(t)^2 + \lambda y(t)^2 \leq 2(u+v)(\lambda - u)^{1/2} + (b-a)^{-1}2\lambda.$$

By (4.3), $\lambda \geq u > 0$, so we can divide by λ:

$$y(t)^2 \leq 2(b-a)^{-1} + 2(u+v)\lambda^{-1}(\lambda - u)^{1/2}.$$

Now

$$\lambda^{-1}(\lambda - u)^{1/2} \leq \frac{1}{2}u^{-1/2} \qquad \text{for} \quad \lambda \geq u,$$

so we get

$$y(t)^2 \leq 2(b-a)^{-1} + (u+v)u^{-1/2}.$$

Our $v > 0$ is given but we can choose $u > 0$ as we like. Taking $u = v$ we get the desired estimate. ∎

Lemma 4.4. *We have*

$$\{\mathrm{Ec}_\nu^n(s,k)\}^2 \leq \frac{2}{K} + 2k|\nu|^{1/2}(\nu+1)^{1/2} \qquad \text{for} \quad n \in \mathbb{N}_0, \quad \nu \geq -\frac{1}{2}, \quad s \in \mathbb{R}.$$

The inequality also holds with Es_ν^{n+1} *in place of* Ec_ν^n.

Proof. If $s \in [0, K]$ this follows from Lemma 4.3 with $q(s) = \nu(\nu+1)k^2\operatorname{sn}^2(s,k)$. Then $R(q) = |\nu|(\nu+1)k^2$. Since the simply-periodic Lamé functions are even or odd and have period or half-period $2K$, the stated inequality holds for all $s \in \mathbb{R}$. ∎

Next, we need an estimate for the functions $\mathrm{Ec}_\nu^n(it)$ and $\mathrm{Es}_\nu^{n+1}(it)$ for $-K' < t < K'$. Setting $W(t) = \mathsf{E}(it)$ and using [18, Section 22.6(iv)], (4.1) becomes the modified Lamé equation

$$\frac{\mathrm{d}^2 W}{\mathrm{d}t^2} - \left(h + \nu(\nu+1)k^2\operatorname{sc}^2(t,k')\right)W = 0. \tag{4.6}$$

In (4.6) $\operatorname{sc}(t,k') := \operatorname{sn}(t,k')/\operatorname{cn}(t,k')$ employs Glaisher's notation [18, Section 22.2].

It should be noted that $W(t) = \mathrm{Ec}_\nu^n(it,k)$ is a solution of (4.6) with $h = a_\nu^n$. If n is even then $W(t)$ is a real-valued even solution. If n is odd then $\mathrm{i}W(t)$ is a real-valued odd solution. A similar statement holds for Es_ν^{n+1}.

The following lemma is a minor modification of Lemma 5.7 in [6].

Lemma 4.5. *Let* $u \colon [0,b] \to \mathbb{R}$ *be a solution of the differential equation*

$$u''(t) = q(t)u(t), \qquad t \in [0,b],$$

determined by the initial conditions $u(0) = 1$, $u'(0) = 0$ *or* $u(0) = 0$, $u'(0) = 1$, *where* $q\colon$ *$[0,b] \to \mathbb{R}$ is a continuous function. Suppose that* $q(t) \geq 0$ *on* $[0,b]$ *and* $q(t) \geq \lambda^2$ *on* $[c,b]$ *for some* $\lambda > 0$ *and* $c \in [0,b)$. *Then* $u(t) > 0$ *and* $u'(t) \geq 0$ *for* $t \in (0,b]$, *and*

$$\frac{u(t)}{u(b)} \leq 2\mathrm{e}^{-\lambda(b-c)} \qquad \text{for all} \quad t \in [0,c].$$

Proof. Since $q(t) \geq 0$ it follows from the initial conditions that $u(t) > 0, u'(t) \geq 0$ for $t \in (0, b]$. The function $z = u'/u$ satisfies the Riccati equation $z' + z^2 = q(t)$, and we know that $z(c) \geq 0$. Therefore,

$$z(t) \geq \lambda \tanh(\lambda(t - c)) \qquad \text{for} \quad t \in [c, b].$$

Integrating from $t = c$ to $t = b$ gives

$$\log \frac{u(b)}{u(c)} \geq \log \cosh(\lambda(b - c)) \geq \log \left(\frac{1}{2} e^{\lambda(b-c)} \right),$$

which yields the claim since u is nondecreasing. ∎

Lemma 4.6. *Let $0 < c < b < K'$. Then there is a constant $p \in (0, 1)$ independent of ν, n and t such that*

$$0 \leq \frac{\mathrm{Ec}_\nu^n(it, k)}{\mathrm{Ec}_\nu^n(ib, k)} \leq 2p^{n+\nu} \qquad \text{for} \quad \nu > 0, \quad n \in \mathbb{N}_0, \quad t \in [0, c].$$

The same result is true for Es_ν^{n+1} in place of Ec_ν^n.

Proof. Apply Lemma 4.5 with $q(t) = a_\nu^n(k) + \nu(\nu + 1)\mathrm{sc}^2(t, k')$. Since $\nu > 0$, Lemma 4.2 yields $a_\nu^n(k) \geq 0$ and so $q(t) \geq 0$. Let $\lambda^2 = q(c) > 0$. Then $q(t) \geq \lambda^2$ on $[c, b]$. Since

$$\lambda^2 = q(c) \geq \frac{\pi^2 n^2}{4K^2} + \nu(\nu + 1)k^2 \mathrm{sc}^2(c, k') \geq \gamma^2 (n + \nu)^2,$$

where

$$\gamma = 2^{-1/2} \min \left\{ \frac{\pi}{2K}, \, k\, \mathrm{sc}(c, k') \right\}.$$

This implies the assertion with $p = e^{-\gamma(b-c)}$. ∎

Unfortunately, the case $\nu = -\frac{1}{2}$ is not covered by Lemma 4.6. It is not even clear that $\mathrm{Ec}_{-1/2}^n(it, k) \neq 0$ for $0 < t < K'$. To deal with this case we apply the transformation

$$W(t) = \mathrm{cd}^{1/2}(t, k')\, w(t)$$

to (4.6), and obtain

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - k^2 \frac{\mathrm{sn}(t, k')}{\mathrm{cn}(t, k')\,\mathrm{dn}(t, k')} \frac{\mathrm{d}w}{\mathrm{d}t} - Q(t)w = 0, \tag{4.7}$$

where

$$Q(t) = h - \frac{1}{4}k^2 + \left(\nu + \tfrac{1}{2}\right)^2 k^2 \mathrm{sc}^2(t, k') + \frac{3}{4}k^2 \mathrm{nd}^2(t, k').$$

Equation (4.7) has regular singularities at $t = \pm K'$ with exponents $\pm\left(\nu + \frac{1}{2}\right)$ at both points.

Lemma 4.7. *If $\nu > 0$, $n \in \mathbb{N}_0$, the function $|\mathrm{Ec}_\nu^n(it, k)|$ is increasing for $0 < t < K'$ and diverges to infinity as $t \to K'$. If $-\frac{1}{2} \leq \nu \leq 0$, $n \in \mathbb{N}_0$, the function $\mathrm{dc}^{1/2}(t, k')|\mathrm{Ec}_\nu^n(it, k)|$ is increasing for $0 < t < K'$ and diverges to infinity as $t \to K'$. In particular, $\mathrm{Ec}_\nu^n(it, k) \neq 0$ for $0 < t < K'$. The same statements hold with Es_ν^{n+1} in place of Ec_ν^n.*

Proof. If $\nu > 0$ we consider differential equation (4.6) with $h = a_\nu^n(k)$. By Lemma 4.2, $a_\nu^n(k) + \nu(\nu + 1)k^2 \operatorname{sc}^2(t, k') > 0$ for $0 < t < K'$. This implies that $|\operatorname{Ec}_\nu^n(\mathrm{i}t, k)|$ is increasing. It diverges to infinity because $t = K'$ is a regular singularity of this equation with exponents $-\nu$ and $\nu + 1$. If $-\frac{1}{2} \leq \nu < 0$ we consider differential equation (4.7) with $h = a_\nu^n(k)$. By Lemma 4.2,

$$Q(t) \geq a_\nu^n(k) - \frac{1}{4}k^2 + \frac{3}{4}k^2 \geq \nu(\nu + 1)k^2 + \frac{1}{2}k^2 > 0.$$

It follows that the solutions of (4.7) determined by initial conditions $w(0) = 1$, $w'(0) = 0$ or $w(0) = 0$, $w'(0) = 1$ are positive and increasing on $(0, K')$. If $-\frac{1}{2} < \nu < 0$ we see as before that $w(t)$ diverges to infinity as $t \to K'$. This is also true for $\nu = -\frac{1}{2}$ by noting that the unique solution w of (4.7) with $\nu = -\frac{1}{2}$ which is analytic at $t = K'$ and satisfies $w(K') = 1$ has $w'(K') = 0$ and $w''(K') = \frac{1}{2}a_{-1/2}^n(k) - \frac{1}{8}k^2 + \frac{3}{8} > 0$. ∎

Lemma 4.8. *Let $0 < b < K'$. Then there are constants $C > 0$ and $p \in (0, 1)$ independent of n and t such that*

$$0 \leq \frac{\operatorname{Ec}_{-\frac{1}{2}}^n(\mathrm{i}t, k)}{\operatorname{Ec}_{-\frac{1}{2}}^n(\mathrm{i}b, k)} \leq Cp^n \qquad \text{for} \quad n \in \mathbb{N}_0, \quad t \in [0, b].$$

The same result is true for $\operatorname{Es}_\nu^{n+1}$ in place of Ec_ν^n.

