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The Gauge Group and Perturbation Semigroup of an Operator System

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Abstract. The perturbation semigroup was first defined in the case of $*$-algebras by Chamseddine, Connes and van Suijlekom. In this paper, we take $\mathcal{E}$ as a concrete operator system with unit. We first give a definition of gauge group $\mathcal{G}(\mathcal{E})$ of $\mathcal{E}$, after that we give the definition of perturbation semigroup of $\mathcal{E}$, and the closed perturbation semigroup of $\mathcal{E}$ with respect to the Haagerup tensor norm. We also show that there is a continuous semigroup homomorphism from the closed perturbation semigroup to the collection of unital completely bounded Hermitian maps over $\mathcal{E}$. Finally we compute the gauge group and perturbation semigroup of the Toeplitz system as an example.

Key words: operator algebras; operator systems; functional analysis; noncommutative geometry

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1 Introduction

An operator system $\mathcal{E}$ is a matrix-normed vector space equipped with a conjugate linear map $x \mapsto x^*$ on $\mathcal{E}$ such that $(x^*)^* = x$ for all $x \in \mathcal{E}$. Although there is not a well-defined product of elements in $\mathcal{E}$, we can embed the operator system $\mathcal{E}$ into some C^*-algebra $\mathcal{A}$, and then take the gauge group of $\mathcal{E}$ as the collection of unitary elements of $\mathcal{A}$ that keep $\mathcal{E}$ invariant under the unitary transformation, i.e.,

$$\mathcal{G}(\mathcal{E}) := \{u \in \mathcal{A} \colon u^*\mathcal{E}u = \mathcal{E}\}.$$

There are several different approaches to embed $\mathcal{E}$ into a C^*-algebra, for instance, we can embed $\mathcal{E}$ into the C^*-envelope $C^*_{\rm en}(\mathcal{E})$, the injective envelope $C^*_{\rm in}(\mathcal{E})$, or simply the C^*-algebra $C^*(\mathcal{E})$ generated by $\mathcal{E}$ when $\mathcal{E}$ is a concrete operator system. In this paper, we take $\mathcal{E}$ to be a concrete closed operator system with unit, i.e., a closed linear subspace of bounded operators on some Hilbert space $\mathcal{H}$ with $\mathrm{Id} \in \mathcal{E} \subset B(\mathcal{H})$, and we embed $\mathcal{E}$ into $C^*(\mathcal{E})$. In Section 2, we show that there is a group homomorphism from $\mathcal{G}(\mathcal{E})$ to the set of unital completely positive maps on $\mathcal{E}$. In Section 4.1, we show that the gauge group $\mathcal{G}(\mathrm{Toep}_n)$ of Toeplitz system Toep_n is independent of n, and

$$\mathcal{G}(\mathrm{Toep}_n) \cong U(1) \times (U(1) \rtimes \mathbb{Z}_2).$$

Inspired by the definition of perturbation semigroup of $*$-algebras given in [3], the perturbation semigroup of matrix algebras [10] and C^*-algebras [8], in Section 3, we give the definition of the perturbation semigroup $\mathrm{Pert}(\mathcal{E})$ of an operator system $\mathcal{E}$. More than that, since the perturbation semigroup $\mathrm{Pert}(\mathcal{E})$ is a subset of $\mathcal{A} \otimes \mathcal{A}^\circ$, we can take the closure of $\mathrm{Pert}(\mathcal{E})$ with respect to the Haagerup tensor norm, and we can show that there is a continuous semigroup homomorphism from this closure of $\mathrm{Pert}(\mathcal{E})$ to the collection of unital completely bounded Hermitian maps on $\mathcal{E}$.

In Section 4.2, we discuss the perturbation semigroups $\mathrm{Pert}(\mathrm{Toep}_n)$ of Toeplitz system Toep_n in more detail. We show the relationship between an element $\omega \in \mathrm{Pert}(\mathrm{Toep}_n)$ and the corresponding $(2n-1)\times(2n-1)$ transformation matrix of Toeplitz system Toep_n under the fundamental basis $\{\tau_{-n+1},\dots,\tau_0,\dots,\tau_{n-1}\}$ of Toep_n.

2 Gauge group of an operator system

Let $\mathcal{H}$ be a separable Hilbert space, we denote by $B(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$, $\mathcal{E}\subset B(\mathcal{H})$ an operator system,[1] and $C^*(\mathcal{E})$ the C^*-algebra generated by $\mathcal{E}$. We are mainly interested in the unital completely positive (UCP) maps over $\mathcal{E}$. According to Arveson's extension theorem [1, 11], if $\varphi\colon \mathcal{E}\to\mathcal{E}$ is a UCP map, then there is a UCP map $\widetilde{\varphi}\colon B(\mathcal{H})\to B(\mathcal{H})$ such that $\widetilde{\varphi}\big|_{\mathcal{E}}=\varphi$. In addition, if $\widetilde{\varphi}$ is normal,[2] according to Kraus [9, Theorem 3.3 or Theorem 4.1], the map $\widetilde{\varphi}$ can be written as

$$\widetilde{\varphi}(x)=\sum_k V_k^* x V_k, \qquad \forall x\in B(\mathcal{H}),$$

for some operators $\{V_k\}_{k\in K}\subset B(\mathcal{H})$ such that $\sum V_k^*V_k=\mathrm{Id}$. Hence especially when $U\in C^*(\mathcal{E})$ is a unitary element satisfying $U^*\mathcal{E}U\subset\mathcal{E}$ the corresponding map $\varphi\colon x\mapsto U^*xU$ is a UCP map over $\mathcal{E}$.

We denote by $\mathrm{UCP}(\mathcal{E})$ the collection of all the unital completely positive maps, and $\mathrm{UCP}_{\mathrm{rank}=1}(\mathcal{E})$ the collection of rank-1 unital completely positive maps, i.e.,

$$\mathrm{UCP}_{\mathrm{rank}=1}(\mathcal{E}):=\big\{\varphi\colon\mathcal{E}\to\mathcal{E}\mid \varphi(\cdot)=V^*(\cdot)V \text{ for some } V\in B(\mathcal{H}) \text{ with } V^*V=\mathrm{Id}\big\}.$$

We realize that both $\mathrm{UCP}(\mathcal{E})$ and $\mathrm{UCP}_{\mathrm{rank}=1}(\mathcal{E})$ are semigroups with respect to the map composition.

Definition 2.1. We define the gauge group $\mathcal{G}(\mathcal{E})$ of $\mathcal{E}$ as

$$\mathcal{G}(\mathcal{E}):=\{U\in\mathcal{U}(C^*(\mathcal{E}))\mid U^*\mathcal{E}U=\mathcal{E}\},$$

here $\mathcal{U}(C^*(\mathcal{E}))$ denotes the group of all the unitary elements in $C^*(\mathcal{E})$.

Remark 2.2. If $\varphi(\cdot)=V^*(\cdot)V\in\mathrm{UCP}_{\mathrm{rank}=1}(\mathcal{E})$, then $V\in B(\mathcal{H})$ is an isometry. In particular, if $\mathcal{E}\subset M_n(\mathbb{C})$ is a finite dimensional operator system, then V is a unitary matrix and $\mathrm{UCP}_{\mathrm{rank}=1}(\mathcal{E})$ is a group.

Proposition 2.3. *There is a multiplicative map* $\Psi\colon\mathcal{G}(\mathcal{E})\to\mathrm{UCP}_{\mathrm{rank}=1}(\mathcal{E})$ *defined as*

$$\Psi\colon\ U\mapsto U^*(\cdot)U,\quad U\in\mathcal{G}(\mathcal{E}).$$

We observe that the image of Ψ forms a group and the map $\Psi\colon\mathcal{G}(\mathcal{E})\to\mathrm{Image}(\Psi)$ is a surjective group homomorphism.

3 Perturbation semigroup of an operator system

In this section, we discuss unital completely bounded Hermitian(UCBH) maps and the perturbation semigroup of a concrete unital operator system $\mathcal{E}\subset B(\mathcal{H})$.

[1] Please check Appendix A for more details.
[2] Please check Appendix A for the definition of normal map.

Definition 3.1. We say $\Psi\colon \mathcal{E} \to \mathcal{E}$ is a Hermitian unital map if $\Psi(x^*) = \Psi(x)^*$ for all $x \in \mathcal{E}$ and $\Psi(\mathrm{Id}) = \mathrm{Id}$ for the unital element $\mathrm{Id} \in \mathcal{E}$. We denote by $\mathrm{UCBH}(\mathcal{E})$ the collection of all unital completely bounded Hermitian maps over $\mathcal{E}$, i.e.,

$$\mathrm{UCBH}(\mathcal{E}) := \big\{\Psi\colon \mathcal{E} \to \mathcal{E} \mid \Psi(x^*) = \Psi(x)^*,\ \Psi(\mathrm{Id}) = \mathrm{Id},\ \Psi \text{ is completely bounded}\big\}.$$

Inspired by the definition of perturbation semigroups introduced in [3, 8, 10], we define the perturbation semigroup $\mathrm{Pert}(\mathcal{E})$ of an operator system as follows:

Definition 3.2. Let $\mathcal{E}$ be an operator system, $C^*(\mathcal{E})$ be the C^*-algebra generated by $\mathcal{E}$ and $C^*(\mathcal{E})^\circ$ be the opposite algebra of $C^*(\mathcal{E})$. We define the perturbation semigroup $\mathrm{Pert}(\mathcal{E})$ as the collection of all the finite sums of the form $\sum a_i \otimes b_i^\circ \in C^*(\mathcal{E}) \otimes C^*(\mathcal{E})^\circ$ satisfying the following requirements:

1) $\sum a_i b_i = \mathrm{Id}$,
2) $\sum a_i \mathcal{E} b_i \subset \mathcal{E}$,
3) $\sum a_i \otimes b_i^\circ = \sum b_i^* \otimes a_i^{*\circ}$.