Proof. We apply Lemma 4.5 with $q(t) = a_\nu^n(k) - \frac{1}{4}k^2 \operatorname{sc}^2(t, k')$. By Lemma 4.2,

$$q(t) \geq -\frac{1}{4}k^2 + \frac{\pi^2 n^2}{4K^2} - \frac{1}{4}k^2 \operatorname{sc}^2(b, k') \qquad \text{for} \quad 0 \leq t \leq b.$$

If n is sufficiently large then

$$q(t) \geq \left(\frac{\pi}{2K}n - B\right)^2 \qquad \text{for} \quad 0 \leq t \leq b,$$

where B is a positive constant independent of n and t. Then Lemma 4.5 implies the desired estimate for large n. Using Lemma 4.7, we see that by choosing C large enough the estimate is true for all $n \in \mathbb{N}_0$. ∎

The Lamé equation (4.1) has regular singularities at $s = \pm \mathrm{i}K'$ with exponents $\nu + 1$, $-\nu$. The Lamé function of the second kind $\operatorname{Fc}_\nu^n(s, k)$, $|\operatorname{Im} s| < K'$, is defined as a solution of (4.1) with $h = a_\nu^n(k)$ belonging to the exponent $\nu + 1$, that is,

$$\operatorname{Fc}_\nu^n(s, k) = \sum_{p=0}^\infty d_p(s - \mathrm{i}K')^{2p+\nu+1}, \qquad d_0 \neq 0, \tag{4.8}$$

for s close to $\mathrm{i}K'$. Similarly, $\operatorname{Fs}_\nu^{n+1}$ is the solution of (4.1) with $h = b_\nu^{n+1}$ which belongs to the exponent $\nu + 1$ at $s = \mathrm{i}K'$. It follows from Lemma 4.7 that Ec_ν^n, Fc_ν^n and Es_ν^n, Fs_ν^n are linearly independent. Therefore, we can normalize the Lamé functions of the second kind such that

$$\mathsf{F}(\mathrm{i}t)\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}(\mathrm{i}t) - \mathsf{E}(\mathrm{i}t)\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{F}(\mathrm{i}t) = 1, \tag{4.9}$$

where $\mathsf{E} = \operatorname{Ec}$, $\mathsf{F} = \operatorname{Fc}$ or $\mathsf{E} = \operatorname{Es}$, $\mathsf{F} = \operatorname{Fs}$.

5   Flat-ring harmonics

5.1   Internal flat-ring harmonics

We use the first variant of flat-ring coordinates s, t, ϕ with $s \in (-2K, 2K)$, $t \in (0, K')$ and $\phi \in (-\pi, \pi)$. If $t_0 \in (0, K')$ is a fixed value, then the coordinate surface (2.12) describes a closed surface, a flat-ring as depicted in Figure 3. Also, in Figure 4 we present a three-dimensional visualization of the flat-rings and their orthogonal surfaces which we refer to as peanuts.
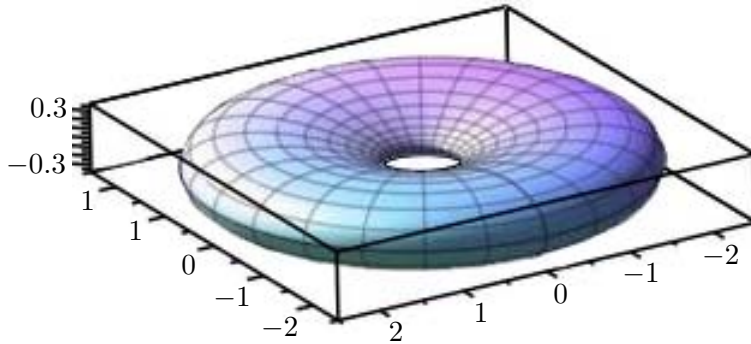
Figure 3. Flat-ring, $k = 0.7$, $t = 0.2K'$.
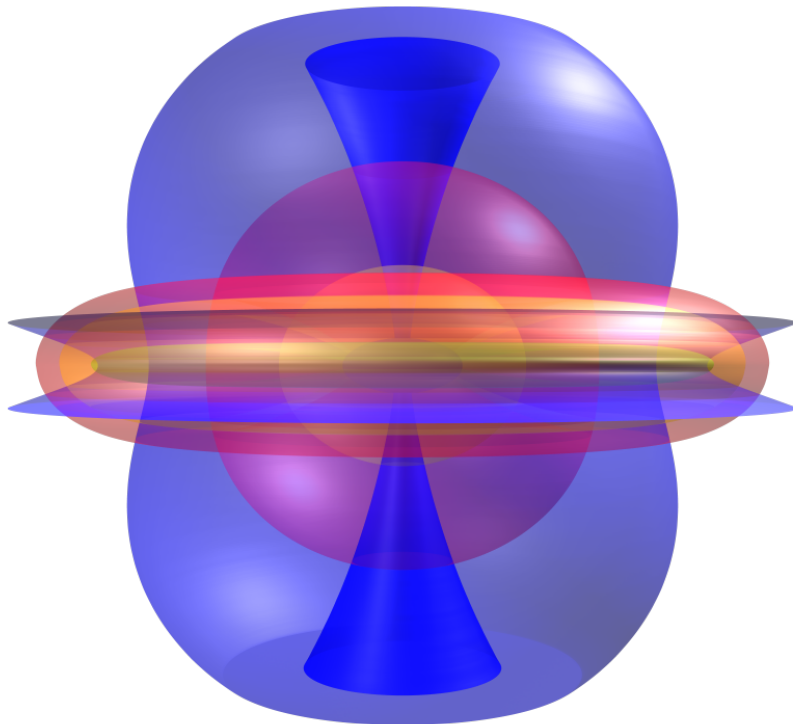
Figure 4. For $k = 0.7$ this figure depicts a three-dimensional visualization of rotationally-invariant flat-ring cyclides for $t \in \{\pm 0.10K', \pm 0.28K', \pm 0.35K', \pm 0.90K'\}$ and orthogonal peanut shaped cyclides $s \in \{0.369K, 0.923K, 1.26K, 1.48K, 1.75K\}$. Note that the ring cyclides are only partially shown for $t = \pm 0.1K'$ and the peanut cyclide is only partially shown for $s = 1.75K$ (since these would extend beyond the figure).

The interior of this flat-ring is given by

$$D_1 = \left\{ \mathbf{r} \in \mathbb{R}^3 \colon \frac{k^2(\|\mathbf{r}\|^2 + 1)^2}{\mathrm{dn}^2(\mathrm{i}t_0, k)} - \frac{(\|\mathbf{r}\|^2 - 1)^2}{\mathrm{cn}^2(\mathrm{i}t_0, k)} + \frac{4z^2}{\mathrm{sn}^2(\mathrm{i}t_0, k)} > 0 \right\}, \tag{5.1}$$

where $\mathbf{r} = (x, y, z)$, $\|\mathbf{r}\| = \left(x^2 + y^2 + z^2\right)^{1/2}$. Internal flat-ring harmonics are harmonic functions of the separated form (3.3) which are harmonic in the region D_1 for every $t_0 \in (0, K')$. Therefore, internal flat-ring harmonics have to be harmonic on all of $\mathbb{R}^3$ except the z-axis.

Suppose that $u_1(\zeta)$ and $u_2(\zeta)$ are solutions of the Lamé equation (3.4) for $|\operatorname{Im}\zeta| < K'$, and u_3 is a solution of (3.5). By Theorem 3.1, $u(x, y, z)$ defined by (3.3) is a harmonic function in $\tilde{Q}_1$ as defined in (3.1). We want this function to be harmonic on $\mathbb{R}^3$ except the z-axis. Clearly, we need $m \in \mathbb{Z}$, and we choose $u_3(\phi) = \mathrm{e}^{\mathrm{i}m\phi}$ (alternatively, we could use $\cos(m\phi)$, $m = 0, 1, 2, \ldots$ and $\sin(m\phi)$, $m = 1, 2, 3, \ldots$). Then we have to require that the function $u_1(s)u_2(\mathrm{i}t)$ is analytic in the right-hand half plane $x > 0$, $z \in \mathbb{R}$. We know that $u_1(s)u_2(\mathrm{i}t)$ is always analytic in the quadrant $x > 0$, $z > 0$. When we analytically extend this function to the quadrant $x > 0, z < 0$ across the segment $(x, 0)$, $0 < x < b$, by using the first variant of flat-ring coordinates then this extension has the value $u_1(-s)u_2(\mathrm{i}t)$ at the point $(x, -z)$ when (x, z) has coordinates s, t. When we extend across the segment $(x, 0)$, $b^{-1} < x$, using the third variant we obtain $u_1(4K - s)u_2(\mathrm{i}t)$. We want these extensions to be the same, so we need u_1 to be periodic with period $4K$ (provided u_2 is not identically zero). Therefore, we take $u_1 = \mathrm{Ec}^n_{|m|-\frac{1}{2}}$ or $u_1 = \mathrm{Es}^{n+1}_{|m|-\frac{1}{2}}$. When we analytically extend across the segment $(x, 0)$, $b < x < b^{-1}$, using the second variant of flat-ring coordinates we obtain $u_1(s)u_2(-\mathrm{i}t)$. This extension should also be the same as the previous two, so we have to require that u_1 and u_2 have the same parity. Therefore, u_1 has to be a constant multiple of u_2. We can take $u_1 = u_2$. Thus we are led to define internal flat-ring harmonics by

$$\mathrm{Gc}^n_m(x, y, z) = \left(x^2 + y^2\right)^{-1/4} \mathrm{Ec}^n_{|m|-\frac{1}{2}}(s, k)\, \mathrm{Ec}^n_{|m|-\frac{1}{2}}(\mathrm{i}t, k)\mathrm{e}^{\mathrm{i}m\phi}, \tag{5.2}$$

$$\mathrm{Gs}^{n+1}_m(x, y, z) = \left(x^2 + y^2\right)^{-1/4} \mathrm{Es}^{n+1}_{|m|-\frac{1}{2}}(s, k)\, \mathrm{Es}^{n+1}_{|m|-\frac{1}{2}}(\mathrm{i}t, k)\mathrm{e}^{\mathrm{i}m\phi}, \tag{5.3}$$

where $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$.

We collect some properties of internal harmonics in the following theorem. We will use the Kelvin transformation [12, Chapter IX.2]. If $u(\mathbf{r})$ is a harmonic function then its Kelvin transformation

$$\tilde{u}(\mathbf{r}) = \|\mathbf{r}\|^{-1} u(\sigma(\mathbf{r}))$$

is also harmonic, where σ denotes the inversion at the unit sphere

$$\sigma(\mathbf{r}) = \|\mathbf{r}\|^{-2}\mathbf{r}.$$

Theorem 5.1. *The internal flat-ring harmonics Gc^n_m and Gs^{n+1}_m are harmonic functions defined in all of $\mathbb{R}^3$ except for the z-axis. They have the following symmetry properties*

$$\mathrm{Gc}^n_m(\sigma(\mathbf{r})) = \|\mathbf{r}\|\, \mathrm{Gc}^n_m(\mathbf{r}),$$
$$\mathrm{Gs}^{n+1}_m(\sigma(\mathbf{r})) = -\|\mathbf{r}\|\, \mathrm{Gs}^{n+1}_m(\mathbf{r}),$$
$$\mathrm{Gc}^n_m(x, y, -z) = (-1)^n\, \mathrm{Gc}^n_m(x, y, z),$$
$$\mathrm{Gs}^{n+1}_m(x, y, -z) = (-1)^n\, \mathrm{Gs}^{n+1}_m(x, y, z).$$

Proof. From our discussion at the beginning of this section we know that $u = \mathrm{Gc}$ and $u = \mathrm{Gs}$ are harmonic functions on $\mathbb{R}^3$ except for the z-axis and the two circles centered at the origin with radii b^{-1} and b in the xy-plane, respectively. The function u is bounded in a neighborhood of the

two circles. Therefore, the two circles are removable singularities of u (see [12, Theorem XIII, p. 271]). Hence, u is harmonic on $\mathbb{R}^3$ except for along the z-axis.

The inversion σ in the upper half-space $z > 0$ is expressed in flat-ring coordinates by the map $s \mapsto 2K - s$. The Lamé functions Ec remain unchanged under this reflection while the Lamé functions Es change sign. Similarly, the reflection $z \mapsto -z$ is expressed by the map $s \mapsto -s$. The Lamé functions Ec_ν^n, Es_ν^{n+1} remain unchanged under this reflection if n is even but change sign when n is odd. ∎

5.2   The Dirichlet problem

Theorem 4.1 implies the following theorem.

Theorem 5.2. *The system of functions*

$$(8\pi)^{-1/2}\,\text{Ec}_{|m|-\frac{1}{2}}^n(s,k)\text{e}^{\text{i}m\phi}, \qquad (8\pi)^{-1/2}\,\text{Es}_{|m|-\frac{1}{2}}^{n+1}(s,k)\text{e}^{\text{i}m\phi}, \qquad m \in \mathbb{Z}, \quad n \in \mathbb{N}_0,$$

is an orthonormal basis in the Hilbert space

$$H_1 = L^2((-2K, 2K) \times (-\pi, \pi)).$$

We use internal flat-ring harmonics to solve the following Dirichlet problem. For a fixed $t_0 \in (0, K')$ consider the region D_1 defined in (5.1). For a given function f defined on the boundary ∂D_1 of D_1 we find the function u which is harmonic on D_1 and attains the boundary values f on ∂D_1 in the weak sense. The latter means that $(x^2 + y^2)^{1/4}u$ (expressed in terms of flat-ring coordinates s, t, ϕ) evaluated at $t_1 \in (0, t_0)$ converges to $(x^2 + y^2)^{1/4}f$ in the Hilbert space H_1 as $t_1 \to t_0$. Since D_1 has a positive distance to the z-axis the factor $(x^2 + y^2)^{1/4}$ can be omitted in this definition.

We note that the solution of this Dirichlet problem is unique (if it exists). To see this, assume that u is a harmonic function in D_1 that attains the boundary values 0 in the weak sense. Let $g(s,t,\phi)$ be the function obtained from $(x^2 + y^2)^{1/4}u(x,y,z)$ by expressing x, y, z in flat-ring coordinates (2.11). This function is analytic for $s, \phi \in \mathbb{R}$ and $-t_0 < t < t_0$. For $t \in (0, t_0)$, $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$ define

$$v(\text{i}t) = \int_{-\pi}^{\pi}\int_{-2K}^{2K} g(s,t,\phi)\,\text{Ec}_{|m|-\frac{1}{2}}^n(s,k)\text{e}^{-\text{i}m\phi}\,\text{d}s\,\text{d}\phi.$$

By differentiation under the integral sign, integration by parts and using (3.2) we find that v satisfies Lamé's equation (4.1) with $h = a_{|m|-\frac{1}{2}}^n(k)$. Since $g(s,t,\phi) = g(-s,-t,\phi)$, v is even if E is even and odd if E is odd. Therefore, v is a constant multiple of E. By assumption, $v(\text{i}t) \to 0$ as $t \to t_0$ but this is impossible by Lemma 4.7 unless $v(\text{i}t)$ is identically zero. Using the same argument with Es in place of Ec we conclude that u is identically zero by Theorem 5.2.

Theorem 5.3. *Let f be a function defined on the boundary ∂D_1 of the region D_1 for some $t_0 \in (0, K')$. Suppose that f is represented in the first variant of flat-ring coordinates as*

$$(x^2 + y^2)^{1/4}f(x,y,z) = g(s,\phi), \qquad s \in (-2K, 2K), \quad \phi \in (-\pi, \pi),$$

such that $g \in H_1$. For all $m \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, define

$$c_m^n := \frac{1}{8\pi\,\text{Ec}_{|m|-\frac{1}{2}}^n(\text{i}t_0, k)}\int_{-\pi}^{\pi}\int_{-2K}^{2K} g(s,\phi)\,\text{Ec}_{|m|-\frac{1}{2}}^n(s,k)\text{e}^{-\text{i}m\phi}\,\text{d}s\,\text{d}\phi$$

$$= \frac{1}{8\pi \{\mathrm{Ec}^n_{|m|-\frac{1}{2}}(it_0, k)\}^2} \int_{\partial D_1} \frac{1}{h_s(\mathbf{r})} f(\mathbf{r}) \, \mathrm{Gc}^n_{-m}(\mathbf{r}) \, \mathrm{d}S(\mathbf{r}),$$

and

$$d^{n+1}_m := \frac{1}{8\pi \, \mathrm{Es}^{n+1}_{|m|-\frac{1}{2}}(it_0, k)} \int_{-\pi}^{\pi} \int_{-2K}^{2K} g(s, \phi) \, \mathrm{Es}^{n+1}_{|m|-\frac{1}{2}}(s, k) \mathrm{e}^{-im\phi} \mathrm{d}s \, \mathrm{d}\phi$$

$$= \frac{1}{8\pi \{\mathrm{Ec}^{n+1}_{|m|-\frac{1}{2}}(it_0, k)\}^2} \int_{\partial D_1} \frac{1}{h_s(\mathbf{r})} f(\mathbf{r}) \, \mathrm{Gs}^{n+1}_{-m}(\mathbf{r}) \mathrm{d}S(\mathbf{r}),$$

where h_s is the metric coefficient (2.14). *Then the function*

$$u(\mathbf{r}) = \sum_{m \in \mathbb{Z}} \sum_{n=0}^{\infty} \left(c^n_m \, \mathrm{Gc}^n_m(\mathbf{r}) + d^{n+1}_m \, \mathrm{Gs}^{n+1}_m(\mathbf{r}) \right) \tag{5.4}$$

is harmonic in D_1 and it attains the boundary values f on ∂D_1 in the weak sense. The infinite series in (5.4) *converges absolutely and uniformly in compact subsets of D_1.*

Proof. Using flat-ring coordinates, we can write surface integrals over ∂D_1 as double integrals:

$$\int_{\partial D_1} f(\mathbf{r}) \, \mathrm{d}S(\mathbf{r}) = \int_{-\pi}^{\pi} \int_{-2K}^{2K} h_s h_\phi f \, \mathrm{d}s \, \mathrm{d}\phi$$

with the metric coefficient h_s from (2.14) and $h_\phi = \left(x^2 + y^2 \right)^{1/2}$. When we replace f by $f \, \mathrm{Gc} / h_s$ or $f \, \mathrm{Gs} / h_s$ this shows that the two formulas given for c^n_m and d^{n+1}_m agree.

Let $s \in (-2K, 2K)$, $\phi \in (-\pi, \pi)$ and $0 < t \le t_1 < t_0$. Using Lemmas 4.4, 4.6 and 4.8 we estimate

$$\left| \frac{\mathrm{Ec}^n_{|m|-\frac{1}{2}}(it, k)}{\mathrm{Ec}^n_{|m|-\frac{1}{2}}(it_0, k)} \, \mathrm{Ec}^n_{|m|-\frac{1}{2}}(s, k) \mathrm{e}^{im\phi} \right| \le Cp^{|m|+n}(1 + |m|)^{1/2},$$

where the constants C and $p \in (0, 1)$ are independent of m, n, s, t, ϕ. We have a similar estimate with Es in place of Ec. Since $\{c^n_m\}$ and $\{d^n_m\}$ are bounded double sequences, this proves that the series in (5.4) is absolutely and uniformly convergent on compact subsets of D_1. Consequently, by Theorem 5.1, u defined by (5.4) is harmonic in D_1.

Let $\tilde{u}(t)$ be the function $(x^2 + y^2)^{1/4}u$ for a given $t \in (0, t_0)$ considered as a function of (s, ϕ). Computing the norm $\|\tilde{u} - g\|$ in the Hilbert space H_1 by the Parseval equality, we obtain

$$\|\tilde{u} - g\|^2 \le 8\pi \sum_{m \in \mathbb{Z}} \sum_{n=0}^{\infty} \left(|c^n_m|^2 \left(1 - \frac{\mathrm{Ec}^n_{|m|-\frac{1}{2}}(it, k)}{\mathrm{Ec}^n_{|m|-\frac{1}{2}}(it_0, k)} \right)^2 + |d^{n+1}_m|^2 \left(1 - \frac{\mathrm{Es}^{n+1}_{|m|-\frac{1}{2}}(it, k)}{\mathrm{Es}^{n+1}_{|m|-\frac{1}{2}}(it_0, k)} \right)^2 \right).$$

It is easy to see that the right-hand side converges to 0 as $t \to t_0$. It follows that u attains the boundary values f in the weak sense. ∎

It is well-known [12, Chapter XI] that the classical Dirichlet problem on the flat-ring domain D_1 has a unique solution, that is, for a given continuous function f on ∂D_1 there is a unique continuous function u defined on $\overline{D}_1$ which is harmonic in D_1 and agrees with f on ∂D_1. Obviously, for continuous f, the classical solution of the Dirichlet problem agrees with the solution furnished by Theorem 5.3.