Remark 3.3. In the definition above, the opposite algebra $C^*(\mathcal{E})^\circ$ contains the same elements and addition operation as $C^*(\mathcal{E})$, while the multiplication order is reversed. And it is worth to observe that (1) and (3) inherit from the original definition of perturbation semigroup in [3], while (2) is an extra condition we need to assume in our case of operator system.

For each $(a, b^\circ) \in C^*(\mathcal{E}) \times C^*(\mathcal{E})^\circ$, let $\delta_{(a,b^\circ)}$ denote the completely bounded linear map on $C^*(\mathcal{E})$ in which $\delta_{(a,b^\circ)}(\xi) = a\xi b$, for all $\xi \in C^*(\mathcal{E})$. Let $\mathrm{CB}(C^*(\mathcal{E}))$ denote the set of all completed bounded maps over $C^*(\mathcal{E})$. The map $C^*(\mathcal{E}) \times C^*(\mathcal{E})^\circ \to \mathrm{CB}(C^*(\mathcal{E}))$ that sends each $(a, b^\circ) \in C^*(\mathcal{E}) \times C^*(\mathcal{E})^\circ$ to $\delta_{(a,b^\circ)} \in \mathrm{CB}(C^*(\mathcal{E}))$ is bilinear and therefore extends to a linear map $\Psi\colon C^*(\mathcal{E}) \otimes_{\mathrm{alg}} C^*(\mathcal{E})^\circ \to \mathrm{CB}(C^*(\mathcal{E}))$.

The perturbation semigroup $\mathrm{Pert}(\mathcal{E})$ is a subset of $C^*(\mathcal{E}) \otimes_{\mathrm{alg}} C^*(\mathcal{E})^\circ$, and so we define the map $\Phi\colon \mathrm{Pert}(\mathcal{E}) \to \mathrm{CB}(C^*(\mathcal{E}))$ by $\Phi = \Psi\big|_{\mathrm{Pert}(\mathcal{E})}$. Proposition 3.4 below shows that Φ is a semigroup homomorphism of $\mathrm{Pert}(\mathcal{E})$ into $\mathrm{UCBH}(\mathcal{E})$.

Proposition 3.4. *There is a semigroup homomorphism Φ from $\mathrm{Pert}(\mathcal{E})$ to $\mathrm{UCBH}(\mathcal{E})$ defined by*

$$\begin{aligned} \Phi\colon\ & \mathrm{Pert}(\mathcal{E}) \to \mathrm{UCBH}(\mathcal{E}), \\ & \omega \mapsto \sum a_i(\cdot)b_i \end{aligned}$$

with $\omega = \sum a_i \otimes b_i^\circ \in \mathrm{Pert}(\mathcal{E})$.

Proof. According to the definition of $\mathrm{Pert}(\mathcal{E})$ any element $\omega \in \mathrm{Pert}(\mathcal{E})$ can be written as $\omega = \sum a_i \otimes b_i^\circ = \sum b_i^* \otimes a_i^{*\circ}$, thus $\Phi(\omega)$ is a Hermitian map. The assumption that $\sum a_i b_i = \mathrm{Id}$ confirms $\Phi(\omega)$ is unital. Since there are only finitely many terms in the expression of the sum

$$\Phi(\omega)\colon\ x \mapsto \sum a_i x b_i, \qquad \forall x \in \mathcal{E},$$

hence it is completely bounded due to [11, Chapter 8].

Finally we shall show that the map $\Phi\colon \mathrm{Pert}(\mathcal{E}) \to \mathrm{UCBH}(\mathcal{E})$ is a semigroup homomorphism. Let $\omega = \sum a_i \otimes b_i^\circ$ and $\widetilde{\omega} = \sum \widetilde{a}_j \otimes \widetilde{b}_j^\circ$ be two elements in $\mathrm{Pert}(\mathcal{E})$, we have that $\omega\widetilde{\omega} = \sum a_i \widetilde{a}_j \otimes \big(\widetilde{b}_j b_i\big)^\circ$, and by Definition 3.2

$$\Phi(\omega\widetilde{\omega})(x) = \sum a_i \widetilde{a}_j\, x\, \widetilde{b}_j b_i = \sum_i a_i \Big(\sum_j \widetilde{a}_j\, x\, \widetilde{b}_j\Big) b_i \qquad \text{for any} \quad x \in \mathcal{E},$$

thus $\Phi(\omega\widetilde{\omega}) = \Phi(\omega)\Phi(\widetilde{\omega})$ for $\omega, \widetilde{\omega} \in \mathrm{Pert}(\mathcal{E})$. ∎

We can move one step further by equipping the semigroup $\mathrm{Pert}(\mathcal{E})$ with the Haagerup tensor norm so that Φ can be extended to the closure of $\mathrm{Pert}(\mathcal{E})$. Recall that the Haagerup tensor norm[3] $\|u\|_h$ of an element $u \in C^*(\mathcal{E}) \otimes C^*(\mathcal{E})^\circ$ is defined as

$$\|u\|_h = \inf \left\{ \Big\| \sum a_i a_i^* \Big\|^{1/2} \Big\| \sum b_i^* b_i \Big\|^{1/2} \right\},$$

where the infimum is taken over all the expressions of $u = \sum a_i \otimes b_i^\circ$ for $a_i, b_i \in C^*(\mathcal{E})$. Here we omit the opposite algebra structure. Since $\mathrm{Pert}(\mathcal{E})$ is a subset of $C^*(\mathcal{E}) \otimes C^*(\mathcal{E})^\circ$, we can endow $\mathrm{Pert}(\mathcal{E})$ with the metric topology induced by the Haagerup tensor norm $\|\cdot\|_h$.

Definition 3.5. We define the closed perturbation semigroup $\overline{\mathrm{Pert}(\mathcal{E})}$ as the closure of $\mathrm{Pert}(\mathcal{E})$ with respect to the topology induced by Haagerup tensor norm $\|\cdot\|_h$.

Proposition 3.6. *Let $\mathcal{E} \subset B(\mathcal{H})$ be a unital operator system, the map $\Phi\colon \mathrm{Pert}(\mathcal{E}) \to \mathrm{UCBH}(\mathcal{E})$ can be extended to a map*

$$\widetilde{\Phi}\colon\ \overline{\mathrm{Pert}(\mathcal{E})} \to \mathrm{UCBH}(\mathcal{E}),$$

such that $\widetilde{\Phi}\big|_{\mathrm{Pert}(\mathcal{E})} = \Phi$. Moreover, if we equip $\overline{\mathrm{Pert}(\mathcal{E})}$ and $\mathrm{UCBH}(\mathcal{E})$ with the metric topology induced by Haagerup tensor norm $\|\cdot\|_h$ and complete bound norm $\|\cdot\|_{cb}$ respectively, the map $\widetilde{\Phi}$ is contractive.

Proof. By Definition 3.2 $\mathrm{Pert}(\mathcal{E})$ is a subset of $C^*(\mathcal{E}) \otimes_{\mathrm{alg}} C^*(\mathcal{E})$. Take an element $\omega = \sum a_i \otimes b_i^\circ \in \mathrm{Pert}(\mathcal{E})$, we define a map $\widetilde{\Phi}\colon \mathrm{Pert}(\mathcal{E}) \to \mathrm{CB}(B(\mathcal{H}))$ as $\widetilde{\Phi}(\omega)\colon T \mapsto \sum a_i T b_i$ for $T \in B(\mathcal{H})$. According to [12, Theorem 5.12], the map $\widetilde{\Phi}$ is completely isometric if we equip with ω the Haagerup norm and $\widetilde{\Phi}(\omega)$ the completely bounded norm. If we can take the closure $\overline{\mathrm{Pert}(\mathcal{E})}$, we get a map from $\overline{\mathrm{Pert}(\mathcal{E})}$ to $\mathrm{CB}(B(\mathcal{H}))$, which we still denote as $\widetilde{\Phi}$. By our definition of $\widetilde{\Phi}$, we observe that $\widetilde{\Phi}\big|_{\mathrm{Pert}(\mathcal{E})} = \Phi$, hence we only need to show that the image of $\widetilde{\Phi}$ is contained in $\mathrm{UCBH}(\mathcal{E})$.

Take a sequence of $\{\omega_n\}_{n\geq 1} \subset \mathrm{Pert}(\mathcal{E})$ that approaches to some $\omega \in \overline{\mathrm{Pert}(\mathcal{E})}$. Since

$$\widetilde{\Phi}(\omega_n)(\mathrm{Id}) = \Phi(\omega_n)(\mathrm{Id}) = \mathrm{Id},$$

we obtain that $\widetilde{\Phi}(\omega)$ is a unital map. Similarly, since for each ω_n the map $\Phi(\omega_n)$ is Hermitian, we conclude that $\widetilde{\Phi}(\omega)$ is Hermitian. Hence we only need to show that for any $x \in \mathcal{E}$, $\widetilde{\Phi}(\omega)(x) \in \mathcal{E}$.