5.3   External flat-ring harmonics

External flat-ring harmonics are harmonic functions u of the form (3.3) which are harmonic outside of all flat-ring regions (5.1). Therefore, they are harmonic on $\mathbb{R}^3$ except for the annulus $b^2 \leq x^2 + y^2 \leq b^{-2}$ in the xy-plane. It is clear that m must be an integer and arguing as at the beginning of Section 5.1, we see that u_1 must have period $4K$. We note that, for fixed $s \in (0, 2K)$,

$$\lim_{t \to K'} (K' - t)T \sim \frac{1}{k'}(\mathrm{dn}(s,k) + \mathrm{cn}(s,k)).$$

Since the function u has to be harmonic along the z-axis, we require that u_2 is a solution of the Lamé equation (3.4) such that $\left(x^2 + y^2\right)^{-1/4} u_2(\mathrm{i}t) = T^{1/2} u_2(\mathrm{i}t)$ stays bounded when t approaches K'. Therefore, u_2 has to be one of the Lamé functions Fc and Fs defined in Section 4. Thus we define external flat-ring harmonics by

$$\mathrm{Hc}_m^n(x, y, z) = \left(x^2 + y^2\right)^{-1/4} \mathrm{Ec}_{|m|-\frac{1}{2}}^n(s, k)\, \mathrm{Fc}_{|m|-\frac{1}{2}}^n(\mathrm{i}t, k)\mathrm{e}^{\mathrm{i}m\phi}, \tag{5.5}$$

$$\mathrm{Hs}_m^{n+1}(x, y, z) = \left(x^2 + y^2\right)^{-1/4} \mathrm{Es}_{|m|-\frac{1}{2}}^{n+1}(s, k)\, \mathrm{Fs}_{|m|-\frac{1}{2}}^{n+1}(\mathrm{i}t, k)\mathrm{e}^{\mathrm{i}m\phi}, \tag{5.6}$$

where $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$. We recall that we use the first variant of the transcendental flat-ring coordinates.

Theorem 5.4. *The external flat-ring harmonics* Hc_m^n, Hs_m^{n+1} *are harmonic functions defined in all of* $\mathbb{R}^3$ *except for the closed annulus in the* xy-plane *centered at the origin with inner radius* $b = \frac{k'}{1+k}$ *and outer radius* b^{-1}. *Moreover,*

$$\mathrm{Hc}_m^n(\sigma(\mathbf{r})) = \|\mathbf{r}\|\, \mathrm{Hc}_m^n(\mathbf{r}), \tag{5.7}$$

$$\mathrm{Hs}_m^{n+1}(\sigma(\mathbf{r})) = -\|\mathbf{r}\|\, \mathrm{Hs}_m^{n+1}(\mathbf{r}), \tag{5.8}$$

$$\mathrm{Hc}_m^n(x, y, -z) = (-1)^n\, \mathrm{Hc}_m^n(x, y, z), \tag{5.9}$$

$$\mathrm{Hs}_m^{n+1}(x, y, -z) = (-1)^n\, \mathrm{Hs}_m^{n+1}(x, y, z), \tag{5.10}$$

and

$$\lim_{\|\mathbf{r}\| \to \infty} \mathrm{Hc}_m^n(\mathbf{r}) = \lim_{\|\mathbf{r}\| \to \infty} \mathrm{Hs}_m^n(\mathbf{r}) = 0. \tag{5.11}$$

Proof. From our preceding discussion we know that the function $u = \mathrm{Hc}_m^n$ (or $u = \mathrm{Hs}_m^{n+1}$) is harmonic on $\mathbb{R}^3$ except for the z-axis and the annulus $b^2 \leq x^2 + y^2 \leq b^{-2}$ in the xy-plane. It follows from (4.8) that u stays bounded when we approach the z-axis. This shows that u is harmonic on all of $\mathbb{R}^3$ except for the annulus $b^2 \leq x^2 + y^2 \leq b^{-2}$ in the xy-plane. The symmetry properties are shown as in the proof of Theorem 5.1. The limits (5.11) follow from (5.7) and (5.8) and the fact that external flat-ring harmonics are harmonic at the origin.   ∎

We now show that external harmonics admit an integral representation in terms of internal harmonics.

Theorem 5.5. *Let* $t_0 \in (0, K')$, $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$, *and let* $\mathbf{r}^*$ *be a point outside* $\overline{D}_1$, *where* D_1 *is given by* (5.1). *Then*

$$\mathrm{Hc}_m^n(\mathbf{r}^*) = \frac{1}{4\pi\left\{\mathrm{Ec}_{|m|-\frac{1}{2}}^n(\mathrm{i}t_0, k)\right\}^2} \int_{\partial D_1} \frac{\mathrm{Gc}_m^n(\mathbf{r})}{h_s(\mathbf{r})\|\mathbf{r} - \mathbf{r}^*\|}\, \mathrm{d}S(\mathbf{r}), \tag{5.12}$$

and

$$\mathrm{Hs}_m^{n+1}(\mathbf{r}^*) = \frac{1}{4\pi\left\{\mathrm{Es}_{|m|-\frac{1}{2}}^{n+1}(\mathrm{i}t_0, k)\right\}^2} \int_{\partial D_1} \frac{\mathrm{Gs}_m^{n+1}(\mathbf{r})}{h_s(\mathbf{r})\|\mathbf{r} - \mathbf{r}^*\|}\, \mathrm{d}S(\mathbf{r}). \tag{5.13}$$

Proof. Let D be an open bounded subset of $\mathbb{R}^3$ with smooth boundary. For $u, v \in C^2(\overline{D})$, Green's formula states that

$$\int_D (u\Delta v - v\Delta u)\,\mathrm{d}\mathbf{r} = \int_{\partial D} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right)\mathrm{d}S, \tag{5.14}$$

where $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of u on the boundary ∂D of D. We apply (5.14) to $D = D_1$, $u = G = \mathrm{Gc}_m^n$, and

$$v(\mathbf{r}) = \frac{1}{4\pi\|\mathbf{r}^* - \mathbf{r}\|}. \tag{5.15}$$

Since $\Delta u = \Delta v = 0$ on D_1 we obtain

$$0 = \int_{\partial D_1} \left(G\frac{\partial v}{\partial \nu} - v\frac{\partial G}{\partial \nu}\right)\mathrm{d}S. \tag{5.16}$$

Let $B_r(\mathbf{q})$ denote the open ball centered at $\mathbf{q} \in \mathbb{R}^3$ with radius $r > 0$. We apply (5.14) a second time with $D = B_R(\mathbf{0}) - \overline{D}_1 - B_\epsilon(\mathbf{r}^*)$ with large R and small $\epsilon > 0$. Choose $u = H = \mathrm{Hc}_m^n$ and v as in (5.15). Note that $\Delta u = \Delta v = 0$ on D. By a standard argument [25, Chapter VII, Proof of Theorem 5.1], taking the limit $\epsilon \to 0$, we obtain

$$H(\mathbf{r}^*) = -\int_{\partial B_R(\mathbf{0})} \left(H\frac{\partial v}{\partial \nu} - v\frac{\partial H}{\partial \nu}\right)\mathrm{d}S + \int_{\partial D_1} \left(H\frac{\partial v}{\partial \nu} - v\frac{\partial H}{\partial \nu}\right)\mathrm{d}S, \tag{5.17}$$

where, in the second integral, $\frac{\partial}{\partial \nu}$ denotes again the derivative in the direction of the outward normal as in (5.16). As $r = \|\mathbf{r}\| \to \infty$, we have

$$v(\mathbf{r}) = O(r^{-1}), \qquad \frac{\partial v}{\partial \nu} = O(r^{-2})$$

and, from (5.7),

$$H(\mathbf{r}) = O(r^{-1}), \qquad \frac{\partial H}{\partial \nu} = O(r^{-2}).$$

Therefore, the first integral on the right-hand side of (5.17) tends to 0 as $R \to \infty$, so we obtain

$$H(\mathbf{r}^*) = \int_{\partial D_1} \left(H\frac{\partial v}{\partial \nu} - v\frac{\partial H}{\partial \nu}\right)\mathrm{d}S. \tag{5.18}$$

Set $\mathsf{E}(\zeta) = \mathrm{Ec}_{|m|-\frac{1}{2}}^n(\zeta)$ and $\mathsf{F}(\zeta) = \mathrm{Fc}_{|m|-\frac{1}{2}}^n(\zeta)$. We multiply (5.16) by $-\mathsf{F}(\mathrm{i}t_0)$, then multiply (5.18) by $\mathsf{E}(\mathrm{i}t_0)$ and add these equations. Using the definitions (5.2) and (5.5) of internal and external flat-ring harmonics and cancelling terms, we find

$$\mathsf{E}(\mathrm{i}t_0)H(\mathbf{r}^*) = \int_{\partial D_1} v\left(\mathsf{F}(\mathrm{i}t_0)\frac{\partial G}{\partial \nu} - \mathsf{E}(\mathrm{i}t_0)\frac{\partial H}{\partial \nu}\right)\mathrm{d}S. \tag{5.19}$$