In fact, for any $\epsilon > 0$, there exists an $N > 0$ such that when $n \geq N$ we have $\|\omega_n - \omega\|_h < \epsilon$. Besides that, according to [12, Theorem 5.12], if we regard $\widetilde{\Phi}(\omega_n) - \widetilde{\Phi}(\omega)$ as a map on $B(\mathcal{H})$ we can obtain that $\big\|\widetilde{\Phi}(\omega_n) - \widetilde{\Phi}(\omega)\big\|_{cb} = \|\omega_n - \omega\|_h$, since $\mathcal{E} \subset B(\mathcal{H})$. For the restriction of $\widetilde{\Phi}(\omega_n) - \widetilde{\Phi}(\omega)$ to $\mathcal{E}$ we obtain $\big\|\widetilde{\Phi}(\omega_n) - \widetilde{\Phi}(\omega)\big\|_{cb} \leq \|\omega_m - \omega\|_h$. Hence

$$\big\|\widetilde{\Phi}(\omega_n) - \widetilde{\Phi}(\omega)\big\| \leq \big\|\widetilde{\Phi}(\omega_n) - \widetilde{\Phi}(\omega)\big\|_{cb} \leq \|\omega_n - \omega\|_h < \epsilon.$$

Thus if we take an $x \in \mathcal{E}$, we have

$$\frac{\big\|\widetilde{\Phi}(\omega_n)(x) - \widetilde{\Phi}(\omega)(x)\big\|}{\|x\|} < \epsilon.$$

Therefore $\widetilde{\Phi}(\omega_n)(x) \to \widetilde{\Phi}(\omega)(x)$. So that by closedness of $\mathcal{E}$ we obtain that $\widetilde{\Phi}(\omega)(x) \in \mathcal{E}$.

Hence for an element $\omega \in \overline{\mathrm{Pert}(\mathcal{E})}$, we can consider $\widetilde{\Phi}(\omega)$ as either an element of $\mathrm{UCBH}(B(\mathcal{H}))$ or an element of $\mathrm{UCBH}(\mathcal{E})$. However, since $\mathcal{E} \subset B(\mathcal{H})$ is a subset, if we regard $\widetilde{\Phi}(\omega)$ as a element in $\mathrm{UCBH}(\mathcal{E})$, the completely bounded norm of $\widetilde{\Phi}(\omega)$ is less than or equal to the completely bounded norm of $\widetilde{\Phi}(\omega)$ as an element of $\mathrm{UCBH}(B(\mathcal{H}))$. Therefore the map $\widetilde{\Phi}$ is contractive. ∎

[3]Please see Appendix B for more details.

For a general operator system $\mathcal{E}$ we can only conclude the map $\widetilde{\Phi}\colon \overline{\mathrm{Pert}(\mathcal{E})} \to \mathrm{UCBH}(\mathcal{E})$ is completely contractive rather than completely isometric.

Example 3.7. Let $\{E_{ij}\}$, $1 \leq i, j \leq 2$ be the standard matrix units for $M_2(\mathbb{C})$. Define

$$\mathrm{Toep}_2 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \subset M_2(\mathbb{C}) \right\}.$$

Take $\omega_1, \omega_2 \in \mathrm{Pert}(\mathrm{Toep}_2)$ given as

$$\begin{aligned} \omega_1 &= E_{12} \otimes E_{12}^\circ + E_{21} \otimes E_{21}^\circ + E_{11} \otimes E_{11}^\circ + E_{22} \otimes E_{22}^\circ, \\ \omega_2 &= (E_{12} + E_{21}) \otimes (E_{12} + E_{21})^\circ. \end{aligned}$$

By a direct computation we obtain that $\Phi(\omega_1) = \Phi(\omega_2)$ on Toep_2, both give rise to the transposition map on Toep_2, and we observe that $E_{12} + E_{21}$ is a 2×2 unitary matrix, thus $\|\Phi(\omega_2)\|_{cb} = 1$, and therefore we obtain that $\|\Phi(\omega_1)\|_{cb} = 1$.

However, according to [11, Theorem 17.4], the Haagerup tensor norm $\|\omega_1\|_h$ is equal to the completely bounded norm of the transposition transformation over $M_2(\mathbb{C})$, which is equal to 2. Therefore, $\|\Phi(\omega_1)\|_{cb} = 1 < \|\omega_1\|_h = 2$.

Definition 3.8. We denote by $\mathrm{Pert}^+(\mathcal{E})$ the sub-semigroup of $\mathrm{Pert}(\mathcal{E})$ containing all the $\omega \in \mathrm{Pert}(\mathcal{E})$ of the form $\omega = \sum a_i \otimes a_i^{*\circ}$ for some $a_i \in C^*(\mathcal{E})$, i.e.,

$$\mathrm{Pert}^+(\mathcal{E}) := \left\{ \omega \in \mathrm{Pert}(\mathcal{E}) \,\middle|\, \omega = \sum a_i \otimes a_i^{*\circ} \text{ for some } a_i \in C^*(\mathcal{E}) \right\}.$$

To simplify the notation we still denote the restriction $\Phi|_{\mathrm{Pert}^+(\mathcal{E})}$ to $\mathrm{Pert}^+(\mathcal{E})$ by Φ.

Corollary 3.9. *Let $\omega = \sum a_i \otimes a_i^{*\circ} \in \mathrm{Pert}^+(\mathcal{E})$. We have $\Phi(\omega) \in \mathrm{UCP}(\mathcal{E})$, namely*

$$\begin{aligned} \Phi\colon \ \mathrm{Pert}^+(\mathcal{E}) &\to \mathrm{UCP}(\mathcal{E}), \\ \omega &\mapsto \sum a_i (\cdot) a_i^*. \end{aligned}$$

Proof. By Proposition 3.4 we have that $\Phi(\omega) \in \mathrm{UCBH}(\mathcal{E})$ for $\omega \in \mathrm{Pert}^+(\mathcal{E})$, and $\Phi(\omega)(\cdot) = \sum a_i(\cdot)a_i^*$, which is a completely positive map. ■

As in the case of $\mathrm{Pert}(\mathcal{E})$, we can take the closure of $\mathrm{Pert}^+(\mathcal{E})$ with respect to Haagerup tensor norm, which we denote as $\overline{\mathrm{Pert}^+(\mathcal{E})}$.

Proposition 3.10. *Let $\overline{\mathrm{Pert}^+(\mathcal{E})}$ be the closure of $\mathrm{Pert}^+(\mathcal{E})$ with respect to Haagerup tensor norm. We can extend the map $\Phi\colon \mathrm{Pert}^+(\mathcal{E}) \to \mathrm{UCP}(\mathcal{E})$ to a map*

$$\widetilde{\Phi}\colon \ \overline{\mathrm{Pert}^+(\mathcal{E})} \to \mathrm{UCP}(\mathcal{E}),$$

such that $\widetilde{\Phi}|_{\mathrm{Pert}^+(\mathcal{E})} = \Phi$. Moreover, we have $\|\omega\|_h = 1$ and $\|\widetilde{\Phi}(\omega)\|_{cb} = 1$ for every $\omega \in \overline{\mathrm{Pert}^+(\mathcal{E})}$.

Proof. Take an element $\omega \in \overline{\mathrm{Pert}^+(\mathcal{E})}$, according to Proposition 3.6, the map $\widetilde{\Phi}(\omega) \in \mathrm{UCBH}(\mathcal{E})$. we then need to show that $\widetilde{\Phi}(\omega)$ is completely positive. Indeed, if we take a sequence $\{\omega_n\}_{n\geq 1} \subset \mathrm{Pert}^+(\mathcal{E})$ such that $\omega_n \to \omega$, then for any $\epsilon > 0$, there exists an $N > 0$ such that when $n \geq N$

$$\left\| \widetilde{\Phi}(\omega_n) - \widetilde{\Phi}(\omega) \right\|_{cb} \leq \|\omega_n - \omega\|_h < \epsilon. \tag{3.1}$$

Take a positive element $X_k \in M_k(\mathcal{E})$, then $\widetilde{\Phi}(\omega_n)(X_k) \in M_k(\mathcal{E})$ is also positive for all $n \in \mathbb{N}$. And by the inequality (3.1), we have

$$\frac{\left\| \widetilde{\Phi}(\omega_n)(X_k) - \widetilde{\Phi}(\omega)(X_k) \right\|}{\|X_k\|} < \epsilon,$$

that is to say, $\widetilde{\Phi}(\omega)(X_k)$ is the limit point of the sequence of positive elements $\{\widetilde{\Phi}(\omega_n)(X_k)\}_{n\geq 1}$ in $M_k(\mathcal{E})$, thus $\widetilde{\Phi}(\omega)(X_k) \in M_k(\mathcal{E})$ is positive. Since this is true for all $k \in \mathbb{N}$, $\widetilde{\Phi}(\omega)$ is completely positive and therefore $\widetilde{\Phi}(\omega) \in \mathrm{UCP}(\mathcal{E})$.

Finally, we only need to show that $\|\widetilde{\Phi}(\omega)\|_{cb} = \|\omega\|_h = 1$ for each $\omega \in \overline{\mathrm{Pert}^+(\mathcal{E})}$. Take an element $\omega \in \overline{\mathrm{Pert}^+(\mathcal{E})}$. For any $\epsilon > 0$, there exists an $\omega' \in \mathrm{Pert}^+(\mathcal{E})$ such that

$$\|\Phi(\omega')\|_{cb} - \epsilon \leq \|\widetilde{\Phi}(\omega)\|_{cb} \leq \|\omega\|_h \leq \|\omega'\|_h + \epsilon.$$

Since $\omega' \in \mathrm{Pert}^+(\mathcal{E})$, we can write ω' as $\omega' = \sum_{i=1}^{k} a_i \otimes a_i^{*\circ}$ for some $a_i \in C^*(\mathcal{E})$, and according to Definition 3.2, we obtain that $\sum_{i=1}^{k} a_i a_i^* = \mathrm{Id}$. Thus

$$\|\omega'\|_h \leq \left\| \sum_{i=1}^{k} a_i a_i^* \right\| = 1.$$

On the other hand, we observe the inequality

$$\|\Phi(\omega')\|_{cb} \geq \|\Phi(\omega')\| \geq \|\Phi(\omega')(\mathrm{Id})\| = 1.$$

Hence combine the three inequalities above together we conclude that

$$1 - \epsilon \leq \|\widetilde{\Phi}(\omega)\|_{cb} \leq \|\omega\|_h \leq 1 + \epsilon.$$

Since this is true for every $\epsilon > 0$, we obtain that $\|\widetilde{\Phi}(\omega)\|_{cb} = \|\omega\|_h = 1$ for all $\omega \in \overline{\mathrm{Pert}^+(\mathcal{E})}$. ■

We also observe that there is a map from the gauge group $\mathcal{G}(\mathcal{E})$ to the semigroup $\mathrm{Pert}^+(\mathcal{E})$, as stated in the following proposition.

Proposition 3.11. *There is a multiplicative map from $\mathcal{G}(\mathcal{E})$ to $\mathrm{Pert}^+(\mathcal{E})$ defined by*

$$\begin{aligned} \mathcal{G}(\mathcal{E}) &\to \mathrm{Pert}^+(\mathcal{E}), \\ u &\to u^* \otimes u^\circ. \end{aligned}$$

Remark 3.12. Although for an element $\omega \in \overline{\mathrm{Pert}^+(\mathcal{E})}$ we have $\|\widetilde{\Phi}(\omega)\|_{cb} = \|\omega\|_h = 1$, the completely bounded norm $\|\widetilde{\Phi}(\omega_1) - \widetilde{\Phi}(\omega_2)\|_{cb}$ and the Haagerup norm of $\|\omega_1 - \omega_2\|_h$ for two elements $\omega_1, \omega_2 \in \overline{\mathrm{Pert}^+(\mathcal{E})}$ usually are not equal.

Consider the 2×2 Toeplitz system Toep_2. Take $\omega_1, \omega_2 \in \mathrm{Pert}^+(\mathrm{Toep}_2)$ as

$$\omega_1 = E_{11} \otimes E_{11}^\circ + E_{22} \otimes E_{22}^\circ, \qquad \omega_2 = E_{12} \otimes E_{21}^\circ + E_{21} \otimes E_{12}^\circ,$$

although $\omega_1 \neq \omega_2$, we have $\Phi(\omega_1) = \Phi(\omega_2)$. Indeed,

$$\Phi(\omega_1) = \Phi(\omega_2)\colon \begin{pmatrix} a & b \\ c & a \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Therefore $\|\widetilde{\Phi}(\omega_1) - \widetilde{\Phi}(\omega_2)\|_{cb}$ is equal to 0 while $\|\omega_1 - \omega_2\|_h$ is not.

4 Gauge group and perturbation semigroup of the Toeplitz system

The concept of truncated circle is introduced by Alain Connes and Walter D. van Suijlekom in [4]. Recall that the canonical spectral triple on the circle is in the form of

$$\left(C^\infty(S^1),\, L^2(S^1),\, D = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)$$

as discussed in [15, Chapter 5]. Let $\{e_n\}_{n\in\mathbb{Z}}$ be the set of eigenvectors of D, we consider a spectral truncation defined by the orthogonal projection P_n onto $\operatorname{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$ for some $n > 0$. The truncated circle with respect to P_n is defined as

$$\big(P_n C^\infty(S^1) P_n, P_n L^2(S^1), P_n D P_n\big).$$

Since P_n does not commute with the $*$-algebra $C^\infty(S^1)$, $P_n C^\infty(S^1) P_n$ is only an operator system rather than an algebra. In fact, if $f \in C^\infty(S^1)$ is a smooth function with Fourier coefficients $\{a_n\}_{n\in\mathbb{Z}}$, then the truncation $P_n f P_n$ can be written as a Toeplitz matrix:

$$P_n f P_n = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-n+2} & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-n+2} \\ \vdots & a_1 & a_0 & \ddots & \vdots \\ a_{n-2} & \vdots & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{pmatrix}.$$

Hence it turns out that $P_n C^\infty(S^1) P_n$ is the Toeplitz operator system containing all the $n \times n$ Toeplitz matrices, which we denote as Toep_n.

One interesting question is what are the gauge group and perturbation semigroup of the Toeplitz operator system Toep_n. In this section, we will present the structure of gauge group $\mathcal{G}(\mathrm{Toep}_n)$ and some properties of perturbation semigroup $\mathrm{Pert}(\mathrm{Toep}_n)$. Many properties of Toep_n are different from that of $M_n(\mathbb{C})$, in this section, we will also show that the transpose map on Toep_n is a UCP map, which is absolutely wrong in the case of $M_n(\mathbb{C})$. The readers can refer to [7] for more details and other interesting behaviors about Toeplitz operator system.

4.1 Gauge group of the Toeplitz system

As is shown in [4], the C^*-algebra generated by Toep_n is just $M_n(\mathbb{C})$. The main goal of this section is to figure out $\mathcal{G}(\mathrm{Toep}_n)$. One interesting phenomenon is that $\mathcal{G}(\mathrm{Toep}_n)$ is independent of n. Before proving that we need the following lemma.

Lemma 4.1. *Let $U \in \mathcal{G}(\mathrm{Toep}_n)$, then U is either a diagonal matrix or an anti-diagonal matrix.*

Proof. We take a unitary matrix $U = (u_{ij})_{1\le i,j\le n} \in \mathcal{U}(M_n(\mathbb{C}))$ and a basis $\{\tau_j\}_{j=-n+1,\ldots,n-1}$ of the Toeplitz system Toep_n given by 1's on the j'th diagonal and 0's elsewhere, i.e., for positive k we have

$$\tau_k = \sum_{i=1}^{n-k} E_{i,i+k}, \qquad \tau_{-k} = \sum_{i=1}^{n-k} E_{i+k,i},$$

here $E_{i,j}$ is the $n \times n$ unit matrix with 1 in (i,j)-entry and 0's everywhere else. An element $U \in \mathcal{G}(\mathrm{Toep}_n)$ if and only if $U^*\tau_j U \in \mathrm{Toep}_n$ for all $j \in [-n+1, n-1]$. We observe first that when $k > 0$ the (j,l)-entry of $U^*\tau_k U$ is given by

$$(U^*\tau_k U)_{j,l} = \sum_{i=1}^{n-k} \overline{u}_{i,j} u_{k+i,l}, \qquad 1 \le j, l \le n, \tag{4.1}$$

and

$$\mathrm{Tr}(U^*\tau_k U) = \sum_{j=1}^{n}\sum_{i=1}^{n-k} \overline{u}_{i,j} u_{k+i,j}.$$

Since U is a unitary matrix, we have $\sum_{j=1}^{n} \overline{u}_{i,j} u_{k+i,j} = 0$ for $k > 0$ and $1 \leq i \leq n-k$. Thus we have

$$\mathrm{Tr}(U^*\tau_k U) = \sum_{j=1}^{n}\sum_{i=1}^{n-k} \overline{u}_{i,j} u_{k+i,j} = 0, \qquad k > 0.$$

Due to our assumption that $U^*\tau_k U \in \mathrm{Toep}_n$, we must have all the diagonal entries of $U^*\tau_k U$ are zeros:

$$\sum_{i=1}^{n-k} \overline{u}_{i,j} u_{k+i,j} = 0, \qquad k > 0, \quad 1 \leq j \leq n. \tag{4.2}$$

Take $k = n-1$ and $j = 1$ in formula (4.2), we have that $\overline{u}_{1,1} u_{n,1} = 0$. However $\overline{u}_{1,1}$ and $u_{n,1}$ can not be both equal to 0, otherwise by equation (4.1), $U^*\tau_{n-1}U = 0$. In fact, the equation (4.1) implies that $(U^*\tau_{n-1}U)_{j,l} = \overline{u}_{1,j} u_{n,l}$, suppose if $\overline{u}_{1,1} = u_{n,1} = 0$, then $(U^*\tau_{n-1}U)_{1,l} = (U^*\tau_{n-1}U)_{j,1} = 0$ for all $1 \leq j, l \leq n$. That is, all the entries in the first row and the first column of $U^*\tau_{n-1}U$ are 0's, since we assume the matrix $U^*\tau_{n-1}U$ is a Toeplitz matrix, we conclude that $(U^*\tau_{n-1}U) = 0$. This is a contradiction of the unitary of U.