Since flat-ring coordinates are orthogonal, the normal derivative and the partial derivative with respect to t are related by

$$\frac{\partial}{\partial \nu} = \frac{1}{h_t}\frac{\partial}{\partial t},$$

where h_t is given in (2.14). Let $\mathbf{r} = (x, y, z) \in \partial D_1$ with flat-ring coordinates s, t, ϕ. Then

$$
\mathsf{F}(\mathrm{i}t_0)\frac{\partial G}{\partial \nu}(\mathbf{r}) - \mathsf{E}(\mathrm{i}t_0)\frac{\partial H}{\partial \nu}(\mathbf{r})
$$

$$
= \mathsf{F}(\mathrm{i}t_0)\frac{\partial\big((x^2+y^2)^{-1/4}\big)}{\partial \nu}\mathsf{E}(\mathrm{i}t_0)\mathsf{E}(s)\mathrm{e}^{\mathrm{i}m\phi} + \mathsf{F}(\mathrm{i}t_0)\big(x^2+y^2\big)^{-1/4}h_t^{-1}\mathrm{i}\mathsf{E}'(\mathrm{i}t_0)\mathsf{E}(s)\mathrm{e}^{\mathrm{i}m\phi}
$$

$$
- \mathsf{E}(\mathrm{i}t_0)\frac{\partial\big((x^2+y^2)^{-1/4}\big)}{\partial \nu}\mathsf{F}(\mathrm{i}t_0)\mathsf{E}(s)\mathrm{e}^{\mathrm{i}m\phi} - \mathsf{E}(\mathrm{i}t_0)\big(x^2+y^2\big)^{-1/4}h_t^{-1}\mathrm{i}\mathsf{F}'(\mathrm{i}t_0)\mathsf{E}(s)\mathrm{e}^{\mathrm{i}m\phi}
$$

$$
= h_t^{-1}\big(x^2+y^2\big)^{-1/4}\mathrm{i}\{\mathsf{F}(\mathrm{i}t_0)\mathsf{E}'(\mathrm{i}t_0) - \mathsf{F}'(\mathrm{i}t_0)\mathsf{E}(\mathrm{i}t_0)\}\mathsf{E}(s)\mathrm{e}^{\mathrm{i}m\phi}.
$$

We now use the Wronskian (4.9) and obtain

$$
\mathsf{F}(\mathrm{i}t_0)\frac{\partial G}{\partial \nu}(\mathbf{r}) - \mathsf{E}(\mathrm{i}t_0)\frac{\partial H}{\partial \nu}(\mathbf{r}) = \frac{1}{h_t(\mathbf{r})\mathsf{E}(\mathrm{i}t_0)}G(\mathbf{r}). \tag{5.20}
$$

When we substitute (5.20) in (5.19) we obtain (5.12). The proof of (5.13) is similar. ■

5.4 Expansion of the fundamental solution

We obtain the expansion of (5.15) in internal and external flat-ring harmonics by combining Theorems 5.3 and 5.5.

Theorem 5.6. *Let* $\mathbf{r}, \mathbf{r}^* \in \mathbb{R}^3$ *with flat-ring coordinates* $t, t^* \in (0, K')$, *respectively. If* $t < t^*$ *then*

$$
\frac{1}{\|\mathbf{r} - \mathbf{r}^*\|} = \frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty}\big(\mathrm{Gc}_m^n(\mathbf{r})\,\mathrm{Hc}_{-m}^n(\mathbf{r}^*) + \mathrm{Gs}_m^{n+1}(\mathbf{r})\,\mathrm{Hs}_{-m}^{n+1}(\mathbf{r}^*)\big). \tag{5.21}
$$

Proof. We pick t_0 such that $t < t_0 < t^*$, and consider the domain D_1 defined in (5.1). Therefore, by applying Theorem 5.3 to the harmonic function $f(\mathbf{q}) := \frac{1}{\|\mathbf{q}-\mathbf{r}^*\|}$, we find

$$
f(\mathbf{r}) = \sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty}\big(c_m^n\,\mathrm{Gc}_m^n(\mathbf{r}) + d_m^{n+1}\,\mathrm{Gs}_m^{n+1}(\mathbf{r})\big),
$$

where c_m^n and d_m^{n+1} can be evaluated by Theorem 5.5:

$$
c_n^m = \frac{1}{2}\,\mathrm{Hc}_{-m}^n(\mathbf{r}^*), \qquad d_m^{n+1} = \frac{1}{2}\,\mathrm{Hs}_{-m}^{n+1}(\mathbf{r}^*).
$$

We use $h_s = h_t$. Thus we obtain (5.21). ■

5.5 Addition theorem for the azimuthal Fourier coefficients

Then, the azimuthal Fourier expansion of $\|\mathbf{r} - \mathbf{r}^*\|^{-1}$ is well-known [5, equation (15)]

$$
\frac{1}{\|\mathbf{r} - \mathbf{r}^*\|} = \frac{1}{\pi\sqrt{RR^*}}\sum_{m\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}m(\phi-\phi^*)}Q_{m-\frac{1}{2}}(\chi), \tag{5.22}
$$

where

$$
\chi = \frac{R^2 + R^{*2} + (z - z^*)^2}{2RR^*}, \tag{5.23}
$$

and R, R^*, z, z^* are the radial and vertical rotationally-invariant cylindrical coordinates of $\mathbf{r}$ and $\mathbf{r}^*$ respectively. It is understood that

$$\frac{1}{R} = \frac{1}{k'} \operatorname{dn}(s,k) \operatorname{dn}(\mathrm{i}t,k) + \frac{k}{k'} \operatorname{cn}(s,k) \operatorname{sn}(\mathrm{i}t,k), \qquad z = -\mathrm{i}kR\operatorname{sn}(s,k)\operatorname{sn}(\mathrm{i}t,k),$$

and, similarly, R^*, z^* are expressed in terms of s^*, t^*. One can express χ in a different way as follows.

Lemma 5.7. *Let $k \in (0,1)$, $s, s^* \in \mathbb{R}$, $t, t^* \in (-K', K')$ such that $0 < t < t^* < K'$. Then χ from (5.23) expressed in terms of flat-ring cyclide coordinates (2.1) is given by*

$$\chi = k^2 \operatorname{sn}(s,k) \operatorname{sn}(\mathrm{i}t,k) \operatorname{sn}(s^*,k) \operatorname{sn}(\mathrm{i}t^*,k) - \frac{k^2}{k'^2} \operatorname{cn}(s,k) \operatorname{cn}(\mathrm{i}t,k) \operatorname{cn}(s^*,k) \operatorname{cn}(\mathrm{i}t^*,k)$$

$$+ \frac{1}{k'^2} \operatorname{dn}(s,k) \operatorname{dn}(\mathrm{i}t,k) \operatorname{dn}(s^*,k) \operatorname{dn}(\mathrm{i}t^*,k).$$

Proof. Setting

$$\frac{1}{\tilde{R}} = \frac{1}{k'} \operatorname{dn}(s,k) \operatorname{dn}(\mathrm{i}t,k) - \frac{k}{k'} \operatorname{cn}(s,k) \operatorname{sn}(\mathrm{i}t,k),$$

then a short calculation shows that $R = \tilde{R}(R^2 + z^2)$. Now

$$\chi = \frac{R^2 + R^{*2} + (z - z^*)^2}{2RR^*} = \frac{(R^2 + z^2) + (R^{*2} + z^{*2}) - 2zz^*}{2RR^*} = \frac{1}{2\tilde{R}R^*} + \frac{1}{2R\tilde{R}^*} - \frac{zz^*}{RR^*}.$$

If we express the last expression in terms of s, t, s^*, t^*, the desired formula is obtained. $\blacksquare$

Note that the Legendre functions $Q_\nu := Q_\nu^0$, $P_\nu := P_\nu^0$ are the associated Legendre functions Q_ν^μ, P_ν^μ with $\mu = 0$. These Legendre functions $Q_{m-\frac{1}{2}}$, for $m \in \mathbb{Z}$, appear in the separation of variables for the three-variable Laplace equation in toroidal coordinates (see Section 7 below. By starting with Theorem 5.6 and substituting (5.2), (5.3), (5.5) and (5.6), we obtain the following double summation expression for $\|\mathbf{r} - \mathbf{r}^*\|^{-1}$ in flat-ring cyclide coordinates expressed in terms of internal and external flat-ring harmonics

$$\frac{1}{\|\mathbf{r} - \mathbf{r}^*\|} = \frac{1}{2\sqrt{RR^*}} \sum_{m \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}m(\phi - \phi^*)}$$

$$\times \sum_{n=0}^{\infty} \Big(\operatorname{Ec}_{|m|-\frac{1}{2}}^n(s,k) \operatorname{Ec}_{|m|-\frac{1}{2}}^n(s^*,k) \operatorname{Ec}_{|m|-\frac{1}{2}}^n(\mathrm{i}t,k) \operatorname{Fc}_{|m|-\frac{1}{2}}^n(\mathrm{i}t^*,k)$$

$$+ \operatorname{Es}_{|m|-\frac{1}{2}}^{n+1}(s,k) \operatorname{Es}_{|m|-\frac{1}{2}}^{n+1}(s^*,k) \operatorname{Es}_{|m|-\frac{1}{2}}^{n+1}(\mathrm{i}t,k) \operatorname{Fs}_{|m|-\frac{1}{2}}^{n+1}(\mathrm{i}t^*,k) \Big). \qquad (5.24)$$

By comparing the azimuthal Fourier coefficients of (5.24) and (5.22), one can obtain the following addition theorem for the odd-half-integer Legendre function of the second kind $Q_{m-\frac{1}{2}}$, expressed in terms of Lamé functions of the first (simply periodic) and second kind (non-periodic) in flat-ring cyclide coordinates.

Theorem 5.8. *Let $m \in \mathbb{N}_0$, $k \in (0,1)$, $s, s^* \in \mathbb{R}$, $t, t^* \in (-K', K')$ such that $0 < t < t^* < K'$. Then*

$$Q_{m-\frac{1}{2}}(\chi) = \frac{\pi}{2} \sum_{n=0}^{\infty} \Big(\operatorname{Ec}_{m-\frac{1}{2}}^n(s,k) \operatorname{Ec}_{m-\frac{1}{2}}^n(s^*,k) \operatorname{Ec}_{m-\frac{1}{2}}^n(\mathrm{i}t,k) \operatorname{Fc}_{m-\frac{1}{2}}^n(\mathrm{i}t^*,k)$$

$$+ \operatorname{Es}_{m-\frac{1}{2}}^{n+1}(s,k) \operatorname{Es}_{m-\frac{1}{2}}^{n+1}(s^*,k) \operatorname{Es}_{m-\frac{1}{2}}^{n+1}(\mathrm{i}t,k) \operatorname{Fs}_{m-\frac{1}{2}}^{n+1}(\mathrm{i}t^*,k) \Big).$$

Proof. Start with

$$R = \frac{\sqrt{1-k^2}}{k\,\mathrm{cn}(s,k)\,\mathrm{cn}(\mathrm{i}t,k) + \mathrm{dn}(s,k)\,\mathrm{dn}(\mathrm{i}t,k)}, \qquad z = \frac{-\mathrm{i}k\,\sqrt{1-k^2}\,\mathrm{sn}(s,k)\,\mathrm{sn}(\mathrm{i}t,k)}{k\,\mathrm{cn}(s,k)\,\mathrm{cn}(\mathrm{i}t,k) + \mathrm{dn}(s,k)\,\mathrm{dn}(\mathrm{i}t,k)},$$

and similarly for R^*, z^*. Then comparing (5.22) to (5.24) identifies the azimuthal Fourier coefficients as a Legendre function of odd-half-integer degree given in terms of two infinite series of products of Lamé functions of the first and second kind. This completes the proof. ∎

Remark 5.9. *The above addition Theorem 5.8 is a rare instance of an infinite series representation over Lamé functions which reduces to an analytic (Gauss hypergeometric) function.*

The discovery of the new addition Theorem 5.8 leads to an integral relation for products of Lamé functions of the first and second kind.