Since $u_{1,1}$ and $u_{n,1}$ can not both be equal to 0, we first assume that $u_{1,1} = \alpha \neq 0$ and $u_{n,1} = 0$. If we take $k = n-2$ and $j = 1$ in formula (4.2), we obtain that

$$\overline{u}_{1,1} u_{n-1,1} + \overline{u}_{2,1} u_{n,1} = 0. \tag{4.3}$$

Therefore we have $u_{n-1,1} = 0$. We then take $k = n-3$ and $j = 1$ in formula (4.2) again, we obtain the equation

$$\overline{u}_{1,1} u_{n-2,1} + \overline{u}_{2,1} u_{n-1,1} + \overline{u}_{3,1} u_{n,1} = 0, \tag{4.4}$$

since $u_{1,1} \neq 0, u_{n,1} = 0$ and $u_{n-1,1} = 0$, we obtain that $u_{n-2,1} = 0$. By induction, take $j = 1$ and $k = n-4, n-5, \ldots, 2, 1$, we obtain that $u_{i,1} = 0$ for $1 < i \leq n$, namely, all the entries in the first column of U are equal to 0 except that $u_{1,1} = \alpha \neq 0$. Thus we can write U in the matrix form as

$$U = \begin{pmatrix} \alpha & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ 0 & u_{3,2} & u_{3,3} & \cdots & u_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & u_{n,2} & u_{n,3} & \cdots & u_{n,n} \end{pmatrix},$$

and by a simple computation

$$U^*\tau_{n-1}U = \begin{pmatrix} 0 & \overline{\alpha} u_{n,2} & \overline{\alpha} u_{n,3} & \cdots & \overline{\alpha} u_{n,n} \\ 0 & \overline{u}_{1,2} u_{n,2} & \overline{u}_{1,2} u_{n,3} & \cdots & \overline{u}_{1,2} u_{n,n} \\ 0 & \overline{u}_{1,3} u_{n,2} & \overline{u}_{1,3} u_{n,3} & \cdots & \overline{u}_{1,3} u_{n,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \overline{u}_{1,n} u_{n,2} & \overline{u}_{1,n} u_{n,3} & \cdots & \overline{u}_{1,n} u_{n,n} \end{pmatrix}. \tag{4.5}$$

Now we show that $u_{1,2} = u_{1,3} = \cdots = u_{1,n} = 0$. Since the matrix (4.5) is a Toeplitz matrix, the $(2,2)$-entry element in (4.5) must be equal to 0, namely $\overline{u}_{1,2}u_{n,2} = 0$. Suppose $u_{1,2} \neq 0$, then we must have $u_{n,2} = 0$, which implies that the second column of (4.5) is 0. It then implies that the $(2,3)$-entry element in (4.5) is equal to 0, which implies $u_{n,3} = 0$ and thus the third column of (4.5) is 0. By induction, we obtain that $u_{n,1} = u_{n,2} = u_{n,3} = \cdots = u_{n,n-1} = u_{n,n} = 0$, that is, $U^*\tau_{n-1}U = 0$, which is impossible. Therefore we must have $u_{1,2} = 0$, and we deduce that all the entries in the second row of (4.5) are 0's. Hence the only non-zero entry in (4.5) is the $(1,n)$-entry and all the rest entries are 0's. Namely,

$$U^*\tau_{n-1}U = \overline{\alpha}u_{n,n}\tau_{n-1}.$$

Thus we obtain that $u_{1,2} = u_{1,3} = \cdots = u_{1,n} = 0$. That is to say, the unitary matrix U is of the form

$$U = \begin{pmatrix} \alpha & 0 \\ 0 & \widetilde{U} \end{pmatrix}, \qquad |\alpha| = 1, \quad \widetilde{U} \in \mathcal{U}(M_{n-1}(\mathbb{C})).$$

Take a Toeplitz matrix $T = (t_{ij})_{1\leq i,j\leq n} \in \mathrm{Toep}_n$, we write T in the block form as

$$T = \begin{pmatrix} x & X \\ Y & \widetilde{T} \end{pmatrix}, \qquad \widetilde{T} \in \mathrm{Toep}_{n-1},$$

here $x = t_{11}$, $X = (t_{12}, \ldots, t_{1n})$, and $Y = (t_{21}, \ldots, t_{n1})^{\mathrm{T}}$. A simple computation shows that

$$U^*TU = \begin{pmatrix} x & \overline{\alpha}X\widetilde{U} \\ \alpha\widetilde{U}^*Y & \widetilde{U}^*\widetilde{T}\widetilde{U} \end{pmatrix} \in \mathrm{Toep}_n,$$

which implies that $\widetilde{U}^*\widetilde{T}\widetilde{U} \in \mathrm{Toep}_{n-1}$. Apply the same argument to $\widetilde{U} \in \mathrm{Toep}_{n-1}$, we obtain that the $(n-1)\times(n-1)$ unitary matrix $\widetilde{U}$ is of the form

$$\widetilde{U} = \begin{pmatrix} \beta & 0 \\ 0 & \widehat{U} \end{pmatrix}, \qquad |\beta| = 1, \quad \widehat{U} \in \mathcal{U}(M_{n-2}(\mathbb{C})),$$

apply the same argument to $\widehat{U}$ again, by induction we obtain that U is a diagonal matrix when $u_{1,1} \neq 0$.

On the other hand, when $u_{1,1} = 0$ and $u_{n,1} = \alpha \neq 0$, the equation (4.3) then implies that $u_{2,1} = 0$, and the equation (4.4) implies that $u_{3,1} = 0$, by induction, take $k = n-4, n-5, \ldots, 2, 1$ and $j = 1$, we obtain that the first column of U are all zeros except $u_{n,1} \neq 0$. Namely the unitary matrix U is of the form

$$U = \begin{pmatrix} 0 & u_{1,2} & \cdots & u_{1,n-1} & u_{1,n} \\ 0 & u_{2,2} & \cdots & u_{2,n-1} & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & u_{n-1,2} & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ \alpha & u_{n,2} & \cdots & u_{n,n-1} & u_{n,n} \end{pmatrix},$$

and by a direct computation we can write the matrix $U^*\tau_{n-1}U$ as

$$U^*\tau_{n-1}U = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \overline{u}_{1,2}\alpha & \overline{u}_{1,2}u_{n,2} & \cdots & \overline{u}_{1,2}u_{n,n-1} & \overline{u}_{1,2}u_{n,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{u}_{1,n-1}\alpha & \overline{u}_{1,n-1}u_{n,2} & \cdots & \overline{u}_{1,n-1}u_{n,n-1} & \overline{u}_{1,n-1}u_{n,n} \\ \overline{u}_{1,n}\alpha & \overline{u}_{1,n}u_{n,2} & \cdots & \overline{u}_{1,n}u_{n,n-1} & \overline{u}_{1,n}u_{n,n} \end{pmatrix}.$$

Using a similar argument as in the case of $u_{1,1} \neq 0$ and $u_{n,1} = 0$, we can deduce that U is an anti-diagonal matrix if $u_{1,1} = 0$ and $u_{n,1} \neq 0$. ∎

The gauge group $\mathcal{G}(\mathrm{Toep}_n)$ has a more explicit expression as given below.

Proposition 4.2. *The gauge group $\mathcal{G}(\mathrm{Toep}_n)$ is generated by the diagonal matrices $U_{\alpha,\beta}$ and anti-diagonal matrix V of the form*

$$U_{\alpha,\beta} = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & \beta & 0 & \cdots & 0 \\ 0 & 0 & \overline{\alpha}\beta^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \overline{\alpha}^{n-2}\beta^{n-1} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 & 0 \end{pmatrix}, \quad |\alpha| = |\beta| = 1. \tag{4.6}$$

Proof. According to Lemma 4.1, any $U \in \mathcal{G}(\mathrm{Toep}_n)$ is either a diagonal matrix or an anti-diagonal matrix. Suppose U is a diagonal matrix, then U can be expressed as

$$U = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix}$$

with $|\alpha_i| = 1$ for $i = 1, 2, \ldots, n$. We then obtain

$$U^*\tau_1 U = \begin{pmatrix} 0 & \overline{\alpha}_1\alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \overline{\alpha}_2\alpha_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \overline{\alpha}_{n-1}\alpha_n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

since $U^*\tau_1 U \in \mathrm{Toep}_n$, we must have $\overline{\alpha}_1\alpha_2 = \overline{\alpha}_2\alpha_3 = \cdots = \overline{\alpha}_{n-1}\alpha_n$. If we take $\alpha_1 = \alpha$ and $\alpha_2 = \beta$, we must have $\alpha_i = \overline{\alpha}^{i-2}\beta^{i-1}$ for $3 \leq i \leq n$, hence we obtain the unitary matrix $U_{\alpha,\beta}$ as given in (4.6).

Now suppose if the unitary matrix W is an anti-diagonal matrix of the form

$$W = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha_1 \\ 0 & 0 & \cdots & \alpha_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \alpha_{n-1} & \cdots & 0 & 0 \\ \alpha_n & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Using a similar argument we can show that

$$W = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha \\ 0 & 0 & \cdots & \beta & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \overline{\alpha}^{n-3}\beta^{n-2} & \cdots & 0 & 0 \\ \overline{\alpha}^{n-2}\beta^{n-1} & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C} \quad \text{and} \quad |\alpha| = |\beta| = 1.$$

We denote this matrix W as $W_{\alpha,\beta}$, and take $V = W_{1,1}$. Observe that any $W_{\alpha,\beta}$ can be expressed as the product of V and $U_{\alpha,\beta}$, i.e.,

$$W_{\alpha,\beta} = VU_{\alpha,\beta},$$

therefore the gauge group $\mathcal{G}(\mathrm{Toep}_n)$ is generated by $U_{\alpha,\beta}$ and V, with $|\alpha| = |\beta| = 1$. ■

Moreover, if we denote by $\omega = \alpha\overline{\beta}$, let

$$\Omega_\omega = \begin{pmatrix} 1 & \overline{\omega} & \overline{\omega}^2 & \cdots & \overline{\omega}^{n-1} \\ \omega & 1 & \overline{\omega} & \cdots & \overline{\omega}^{n-2} \\ \omega^2 & \omega & 1 & \cdots & \overline{\omega}^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{n-1} & \omega^{n-2} & \omega^{n-3} & \cdots & 1 \end{pmatrix},$$

and we denote by $\Gamma\colon T \mapsto T^{\mathrm{T}}$ the transposition action of $T \in \mathrm{Toep}_n$, we obtain that

$$U^*_{\alpha,\beta} T U_{\alpha,\beta} = \Omega_\omega \odot T, \tag{4.7}$$
$$U^*_{\alpha,\beta} V^* T V U_{\alpha,\beta} = \Omega_\omega \odot \Gamma(T), \tag{4.8}$$
$$V^* U^*_{\alpha,\beta} T U_{\alpha,\beta} V = \Omega_{\overline{\omega}} \odot \Gamma(T), \tag{4.9}$$

here $\Omega_\omega \odot T$ denotes the Schur product of Ω_ω and T, that is, the elementwise product of Ω_ω and T. Hence we obtain the following corollary.