Theorem 5.10. *Let $m, n \in \mathbb{N}_0$, $s^* \in \mathbb{R}$, $0 < t < t^* < K'$. Then*

$$\int_{-2K}^{2K} Q_{m-\frac{1}{2}}(\chi)\,\mathrm{Ec}_{m-\frac{1}{2}}^{n}(s)\,\mathrm{d}s = 2\pi\,\mathrm{Ec}_{m-\frac{1}{2}}^{n}(s^*,k)\,\mathrm{Ec}_{m-\frac{1}{2}}^{n}(\mathrm{i}t,k)\,\mathrm{Fc}_{m-\frac{1}{2}}^{n}(\mathrm{i}t^*,k)$$

and

$$\int_{-2K}^{2K} Q_{m-\frac{1}{2}}(\chi)\,\mathrm{Es}_{m-\frac{1}{2}}^{n+1}(s)\,\mathrm{d}s = 2\pi\,\mathrm{Es}_{m-\frac{1}{2}}^{n+1}(s^*,k)\,\mathrm{Es}_{m-\frac{1}{2}}^{n+1}(\mathrm{i}t,k)\,\mathrm{Fs}_{m-\frac{1}{2}}^{n+1}(\mathrm{i}t^*,k).$$

Proof. Recall that we proved uniform convergence of the series in Theorem 5.8 with respect to s for fixed s^*, t, t^*. Therefore, the integral relations follows from the orthogonality of the function system

$$\mathrm{Ec}_{m-\frac{1}{2}}^{n}(s,k), \qquad \mathrm{Es}_{m-\frac{1}{2}}^{n+1}(s,k), \qquad n \in \mathbb{N}_0,$$

over the interval $-2K \leq s \leq 2K$ and the normalization

$$\int_{0}^{K} \left\{ \mathrm{Ec}_{m-\frac{1}{2}}^{n}(s,k) \right\}^2 \mathrm{d}s = \int_{0}^{K} \left\{ \mathrm{Es}_{m-\frac{1}{2}}^{n+1}(s,k) \right\}^2 \mathrm{d}s = 1$$

which leads to

$$\int_{-2K}^{2K} \left\{ \mathrm{Ec}_{m-\frac{1}{2}}^{n}(s,k) \right\}^2 \mathrm{d}s = \int_{-2K}^{2K} \left\{ \mathrm{Es}_{m-\frac{1}{2}}^{n+1}(s,k) \right\}^2 \mathrm{d}s = 4,$$

which completes the proof. ∎

Theorem 5.10 is a special case of [21, Corollary 2.8]. This reference shows that Theorem 5.10 remains true if m is any nonnegative real number in place of a nonnegative integer. The statement of Theorem 5.10 suggests an alternative proof of the expansion of a fundamental solution of the three-variable Laplace equation in terms of internal and external flat-ring cyclidic harmonics (5.21). One can then start with the known Theorem 5.10 and then derive Theorem 5.8 from it. One can then use Theorem 5.8 in order to obtain the expansion of a fundamental solution using (5.22).

6   Lemmas on second-order linear differential equations

As we show in Section 7, flat-ring coordinates approach toroidal coordinates in the limit $k \to 0$. Therefore, we expect that special functions found by separation of the Laplace equation in flat-ring coordinates will approach corresponding special functions found by separation of the Laplace equation in toroidal coordinates. In order to make this statement precise we introduce the following three lemmas.

Lemma 6.1.

(a) *Let $a \in \mathbb{R}$, and let $\{b_n\}$ be a sequence of real numbers such that $a < b_n \to \infty$ as $n \to \infty$.*

(b) *For $n \in \mathbb{N}$, let $p_n, q_n \colon [a, b_n] \to \mathbb{R}$ be continuous functions such that $q_n(x) < 0$ for all $x \in [a, b_n]$. For every $n \in \mathbb{N}$, let $y_n \colon [a, b_n] \to \mathbb{R}$ be a nontrivial solution of the differential equation*

$$y_n'' + p_n(x)y_n' + q_n(x)y_n = 0$$

such that $y_n(b_n) = 0$.

(c) *Let $p_\infty, q_\infty \colon [a, \infty) \to \mathbb{R}$ be continuous functions such that $p_n(x) \to p_\infty(x)$ and $q_n(x) \to q_\infty(x)$ as $n \to \infty$ uniformly on each compact interval $[a, b]$. Suppose that the differential equation*

$$y_\infty'' + p_\infty(x)y_\infty' + q_\infty(x)y_\infty = 0 \tag{6.1}$$

admits a bounded nontrivial solution $y_\infty \colon [a, \infty) \to \mathbb{R}$, and that every solution of (6.1) which is linearly independent of y_∞ is unbounded as $x \to \infty$.

Under assumptions (a), (b), (c), we have

$$\frac{y_n(x)}{y_n(a)} \to \frac{y_\infty(x)}{y_\infty(a)} \qquad and \qquad \frac{y_n'(x)}{y_n(a)} \to \frac{y_\infty'(x)}{y_\infty(a)} \tag{6.2}$$

as $n \to \infty$ uniformly on every compact interval $[a, b]$. The same result is true if the condition $y_n(b_n) = 0$ is replaced by $y_n'(b_n) = 0$.

Proof. Without loss of generality we assume that $y_n'(b_n) < 0$. Consider y_n for a fixed n. There is $\epsilon > 0$ such that $y_n(x) > 0$, $y_n'(x) < 0$ for $x \in (b_n - \epsilon, b_n)$. Suppose there is $x_0 \in [a, b_n)$ such that $y_n'(x_0) = 0$ and $y_n'(b_n) < 0$ for $x \in (x_0, b_n]$. This implies the contradiction $0 \geq y_n''(x_0) = -q_n(x_0)y_n(x_0) > 0$. Therefore,

$$y_n(x) > 0 \qquad and \qquad y_n'(x) < 0 \qquad for\ all \quad x \in [a, b_n). \tag{6.3}$$

Suppose that the sequence $\frac{y_n'(a)}{y_n(a)}$ converges to $-\infty$. Without loss of generality, we assume that $y_n'(a) = -1$ for all $n \in \mathbb{N}$. Then $y_n(a) \to 0$ as $n \to \infty$. By assumption (c), it follows that $y_n(x) \to y(x)$ uniformly on $[a, b]$, where $y(x)$ is the solution of (6.1) with initial conditions $y(a) = 0$, $y'(a) = -1$. This contradicts (6.3).

Now suppose that the sequence $\frac{y_n'(a)}{y_n(a)}$ converges to $c \in (-\infty, 0]$. Without loss of generality, we assume that $y_n(a) = 1$ for all $n \in \mathbb{N}$. Then $y_n'(a) \to c$ as $n \to \infty$. It follows that $y_n(x) \to y(x)$ and $y_n'(x) \to y'(x)$ as $n \to \infty$ uniformly on compact intervals $[a, b]$, where $y(x)$ is the solution of (6.1) with initial conditions $y(a) = 1$, $y'(a) = c$. It follows from (6.3) that $0 \leq y(x) \leq 1$ for all $x \geq a$. Therefore, y must be a constant multiple of y_∞. This shows that $c = \frac{y_\infty'(a)}{y_\infty(a)}$ is uniquely determined. By taking subsequences if necessary, this proves that $\frac{y_n'(a)}{y_n(a)} \to c$ and so (6.2) is established. The proof in the case $y_n'(b_n) = 0$ is very similar. ∎

The following result is well-known.

Lemma 6.2. *Let D be a simply-connected domain in $\mathbb{C}$, $a \in D$. For $n \in \mathbb{N}$, let $p_n, q_n, p_\infty, q_\infty$: $D \to \mathbb{C}$ be analytic functions such that $p_n(z) \to p_\infty(z)$ and $q_n(z) \to q_\infty(z)$ locally uniformly for $z \in D$. For each $n \in \mathbb{N}$ let $y_n : D \to \mathbb{C}$ be a solution of the differential equation*

$$y_n'' + p_n(z)y_n' + q_n(z)y_n = 0,$$

and let $y_\infty : D \to \mathbb{C}$ be a solution of

$$y_\infty'' + p_\infty(z)y_\infty' + q_\infty(z)y_\infty = 0.$$

If

$$y_n(a) \to y_\infty(a), \qquad y_n'(a) \to y_\infty'(a) \qquad as \quad n \to \infty,$$

then

$$y_n(z) \to y_\infty(z) \qquad as \quad n \to \infty \quad locally \ uniformly \ for \quad z \in D.$$

The following lemma can be considered as known although we do not have a precise reference. We prove it following [4, Section 4.5].

Lemma 6.3. *Let D be a simply-connected domain in the complex plane $\mathbb{C}$ containing 0. Let $p_n : D \to \mathbb{C}$ be analytic functions for each $n \in \mathbb{N} \cup \{\infty\}$ such that $p_n(z) \to p_\infty(z)$ locally uniformly on D, and $p_n(0) = \nu(\nu + 1)$ for all $n \in \mathbb{N} \cup \{\infty\}$, where $\nu \geq -\frac{1}{2}$. For each $n \in \mathbb{N} \cup \{\infty\}$, let $u_n : D \to \mathbb{C}$ be the unique analytic function such that $u_n(0) = 1$ and $y_n(z) := z^{\nu+1}u_n(z)$ solves*

$$z^2 y_n'' = p_n(z)y_n. \tag{6.4}$$

Then $u_n(z) \to u_\infty(z)$ locally uniformly on D.

Proof. Without loss of generality we take $D = \{z \in \mathbb{C} : |z| < R\}$. The point $z = 0$ is a regular singularity of differential equation (6.4). The exponents are $-\nu$ and $\nu + 1$. Since $\nu \geq -\frac{1}{2}$, the solutions y_n as stated in the lemma exist. We write

$$p_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k, \qquad a_{n,0} = \nu(\nu + 1),$$

$$u_n(z) = \sum_{k=0}^{\infty} b_{n,k} z^k, \qquad b_{n,0} = 1.$$

By substituting $y_n(z) = z^{\nu+1}u_n(z)$ in (6.4), we obtain

$$k(k + 2\nu + 1)b_{n,k} = \sum_{j=0}^{k-1} a_{n,k-j} b_{n,j} \qquad for \quad k \in \mathbb{N}.$$

Since $a_{n,k} \to a_{\infty,k}$ as $n \to \infty$ for all $k \in \mathbb{N}_0$, we obtain by induction that

$$b_{n,k} \to b_{\infty,k} \qquad as \quad n \to \infty \qquad for \quad k \in \mathbb{N}_0. \tag{6.5}$$

Let $0 < r_1 < r_2 < R$. By assumption, there is a constant M (independent of n, z) such that $|p_n(z)| \leq M$ for all $n \in \mathbb{N}$ and $|z| \leq r_2$. By Cauchy's estimate, $|a_{n,k}| \leq Mr_2^{-k}$ for all n, k. Set $\gamma_0 = 1$ and define γ_k for $k \in \mathbb{N}$ recursively by

$$k^2 \gamma_k = M \sum_{j=0}^{k-1} r_2^{j-k} \gamma_j. \tag{6.6}$$

By induction, we see that $|b_{n,k}| \le \gamma_k$ for all n, k. Replacing k by $k+1$ in (6.6), we find

$$r_2(k+1)^2 \gamma_{k+1} = \left(M + k^2\right)\gamma_k.$$

The ratio test shows that

$$\sum_{k=0}^{\infty} \gamma_k r_1^k < \infty. \tag{6.7}$$

Since $|b_{n,k} z^k| \le \gamma_k r_1^k$ for $|z| \le r_1$ and all n, k, we obtain from (6.5) and (6.7) that $u_n(z) \to u_\infty(z)$ as $n \to \infty$ uniformly for $|z| \le r_1$. This completes the proof. ∎

7   The limit $k \to 0$ to toroidal coordinates

7.1   Flat-ring coordinates in the limit $k \to 0$

In the following we will frequently use that

$$K(k) \to \frac{1}{2}\pi, \qquad K'(k) \to \infty \qquad \text{as} \quad k \to 0.$$

Moreover,

$$\operatorname{sn}(\zeta, k) \to \sin \zeta, \qquad \operatorname{cn}(\zeta, k) \to \cos \zeta, \qquad \operatorname{dn}(\zeta, k) \to 1 \qquad \text{as} \quad k \to 0 \tag{7.1}$$

locally uniformly on $\mathbb{C}$ [18, Table 22.5.3] and

$$\operatorname{sn}(\zeta, k) \to \tanh \zeta, \qquad \operatorname{cn}(\zeta, k) \to \operatorname{sech} \zeta, \qquad \operatorname{dn}(\zeta, k) \to \operatorname{sech} \zeta \qquad \text{as} \quad k \to 1 \tag{7.2}$$

locally uniformly at all points ζ such that $\cosh \zeta \ne 0$ [18, Table 22.5.4].

We use the first variant of flat-ring coordinates $s \in (-2K, 2K)$, $t \in (0, K')$, and set $\tau = K' - t$, $\psi = 2K - s$. Then

$$T = \frac{\operatorname{dn}(\psi, k) - \operatorname{cn}(\psi, k)\operatorname{dn}(\tau, k')}{k' \operatorname{sn}(\tau, k')}.$$

Therefore,

$$\lim_{k \to 0} T = \frac{\cosh \tau - \cos \psi}{\sinh \tau}.$$

It follows that

$$\lim_{k \to 0} x = \frac{\sinh \tau \cos \phi}{\cosh \tau - \cos \psi}, \qquad \lim_{k \to 0} y = \frac{\sinh \tau \sin \phi}{\cosh \tau - \cos \psi}.$$

Moreover,

$$z = \frac{k' \operatorname{sn}(\psi, k)\operatorname{cn}(\tau, k')}{\operatorname{dn}(\psi, k) - \operatorname{cn}(\psi, k)\operatorname{dn}(\tau, k')}$$

gives

$$\lim_{k \to 0} z = \frac{\sin \psi}{\cosh \tau - \cos \psi}.$$

These limits can be taken for $\tau > 0$ and $\psi \in (0, 2\pi)$. Therefore, flat-ring coordinates become toroidal coordinates (1.3) as $k \to 0$.

7.2   Simply-periodic Lamé functions in the limit $k \to 0$

Theorem 7.1. *For $\nu \geq -\frac{1}{2}$, we have*

$$a_\nu^n(k) \to n^2, \qquad n \in \mathbb{N}_0,$$

$$b_\nu^n(k) \to n^2, \qquad n \in \mathbb{N},$$

$$\mathrm{Ec}_\nu^0(s, k) \to \left(\frac{2}{\pi}\right)^{1/2},$$

$$\mathrm{Ec}_\nu^n(s, k) \to \left(\frac{4}{\pi}\right)^{1/2} \cos\left(n\left(\frac{\pi}{2} - s\right)\right), \qquad n \in \mathbb{N},$$

$$\mathrm{Es}_\nu^n(s, k) \to \left(\frac{4}{\pi}\right)^{1/2} \sin\left(n\left(\frac{\pi}{2} - s\right)\right), \qquad n \in \mathbb{N}$$

as $k \to 0$ locally uniformly for $s \in \mathbb{C}$.

Proof. The limits of the eigenvalues follow from Lemma 4.2. The limits of the eigenfunctions follow from Lemma 6.2 and (7.1). ∎

In the following theorem, Q denotes the Legendre function of the second kind [18, Chapter 14].

Theorem 7.2. *Let $\tau_0 > 0$, $\nu \geq -\frac{1}{2}$, $n \in \mathbb{N}_0$. Then we have*

$$\frac{\mathrm{Ec}_\nu^n(\mathrm{i}(K' - \tau), k)}{\mathrm{Ec}_\nu^n(\mathrm{i}(K' - \tau_0), k)} \to \frac{(\sinh \tau)^{1/2} \, Q_{n-\frac{1}{2}}^{\nu + \frac{1}{2}}(\cosh \tau)}{(\sinh \tau_0)^{1/2} \, Q_{n-\frac{1}{2}}^{\nu + \frac{1}{2}}(\cosh \tau_0)}$$

as $k \to 0$ locally uniformly for $\mathrm{Re}\,\tau > 0$. The same results holds with Es_ν^n in place of Ec_ν^n for $n \in \mathbb{N}$.

Proof. The function $w(\tau) = \mathrm{Ec}_\nu^n(\mathrm{i}(K' - \tau), k)$, $0 < \tau < 2K'$, satisfies the differential equation

$$w'' + \left(\nu(\nu + 1) - a_\nu^n(k) - \nu(\nu + 1)\,\mathrm{ns}^2(\tau, k')\right) w = 0. \tag{7.3}$$

We would like to apply Lemma 6.1 to this differential equation but we cannot show that the coefficient of w is negative if $-\frac{1}{2} \leq \nu < 0$. Instead of w we consider the function

$$v(\tau) := \mathrm{sn}^{-1/2}(\tau, k')w(\tau).$$

It satisfies the differential equation

$$v'' + p(\tau, k)v' + q(\tau, k)v = 0, \tag{7.4}$$

where

$$p(\tau, k) := \mathrm{cn}(\tau, k')\,\mathrm{ds}(\tau, k'),$$

$$q(\tau, k) := \frac{1}{4}k^2 - a_\nu^n(k) - \frac{3}{4}\,\mathrm{dn}^2(\tau, k') - \left(\nu + \frac{1}{2}\right)^2 \mathrm{cs}^2(\tau, k').$$

By Lemma 4.2, $a_\nu^n(k) \geq -\frac{1}{4}k^2$, so $q(\tau, k') \leq -\frac{1}{4}k^2 < 0$. As $k \to 0$,

$$p(\tau, k) \to p(\tau) := (\cosh \tau \sinh \tau)^{-1},$$

$$q(\tau, k) \to q(\tau) := -n^2 - \frac{3}{4}\cosh^{-2}\tau - \left(\nu + \frac{1}{2}\right)^2 \sinh^{-2}\tau$$

locally uniformly for $\operatorname{Re}\tau > 0$. Up to a constant factor, the function

$$u(\tau) = (\cosh\tau)^{1/2}\, Q_{n-\frac{1}{2}}^{\nu+\frac{1}{2}}(\cosh\tau)$$

is the only solution of the differential equation

$$u'' + p(\tau)u' + q(\tau)u = 0,$$

which is bounded on the interval $[\tau_0, \infty)$. Now we apply Lemma 6.1 with $a = \tau_0$ to the differential equation (7.4) as $k \to 0$, and we obtain the statement of the theorem with uniform convergence on $[\tau_0, b]$ for any $b > \tau_0$. The extension to locally uniform convergence for $\operatorname{Re}\tau > 0$ follows from Lemma 6.2 applied to differential equation (7.3). The proof for Es is very similar. ∎

Let $\mathrm{F}_\nu(s, h, k)$ be the solution of Lamé's equation belonging to the exponent $\nu + 1$ at $z = \mathrm{i}K'$, that is,

$$\mathrm{F}_\nu(s, h, k) = \tau^{\nu+1} \sum_{j=0}^{\infty} c_j \tau^j, \qquad \tau = K' + \mathrm{i}s, \quad c_0 = 1. \tag{7.5}$$

By analytic continuation, this function is well-defined in the strip $-K' < \operatorname{Im} s < K'$ using the principal branch of the power $(K' + \mathrm{i}s)^{\nu+1}$. Note that F_ν is real-valued for $s \in (-\mathrm{i}K', \mathrm{i}K')$. Now we find the behavior of the function (7.5) with $h = a_\nu^n$ or $h = b_\nu^n$ as $k \to 0$.