Corollary 4.3. *The group of* $\mathrm{UCP}_{\mathrm{rank}=1}(\mathrm{Toep}_n)$ *is isomorphic to the semidirect product of* $U(1)$ *and* $\mathbb{Z}_2$*, and the gauge group* $\mathcal{G}(\mathrm{Toep}_n)$ *is different from* $\mathrm{UCP}_{\mathrm{rank}=1}(\mathrm{Toep}_n)$ *by a phase factor, that is,*

$$\mathrm{UCP}_{\mathrm{rank}=1}(\mathrm{Toep}_n) = U(1) \rtimes \mathbb{Z}_2 \tag{4.10}$$

and

$$\mathcal{G}(\mathrm{Toep}_n) = U(1) \times (U(1) \rtimes \mathbb{Z}_2)\,. \tag{4.11}$$

Moreover, We have the short exact sequence which is independent of n*:*

$$1 \longrightarrow U(1) \longrightarrow \mathcal{G}(\mathrm{Toep}_n) \longrightarrow \mathrm{UCP}_{\mathrm{rank}=1}(\mathrm{Toep}_n) \longrightarrow 1.$$

Proof. We first show that the group $\mathrm{UCP}_{\mathrm{rank}=1}(\mathrm{Toep}_n)$ is isomorphic to the semidirect product of $U(1)$ and $\mathbb{Z}_2$. In fact, according to the r.h.s.'s of equations (4.7)–(4.9), the group $\mathrm{UCP}_{\mathrm{rank}=1}(\mathrm{Toep}_n)$ is characterized by Ω_ω and the transposition action Γ. We observe that $\Gamma \circ \Omega_\omega \circ \Gamma = \Omega_{\overline{\omega}}$, and if we equip the collection of matrices $\{\Omega_\omega\}_\omega$ with Schur product, it is obvious to see that $\{\Omega_\omega\}_\omega$ is isomorphic to $U(1)$, therefore we obtain the formula (4.10).

Since ω is determined by the product of α and $\overline{\beta}$, while the matrix $U_{\alpha,\beta}$ is determined by α and β, hence compared with $\mathrm{UCP}_{\mathrm{rank}=1}(\mathrm{Toep}_n)$, the gauge group $\mathcal{G}(\mathrm{Toep}_n)$ has one more $U(1)$-factor, therefore we obtain the formula (4.11). ■

Remark 4.4. Although the transposition map is not completely positive on $M_n(\mathbb{C})$, however, it is unital completely positive on the Toeplitz system Toep_n given by $V^*(\cdot)V$, that is to say, for a general $T \in M_n(\mathbb{C})$ we do not have $V^*TV = T^{\mathrm{T}}$, while if $T \in \mathrm{Toep}_n$ this equality does hold, as is also discussed in [7].

4.2 Perturbation semigroup of the Toeplitz system

In this section, we shall characterize the semigroups $\mathrm{Pert}(\mathrm{Toep}_n)$ and $\mathrm{Pert}^+(\mathrm{Toep}_n)$. We first need to recall the definition of the vectorization of a matrix as is defined in [14].

Definition 4.5 ([14, Section 2]). Let $T \in M_{n\times m}(\mathbb{C})$, we define the vectorization $\mathrm{vec}(T)$ of T as

$$\begin{aligned}\mathrm{vec}\colon\ & M_{n\times m}(\mathbb{C}) \to \mathbb{C}^{nm},\\ & T \mapsto \sum_{j=1}^{m} e_j^{(m)} \otimes T e_j^{(m)},\end{aligned}$$

here the tensor notation is in the sense of Kronecker product, corresponding to the standard identification $\mathbb{C}^{nm} \cong \mathbb{C}^m \otimes \mathbb{C}^n$, and $e_j^{(m)}$ denotes the $j-$th basis element in $\mathbb{C}^m$, i.e., $e_j^{(m)} = (0,\dots,1,\dots,0)^{\mathrm{T}}$ with the j-th entry is equal to 1 and 0's otherwise.

For example, if $T = (t_{ij})_{1\le i,j\le 3} \in M_3(\mathbb{C})$, then

$$\mathrm{vec}\colon\ \begin{pmatrix} t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{pmatrix} \mapsto (t_{11}, t_{21}, t_{31}, t_{12}, t_{22}, t_{32}, t_{13}, t_{23}, t_{33})^{\mathrm{T}}.$$

Remark 4.6. As it is shown in [14, Section 2] we have the formula

$$\mathrm{vec}\left(A\, X\, B^{\mathrm{T}}\right) = (B \otimes A)\,\mathrm{vec}(X), \qquad A \in M_{n\times m}(\mathbb{C}), \quad B \in M_{k\times l}(\mathbb{C}), \quad X \in M_{m\times l}(\mathbb{C}).$$

We take a matrix $\Delta \in M_{n^2\times(2n-1)}(\mathbb{C})$ as

$$\Delta = \big(\,\mathrm{vec}(\tau_{-n+1}), \mathrm{vec}(\tau_{-n+2}), \dots, \mathrm{vec}(\tau_0), \mathrm{vec}(\tau_1), \mathrm{vec}(\tau_2), \dots, \mathrm{vec}(\tau_{n-1})\big).$$

Consider the semigroup homomorphism $\Phi\colon \mathrm{Pert}(\mathrm{Toep}_n) \to \mathrm{UCBH}(\mathrm{Toep}_n)$ as is defined in Section 3. We denote the image of $\omega \in \mathrm{Pert}(\mathrm{Toep}_n)$ by φ, i.e., $\varphi = \Phi(\omega) \in \mathrm{UCBH}(\mathrm{Toep}_n)$. Take $\{\tau_i\}_{-n+1\le i\le n-1}$ as the basis of Toep_n, we can identify φ with a $(2n-1)\times(2n-1)$ matrix $W = (w_{ij})_{-n+1\le i,j\le n-1}$ such that

$$\varphi(\tau_j) = \sum_{i=-n+1}^{n-1} w_{ij}\tau_i. \tag{4.12}$$

If we regard the tensor product in the definition of $\mathrm{Pert}(\mathrm{Toep}_n)$ as Kronecker product, we can then treat an element $\omega \in \mathrm{Pert}(\mathrm{Toep}_n)$ as a $n^2 \times n^2$ matrix, which we still denote as ω without confusion. In the case of Toeplitz operator system Toep_n, the C^*-algebra generated by Toep_n is $M_n(\mathbb{C})$. The opposite algebra $M_n(\mathbb{C})^\circ$ is the transposition of $M_n(\mathbb{C})$, and an element $a^\circ \in M_n(\mathbb{C})^\circ$ is just equal to a^{T}. The relationship between ω and φ is described in the following proposition.

Proposition 4.7. *Let $\omega \in \mathrm{Pert}(\mathrm{Toep}_n)$, then we have the equation*

$$\omega\Delta = \Delta\overline{W}, \tag{4.13}$$

here $W \in M_{2n-1}(\mathbb{C})$ is the square matrix associated with $\Phi(\omega) = \varphi \in \mathrm{UCBH}(\mathrm{Toep}_n)$ defined by equation (4.12), and $\overline{W}$ denotes the elementwise complex conjugation of W.

Proof. Let $\omega = \sum a_k \otimes b_k^{\mathrm{T}} \in \mathrm{Pert}(\mathrm{Toep}_n)$. We observe that for $-n+1 \le j \le n-1$, the $j-$th column of $\omega\Delta$ is equal to

$$\sum_i a_i \otimes b_i^{\mathrm{T}}(\mathrm{vec}(\tau_j)) = \mathrm{vec}\left(\sum_i b_i^{\mathrm{T}}\tau_j a_i^{\mathrm{T}}\right) = \mathrm{vec}\left(\sum_i (a_i\tau_{-j}b_i)^{\mathrm{T}}\right) = \mathrm{vec}\left(\varphi(\tau_{-j})^{\mathrm{T}}\right).$$

The equation (4.12) implies that

$$\mathrm{vec}\left(\varphi(\tau_{-j})^{\mathrm{T}}\right) = \sum_{i=-n+1}^{n-1} w_{i,-j}\,\mathrm{vec}(\tau_{-i}) = \sum_{i=-n+1}^{n-1} w_{-i,-j}\,\mathrm{vec}(\tau_i).$$

Since φ is a Hermitian map, we conclude that $w_{ij} = \overline{w_{-i,-j}}$. Indeed, we observe that

$$\varphi(\tau_j) = \varphi(\tau_{-j})^* \Rightarrow \sum w_{ij}\tau_i = \sum \overline{w_{-i,-j}}\tau_i \Rightarrow w_{ij} = \overline{w_{-i,-j}},$$

hence

$$\mathrm{vec}\left(\varphi(\tau_{-j})^{\mathrm{T}}\right) = \sum_{i=-n+1}^{n-1} \overline{w_{ij}}\,\mathrm{vec}(\tau_i), \tag{4.14}$$

notice that the l.h.s. of (4.14) is the j-th column of $\omega\Delta$, and the r.h.s. of (4.14) is the j-th column of $\Delta\overline{W}$ for $-n+1 \le j \le n-1$, therefore we obtain the equation (4.13). ■

Remark 4.8. To simplify the expression we count the rows and columns of the $(2n-1)\times(2n-1)$ matrix W from $-n+1$ to $n-1$. Since φ is a unital map, i.e., $\varphi(\tau_0) = \tau_0$, the $0-$th column of W is $(0,\dots,0,1,0,\dots,0)^{\mathrm{T}}$ with 1 in the central entry and 0's elsewhere.