Theorem 7.3. *For $\nu \geq -\frac{1}{2}$ and $n \in \mathbb{N}_0$, we have*

$$\tau^{-\nu-1}\,\mathrm{F}_\nu(\mathrm{i}(K'-\tau), a_\nu^n(k), k) \to 2^{\nu+\frac{1}{2}}\Gamma\left(\nu+\frac{3}{2}\right)\tau^{-\nu-1}(\sinh\tau)^{1/2}P_{n-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cosh\tau)$$

as $k \to 0$ locally uniformly for $|\operatorname{Im}\tau| < \pi$. The same result holds with $b_\nu^n(k)$ in place of $a_\nu^n(k)$ for $n \in \mathbb{N}$.

Proof. The function $w(\tau) = \mathrm{F}_\nu(\mathrm{i}(K'-\tau), h, k)$ satisfies the differential equation

$$\tau^2 w'' = p(\tau, h, k)w, \qquad p(\tau, h, k) := \tau^2\big(h - \nu(\nu+1) + \nu(\nu+1)\,\mathrm{ns}^2(\tau, k')\big).$$

Now (7.2) and the maximum principle for analytic functions give

$$\tau\,\mathrm{ns}(\tau, k') \to \tau\coth\tau \qquad \text{as} \quad k \to 0 \quad \text{locally uniformly for} \quad |\operatorname{Im}\tau| < \pi.$$

Using Theorem 7.1, we see that

$$p(\tau, a_\nu^n(k), k) \to p_\infty(\tau) := \tau^2\big(n^2 - \nu(\nu+1) + \nu(\nu+1)\coth^2\tau\big)$$
$$= \tau^2\big(n^2 + \nu(\nu+1)\sinh^{-2}\tau\big)$$

as $k \to 0$ uniformly for $|\operatorname{Im}\tau| < \pi$. Note that the function

$$w(\tau) = 2^{\nu+\frac{1}{2}}\Gamma\left(\nu+\frac{3}{2}\right)(\sinh\tau)^{1/2}P_{n-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cosh\tau)$$

solves the differential equation $\tau^2 w'' = p_\infty(\tau)w$ and it has the form $w(\tau) = \tau^{\nu+1}v(\tau)$, where v is analytic at $\tau = 0$ with $v(0) = 1$. The latter equation follows from the representation of the associated Legendre function of the first kind P_ν^μ in terms of the hypergeometric function (1.7). Therefore, the statement of the theorem follows from Lemma 6.3. The proof for $h = b_\nu^n$ is the same. ∎

7.3 The expansion of the fundamental solution in flat-ring harmonics in the limit $k \to 0$

In Theorem 5.6 we found the expansion of the fundamental solution in terms of internal and external flat-ring harmonics. Let us write this expansion in the form

$$\frac{1}{\|\mathbf{r} - \mathbf{r}^*\|} = \sum_{m \in \mathbb{Z}} e^{im(\phi - \phi^*)} \sum_{n=0}^{\infty} A_{m,n}(\tau, \tau^*, \psi, \psi^*, k),$$

where $s = 2K - \psi$, $t = K' - \tau$, ϕ are flat-ring coordinates of $\mathbf{r}$, $s^* = 2K - \psi^*$, $t^* = K' - \tau^*$, ϕ^* are flat-ring coordinates of $\mathbf{r}^*$, $t < t^*$, and

$$A_{m,n} = \frac{1}{2} \left(\frac{\mathrm{dn}(\psi, k) - \mathrm{cn}(\psi, k)\,\mathrm{dn}(\tau, k')}{k'\,\mathrm{sn}(\tau, k')} \right)^{1/2} \left(\frac{\mathrm{dn}(\psi^*, k) - \mathrm{cn}(\psi^*, k)\,\mathrm{dn}(\tau^*, k')}{k'\,\mathrm{sn}(\tau^*, k')} \right)^{1/2}$$
$$\times \mathrm{Ec}^n_{|m| - \frac{1}{2}}(2K - \psi, k)\,\mathrm{Ec}^n_{|m| - \frac{1}{2}}(2K - \psi^*, k)$$
$$\times \mathrm{Ec}^n_{|m| - \frac{1}{2}}(\mathrm{i}(K' - \tau), k)\,\mathrm{Fc}^n_{|m| - \frac{1}{2}}(\mathrm{i}(K' - \tau^*), k) + \cdots,$$

where ... indicates a copy of the previous terms with Ec, Fc replaced by Es, Fs if $n \geq 1$ and represents 0 if $n = 0$.

The expansion (1.9) of the fundamental solution in toroidal harmonics can be written in the form

$$\frac{1}{\|\mathbf{r} - \mathbf{r}^*\|} = \sum_{m \in \mathbb{Z}} e^{im(\phi - \phi^*)} \sum_{n=0}^{\infty} B_{m,n}(\tau, \tau^*, \psi, \psi^*),$$

where τ, ψ, ϕ are toroidal coordinates of $\mathbf{r}$, τ^*, ψ^*, ϕ^* are toroidal coordinates of $\mathbf{r}^*$ with $\tau > \tau^*$, and

$$B_{m,n} = \frac{1}{\pi}(\cosh\tau - \cos\psi)^{1/2}(\cosh\tau^* - \cos\psi^*)^{1/2}\epsilon_n \cos(n(\psi - \psi^*))$$
$$\times (-1)^m \frac{\Gamma\left(n - m + \frac{1}{2}\right)}{\Gamma\left(n + m + \frac{1}{2}\right)} Q^m_{n - \frac{1}{2}}(\cosh\tau)P^m_{n - \frac{1}{2}}(\cosh\tau^*),$$

where $\epsilon_n := 2 - \delta_{n,0}$. We now prove the main result of this section.

Theorem 7.4. *Let $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $\tau, \tau^* > 0$, $\psi, \psi^* \in \mathbb{R}$. Then we have*

$$A_{m,n}(\tau, \tau^*, \psi, \psi^*, k) \to B_{m,n}(\tau, \tau^*, \psi, \psi^*) \qquad as \quad k \to 0. \tag{7.6}$$

Proof. It is known [18, formulas (14.9.13) and (14.9.14)] that

$$P^m_\nu(x) = \frac{\Gamma(\nu + m + 1)}{\Gamma(\nu - m + 1)}P^{-m}_\nu(x), \qquad Q^m_\nu(x) = \frac{\Gamma(\nu + m + 1)}{\Gamma(\nu - m + 1)}Q^{-m}_\nu(x). \tag{7.7}$$

Therefore, it is enough to consider $m \in \mathbb{N}_0$. We have as $k \to 0$,

$$\frac{\mathrm{dn}(\psi, k) - \mathrm{cn}(\psi, k)\,\mathrm{dn}(\tau, k')}{k'\,\mathrm{sn}(\tau, k')} \to \frac{\cosh\tau - \cos\psi}{\sinh\tau}. \tag{7.8}$$

It follows from Theorem 7.1 that as $k \to 0$,

$$\mathrm{Ec}^n_{m - \frac{1}{2}}(2K - \psi, k)\,\mathrm{Ec}^n_{m - \frac{1}{2}}(2K - \psi^*, k) \to \frac{2}{\pi}\epsilon_n \cos\left(n\left(\frac{\pi}{2} - \psi\right)\right)\cos\left(n\left(\frac{\pi}{2} - \psi^*\right)\right). \tag{7.9}$$

If $n \in \mathbb{N}$, we have a similar result with Ec replaced by Es and cos replaced by sin. Set

$$f(\tau, k) := \mathrm{Ec}^n_{m-\frac{1}{2}}(\mathrm{i}(K' - \tau), k), \qquad g(\tau, k) := \mathrm{Fc}^n_{m-\frac{1}{2}}(\mathrm{i}(K' - \tau), k),$$

$$f(\tau) := (\sinh \tau)^{1/2} Q^m_{n-\frac{1}{2}}(\cosh \tau), \qquad g(\tau) := (\sinh \tau)^{1/2} P^{-m}_{n-\frac{1}{2}}(\cosh \tau).$$

Then we obtain from Theorems 7.2 and 7.3, that as $k \to 0$,

$$\frac{f(\tau, k)g(\tau^*, k)}{W[g(\cdot, k), f(\cdot, k)]} \to \frac{f(\tau)g(\tau^*)}{W[g, f]}, \tag{7.10}$$

where $W[g, f]$ denotes the Wronskian of g, f (which is a constant). Now, by definition, $W[g(\cdot, k), f(\cdot, k)] = 1$ and, using (7.7) and [18, formula (14.2.10)]

$$W[P^\mu_\nu(x), Q^\mu_\nu(x)] = \mathrm{e}^{\mathrm{i}\mu\pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \frac{1}{1 - x^2}.$$

Therefore, (7.10) yields as $k \to 0$,

$$\mathrm{Ec}^n_{m-\frac{1}{2}}(\mathrm{i}(K' - \tau), k) \, \mathrm{Fc}^n_{m-\frac{1}{2}}(\mathrm{i}(K' - \tau^*), k)$$
$$\to (-1)^m \frac{\Gamma(n + m + \frac{1}{2})}{\Gamma(n - m + \frac{1}{2})} (\sinh \tau \sinh \tau^*)^{1/2} Q^m_{n-\frac{1}{2}}(\cosh \tau) P^m_{n-\frac{1}{2}}(\cosh \tau^*). \tag{7.11}$$

We have the same result when Ec, Fc are replaced by Es, Fs, respectively. If we combine (7.8), (7.9) and (7.11), and noting that

$$\cos\left(n\left(\frac{\pi}{2} - \psi\right)\right)\cos\left(n\left(\frac{\pi}{2} - \psi^*\right)\right) + \sin\left(n\left(\frac{\pi}{2} - \psi\right)\right)\sin\left(n\left(\frac{\pi}{2} - \psi^*\right)\right) = \cos(n(\psi - \psi^*)),$$

we obtain (7.6). ■

Acknowledgements

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