Remark 4.9. It is not difficult to show that $\mathrm{rank}(\Delta) = 2n-1$ by a direct computation, hence for each $\omega \in \mathrm{Pert}(\mathrm{Toep}_n)$ there is a unique $(2n-1)\times(2n-1)$ matrix W satisfying the equation (4.13). Especially, we have that $\omega\Delta = \Delta$ if and only if $\Phi(\omega) = \mathrm{Id} \in \mathrm{UCP}(\mathrm{Toep}_n)$.

The matrix $\omega \in M_{n^2}(\mathbb{C})$ is not Hermitian in general. However, in [13] it is shown that we can transform ω to become a Hermitian matrix.

Definition 4.10 ([13, Section 1])**.** Let $T = (t_{ij})_{1\le i,j\le n^2} \in M_{n^2}(\mathbb{C})$, we may write T in the block form as $T = (T_{ij})_{1\le i,j\le n}$, where $T_{ij} = (t^{ij}_{rs})_{1\le r,s\le n} \in M_n(\mathbb{C})$. We define $\Gamma\colon M_{n^2}(\mathbb{C}) \to M_n(M_n(\mathbb{C}))$ as follows:

$$\Gamma(T)^{ij}_{rs} = t_{[i,j],[r,s]}, \qquad i,j,r,s = 1,\dots,n,$$

here $[i,j] = (i-1)n + j$.

That is to say, we rearrange each row in $T \in M_{n^2}(\mathbb{C})$ to become a new block and then reorder all blocks together. For example, for n=2,

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix}, \qquad \text{and} \qquad \Gamma(T) = \begin{pmatrix} t_{11} & t_{12} & t_{21} & t_{22} \\ t_{13} & t_{14} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{41} & t_{42} \\ t_{33} & t_{34} & t_{43} & t_{44} \end{pmatrix}.$$

Theorem 4.11 ([13, Theorems 1 and 2])**.** *Let $\mathcal{T}\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map, $\langle\mathcal{T}\rangle$ be the matrix representation of $\mathcal{T}$ with respect to the unit matrices $E_{i,j}$. The following are equivalent:*

- *$\mathcal{T}\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is completely positive (resp. Hermitian-preserving).*
- *There exist $A_1,\dots,A_s \in M_n(\mathbb{C})$ such that $\langle\mathcal{T}\rangle = \sum_{i=1}^s A_i \otimes \overline{A}_i$ (resp. $\langle\mathcal{T}\rangle = \sum_{i=1}^s \epsilon_i A_i \otimes \overline{A}_i$ for $\epsilon_1,\dots,\epsilon_s \in \{\pm 1\}$).*
- *There exist $A_1,\dots,A_s \in M_n(\mathbb{C})$ and a $s\times s$ positive semidefinite (resp. Hermitian) matrix (d_{ij}) such that $\langle\mathcal{T}\rangle = \sum_{i,j=1}^s d_{ij} A_i \otimes \overline{A}_j$.*

- $\Gamma(\langle\mathcal{T}\rangle)$ *is positive semidefinite (resp. Hermitian).*
- $\Gamma\big(\langle\mathcal{T}\rangle^{\mathrm{T}}\big)$ *is positive semidefinite (resp. Hermitian).*

In our case, we notice that if we regard ω as a matrix in $M_{n^2}(\mathbb{C})$, then ω plays the role of $\langle\mathcal{T}\rangle$ above. Hence we have the following result:

Theorem 4.12. *If* $\omega = \sum a_i \otimes b_i^\circ \in \mathrm{Pert}(\mathrm{Toep}_n)$ *(resp.* $\mathrm{Pert}^+(\mathrm{Toep}_n)$*), then we have*

- $\Gamma(\omega)$ *is a Hermitian (resp. positive semidefinite)* $n^2 \times n^2$ *matrix,*
- $\varphi = \Phi(\omega)$ *can be extended as a Hermitian-preserving (resp. completely positive) map from* $M_n(\mathbb{C})$ *to* $M_n(\mathbb{C})$*, where* Φ *is defined as* $\Phi(\omega)\colon X \mapsto \sum a_i(X)b_i$ *for* $X \in M_n(\mathbb{C})$*, and hence we obtain the following two semigroup homomorphisms:*

$$\begin{aligned}\mathrm{Pert}(\mathrm{Toep}_n) &\xrightarrow{\Phi} \mathrm{UCBH}(\mathrm{Toep}_n),\\ \mathrm{Pert}^+(\mathrm{Toep}_n) &\xrightarrow{\Phi} \mathrm{UCP}(\mathrm{Toep}_n).\end{aligned}$$

Example 4.13. We now characterize the semigroup $\mathrm{Pert}(\mathrm{Toep}_2)$ and $\mathrm{Pert}^+(\mathrm{Toep}_2)$. Since the basis of Toep_2 is $\{\tau_{-1}, \tau_0, \tau_1\}$, we take

$$\Delta = (\mathrm{vec}(\tau_{-1}), \mathrm{vec}(\tau_0), \mathrm{vec}(\tau_1)) = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}.$$

Let $\varphi \in \mathrm{UCBH}(\mathrm{Toep}_2)$, then φ is determined by a 3×3 matrix

$$W = \begin{pmatrix} a & 0 & \bar{c}\\ b & 1 & \bar{b}\\ c & 0 & \bar{a}\end{pmatrix} \in M_3(\mathbb{C})$$

given by equation (4.12), more explicitly,

$$\begin{aligned}\varphi\colon\ \mathrm{Toep}_2 &\to \mathrm{Toep}_2,\\ \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix} &\mapsto \begin{pmatrix} b & c\\ a & b\end{pmatrix},\\ \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} &\mapsto \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix},\\ \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix} &\mapsto \begin{pmatrix} \bar{b} & \bar{a}\\ \bar{c} & \bar{b}\end{pmatrix}.\end{aligned}$$

Let

$$T = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix},$$

a direct calculation shows that $\Delta = T\,I$. Let ω be an element in $\mathrm{Pert}(\mathrm{Toep}_n)$ such that $\Phi(\omega) = \varphi$, the Proposition 4.7 implies that

$$T^{-1}\,\omega\,T\,I = I\,\overline{W},$$

thus $T^{-1}\,\omega\,T$ can be expressed as

$$T^{-1}\,\omega\,T = \begin{pmatrix} \overline{a} & 0 & c & z_1 \\ \overline{b} & 1 & b & z_2 \\ \overline{c} & 0 & a & z_3 \\ 0 & 0 & 0 & z_4 \end{pmatrix}$$

for some $z_1, \dots, z_4 \in \mathbb{C}$, and therefore

$$\omega = \begin{pmatrix} 1-z_2 & \overline{b} & b & z_2 \\ -z_1 & \overline{a} & c & z_1 \\ -z_3 & \overline{c} & a & z_3 \\ -z_2-z_4+1 & \overline{b} & b & z_2+z_4 \end{pmatrix}, \quad \Gamma(\omega) = \begin{pmatrix} 1-z_2 & \overline{b} & -z_1 & \overline{a} \\ b & z_2 & c & z_1 \\ -z_3 & \overline{c} & -z_2-z_4+1 & \overline{b} \\ a & z_3 & b & z_2+z_4 \end{pmatrix}.$$

According to Theorem 4.12, $\Gamma(\omega)$ is a Hermitian matrix; thus we must have $z_2, z_4 \in \mathbb{R}$ and $z_3 = \overline{z_1}$. Hence $\omega \in \mathrm{Pert}(\mathrm{Toep}_2)$ if and only if ω and $\Gamma(\omega)$ are of the forms

$$\omega = \begin{pmatrix} 1-z_2 & \overline{b} & b & z_2 \\ -z_1 & \overline{a} & c & z_1 \\ -\overline{z}_1 & \overline{c} & a & \overline{z}_1 \\ 1-z_2-z_4 & \overline{b} & b & z_2+z_4 \end{pmatrix}, \qquad \Gamma(\omega) = \begin{pmatrix} 1-z_2 & \overline{b} & -z_1 & \overline{a} \\ b & z_2 & c & z_1 \\ -\overline{z}_1 & \overline{c} & -z_2-z_4+1 & \overline{b} \\ a & \overline{z}_1 & b & z_2+z_4 \end{pmatrix}$$

with $z_2, z_4 \in \mathbb{R}$ and $z_1 \in \mathbb{C}$. Moreover, if $\Gamma(\omega)$ is positive semidefinite then $\omega \in \mathrm{Pert}^+(\mathrm{Toep}_n)$.

We also obtain the positive definite matrix $\Gamma(\omega)$:

$$\Gamma(\omega) = \begin{pmatrix} 1-z_2 & 0 & -z_1 & 1 \\ 0 & z_2 & 0 & z_1 \\ -\overline{z_1} & 0 & 1-z_2-z_4 & 0 \\ 1 & \overline{z_1} & 0 & z_2+z_4 \end{pmatrix}.$$

In the case of Toeplitz system, since the C^*-algebra generated by Toep_n is $M_n(\mathbb{C})$, which is a nuclear C^*-algebra, and since the Haagerup tensor norm is a C^*-cross norm [5, Corollary 2.2], we conclude that $\|\omega\| = \|\omega\|_h$ for an element $\omega \in \mathrm{Pert}(\mathrm{Toep}_n)$. According to Proposition 3.10, for $\omega \in \mathrm{Pert}^+(\mathrm{Toep}_n)$ we have $\|\omega\| = 1$. We then obtain the following proposition.

Proposition 4.14. *Let $\varphi \in \mathrm{UCBH}(\mathrm{Toep}_n)$, $W \in M_{2n-1}(\mathbb{C})$ be the corresponding matrix, and $\Delta = \big(\mathrm{vec}(\tau_i)\big)_{-n+1\le i\le n-1} \in M_{n^2\times(2n-1)}(\mathbb{C})$. A necessary condition for $\varphi \in \mathrm{UCP}(\mathrm{Toep}_n)$ is that $\|\Delta\overline{W}\| \le \|\Delta\|$.*

Proof. If $\varphi \in \mathrm{UCP}(\mathrm{Toep}_n)$, i.e., the map $\mathrm{Toep}_n \xrightarrow{\varphi} \mathrm{Toep}_n$ is a UCP map, according to Arveson's extension theorem [1, 11], we can always extend φ to a UCP map $\widetilde{\varphi}$ over $M_n(\mathbb{C})$, i.e., $M_n(\mathbb{C}) \xrightarrow{\widetilde{\varphi}} M_n(\mathbb{C})$, and since any UCP map $\widetilde{\varphi}$ over $M_n(\mathbb{C})$ can be expressed as $\widetilde{\varphi}(X) = \sum V_i^* X V_i$ for finitely many $V_i \in M_n(\mathbb{C})$, we can take $\omega = \sum V_i \otimes V_i^\circ$ such that $\Phi(\omega) = \varphi$. By Proposition 4.7 we have the equality $\omega\Delta = \Delta\overline{W}$. Hence

$$\|\Delta\overline{W}\| = \|\omega\Delta\| \le \|\omega\|\,\|\Delta\|,$$

and since $\|\omega\| = 1$, we obtain that $\|\Delta\overline{W}\| \le \|\Delta\|$. ∎

A Operator systems

This appendix contains some basic definitions and results about operator systems. In our case we only consider the concrete operator systems, i.e., $\mathcal{E} \subset B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We refer the reader [2, 6, 11] for more details about operator systems.

Definition A.1. Let $\mathcal{H}$ be a Hilbert space, $B(\mathcal{H})$ be the set of all bounded operators on $\mathcal{H}$. A concrete operator system is a (closed) linear subspace $\mathcal{E}$ of $B(\mathcal{H})$. If $\mathcal{E}$ is closed under the involution, i.e., $x \in \mathcal{E}$ implies $x^* \in \mathcal{E}$, then $\mathcal{E}$ is called an operator system. In this paper, we always assume the identity element $\mathrm{Id} \in \mathcal{E} \subset B(\mathcal{H})$.

Let $\mathcal{H}^{(n)}$ be the direct sum of n copies of $\mathcal{H}$, $M_n(\mathcal{E})$ be the set of all $n \times n$ matrices with entries in $\mathcal{E}$. Since we have the C^*-isomorphism $M_n(B(\mathcal{H})) \cong B\big(\mathcal{H}^{(n)}\big)$, thus we can identify each element $(x_{ij}) \in M_n(\mathcal{E})$ as an operator in $B\big(\mathcal{H}^{(n)}\big)$, and (x_{ij}) inherits a norm $\|\cdot\|_n$ from $B\big(\mathcal{H}^{(n)}\big)$, thus $M_n(\mathcal{E})$ turns out to be a normed vector space.

Let $\mathcal{E} \subset B(\mathcal{H})$ for be an operator system, if there is a linear map $\varphi\colon \mathcal{E} \to \mathcal{E}$, then we define $\varphi_n\colon M_n(\mathcal{E}) \to M_n(\mathcal{E})$ by sending (x_{ij}) to $(\varphi(x_{ij}))$.

Definition A.2. Let $\mathcal{E}$ be an operator system, $\varphi\colon \mathcal{E} \to \mathcal{E}$ be a linear map, and φ_n be the induced map $\varphi_n\colon M_n(\mathcal{E}) \to M_n(\mathcal{E})$.

1. The map φ is called completely bounded if $\sup_{n>0} \|\varphi_n\| < \infty$, and we set
$$\|\varphi\|_{cb} = \sup_{n>0} \|\varphi_n\|.$$
2. The map φ is called $n-$positive if φ_n is positive, and φ is called completely positive if φ_n is $n-$positive for all $n > 0$.

If a completely positive map φ preserves the unit, i.e., $\varphi(\mathrm{Id}) = \mathrm{Id}$, then φ is called a UCP map(unital completely positive), and we denote the collection of all UCP maps over $\mathcal{E}$ by $\mathrm{UCP}(\mathcal{E})$.

Theorem A.3 (Arveson's extension theorem)**.** *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{E}$ an operator system contained in $\mathcal{A}$, and $\varphi\colon \mathcal{E} \to B(\mathcal{H})$ a completely positive map. Then there exists a completely positive map, $\psi\colon \mathcal{A} \to B(\mathcal{H})$, extending φ.*

According to Arveson's extension theorem we can always extend a map $\varphi \in \mathrm{UCP}(\mathcal{E})$ to a map $\psi \in \mathrm{UCP}(B(\mathcal{H}))$. In addition, if ψ is normal, according to Kraus, we can obtain a more explicit description of ψ.

Definition A.4. We say a map $\psi\colon B(\mathcal{H}) \to B(\mathcal{H})$ is normal if ψ is ultraweakly continuous. Equivalently, for any trace class operator $T \in B_1(\mathcal{H})$, take a sequence or more generally a net $\{x_i\}_{i\in I} \subset B(\mathcal{H})$ and an $x \in B(\mathcal{H})$, if $\mathrm{Tr}(T\,x_i) \to \mathrm{Tr}(T\,x)$ then we have $\mathrm{Tr}(T\,\psi(x_i)) \to \mathrm{Tr}(T\,\psi(x))$.

Theorem A.5 ([9, Theorem 3.3])**.** *Any linear mapping T of $B(\mathcal{H})$ into itself with $\|TB\| \le \|B\|$, which is completely positive and ultraweakly continuous, is of the form*
$$TB = \sum_{k\in K} A_k^* B A_k \qquad \text{with} \qquad \sum_{k\in K} A_k^* A_k \le 1.$$

Theorem A.6 ([9, Theorem 4.1])**.** *Any completely positive ultraweakly continuous linear mapping T of a von Neumann algebra $\mathfrak{U}$ into itself with $\|TB\| \le \|B\|$ is of the form*
$$TB = \sum_{k\in K} A_k^* B A_k \qquad \text{with} \qquad \sum_{k\in K} A_k^* A_k \le 1.$$

Remark A.7. In Theorems A.5 and A.6 above, the sum is in the sense of ultraweakly convergence for infinite K.

B Haagerup tensor product

In this appendix, we review some fundamental results about Haagerup tensor product of operator systems; we refer to [6, 11, 12] for more details.

Let $\mathcal{H}$ be a Hilbert space, $B(\mathcal{H})$ the set of bounded operators over $\mathcal{H}$, and let $\mathcal{E}$, $\mathcal{F} \subset B(\mathcal{H})$ be two operator systems. We denote by $\mathcal{E} \otimes \mathcal{F}$ the space of algebraic tensor product, i.e.,

$$\mathcal{E} \otimes \mathcal{F} = \left\{ \sum_{i=1}^{k} a_i \otimes b_i \mid a_i \in \mathcal{E},\, b_i \in \mathcal{F},\, k \in \mathbb{N} \right\}.$$

We define the Haagerup tensor norm $\|x\|_h$ of $x \in \mathcal{E} \otimes \mathcal{F}$ as

$$\|x\|_h := \inf \left\{ \left\| \sum a_i a_i^* \right\|^{1/2} \left\| \sum b_i^* b_i \right\|^{1/2} \right\},$$

here the infimum runs over all the expressions of $x = \sum a_i \otimes b_i$.

Definition B.1. We denote by $\mathcal{E} \otimes_h \mathcal{F}$ the completion of $\mathcal{E} \otimes \mathcal{F}$ with respect to the Haagerup tensor norm $\| \cdot \|_h$.

Theorem B.2 ([11, Theorem 17.4])**.** *Let $\mathcal{E} \subset \mathcal{E}_1$ and $\mathcal{F} \subset \mathcal{F}_1$ be operator systems. Then the inclusion of $\mathcal{E} \otimes_h \mathcal{F}$ into $\mathcal{E}_1 \otimes_h \mathcal{F}_1$ is a complete isometry.*

Theorem B.3 ([12, Theorem 5.12])**.** *Let $\mathcal{A} \subset B(\mathcal{H})$ and $\mathcal{B} \subset B(\mathcal{K})$ be C^*-algebras. We have a natural completely isometric embedding*

$$J\colon\ \mathcal{A} \otimes_h \mathcal{B} \to \mathrm{CB}(B(\mathcal{K}, \mathcal{H}))$$

defined by

$$J(a \otimes b)\colon\ T \to aTb,$$

here $\mathrm{CB}\,(B(\mathcal{K}, \mathcal{H}))$ denotes the collection of all the completely bounded maps over $B(\mathcal{K}, \mathcal{H})$.

According to [5] the Haagerup tensor norm is a C^*-cross norm:

Theorem B.4 ([5, Corollary 2.2])**.** *Suppose A and B are C^*-algebras. For any $a \in A$, $b \in B$, $\|a \otimes b\|_h = \|a\|\,\|b\|$.*

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