A Clarification about Hitting Times Densities for Ornstein–Uhlenbeck Processes

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Let $(U_t,\, t \geq 0)$ be an Ornstein–Uhlenbeck process with parameter $\lambda > 0$, starting from $a \in \mathbb{R}$, that is the solution of:

$$U_t = a + B_t - \lambda \int_0^t U_s\, ds = e^{-\lambda t} \left(a + \int_0^t e^{\lambda s}\, dB_s \right),\ t \geq 0, \tag{1}$$

where $(B_t, t \geq 0)$ denotes a Brownian motion starting from 0.
Below, we give an expression of the density of $T_b^U = \inf\{t : U_t = b\}$, for $b \in \mathbb{R}$, in terms of some integrals involving the $\mathrm{BES}^{(3)}$ bridges of lengths $t \geq 0$, starting at $(b - a)$, for $b \geq a$, and conditioned to end at 0 at time t. It is seen on this expression that, in the discussion in Leblanc *et al.* [7], one term has been omitted, which explains why formula (3) in Leblanc *et al.* [7] is incorrect, and how to correct it, at least in terms of $\mathrm{BES}^{(3)}$ bridges.
For general discussions of first hitting times of diffusions we refer to Arbib [1], Breiman [2], Horowitz [5], Kent [6], Nobile *et al.* [8], Novikov [9, 10, 11], Pitman and Yor [13, 14], Ricciardi and Sato [15], Rogers [16], Salminen [17], Shepp [18], Siegert [19], Truman and Williams [21] and Yor [23]. More general discussions of inverse local times and occupation times $\int_0^T 1_{(X_s \leq y)} ds$, when X is a diffusion and T a particular stopping time, are dealt with in Hawkes and Truman [4], Truman [20] and Truman *et al.* [22], with particular emphasis on the Ornstein–Uhlenbeck case.
Denote by $Q_a^{(\lambda)}$ the law of U, solution of (1), and by $W_a = Q_a^{(0)}$ the law of $\{(a + B_t), t \geq 0\}$, both laws being defined on the canonical space $C(\mathbb{R}_+, \mathbb{R})$, where $X_t(\omega) = \omega(t)$, and $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$.

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The absolute continuity relationship:

$$Q_a^{(\lambda)}\Big|_{\mathcal{F}_t} = \exp\left[-\frac{\lambda}{2}(X_t^2 - a^2 - t) - \frac{\lambda^2}{2}\int_0^t X_s^2 ds\right] \cdot W_a|_{\mathcal{F}_t} \tag{2}$$

is well-known (see, e.g. Yor [24], Chapter 2). It also holds with t being replaced by T, any stopping time being assumed to be finite both under $Q_a^{(\lambda)}$ and W_a. Consequently, the following holds:

$$Q_a^{(\lambda)}(T_b \in dt) = \exp\left[-\frac{\lambda}{2}(b^2 - a^2 - t)\right] W_a\left[\exp\left(-\frac{\lambda^2}{2}\int_0^t X_s^2 ds\right); T_b \in dt\right]. \tag{3}$$

In Leblanc *et al.* [7], the authors use the (obvious, but important) fact that $(X_s, s \geq 0)$ under W_a is distributed as $(b - X_s, s \geq 0)$ under $W_{(b-a)}$. Thus, (3) may be written as:

$$\begin{aligned} &Q_a^{(\lambda)}(T_b \in dt) = \\ &\quad \exp\left[-\frac{\lambda}{2}(b^2 - a^2 - t)\right] W_{b-a}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^t (b - X_s)^2 ds\right); T_0 \in dt\right]. \end{aligned} \tag{4}$$

Now, recall that for $c > 0$, under W_c, the process $(X_s, s \leq T_0)$ conditioned by $(T_0 = t)$, is a $\mathrm{BES}^{(3)}$ bridge of length t, starting at c, and ending at 0, whose law we denote by $P_c^{(3)}(\,\cdot\,|X_t = 0)$. Thus, denoting $c = |b - a|$, we obtain:

$$W_{b-a}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^t (b - X_s)^2 ds\right); T_0 \in dt\right] \tag{5}$$

$$= \begin{cases} E_c^{(3)}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^t (b - X_s)^2 ds\right)|X_t = 0\right] W_c(T_0 \in dt), & if\ \ b \geq a, \\ E_c^{(3)}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^t (b + X_s)^2 ds\right)|X_t = 0\right] W_c(T_0 \in dt), & if\ \ b \leq a\,. \end{cases} \tag{6}$$

For $b \neq 0$, the conditional expectations $E_c^{(3)}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^t (b \mp X_s)^2 ds\right)|X_t = 0\right]$ differ from that written in Leblanc *et al.* [7] at the bottom of p.110, and on top of p.111, where we find instead:

$$E_c^{(3)}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^t X_s^2 ds\right)|X_t = 0\right], \tag{7}$$

which is known to be equal to (see Pitman and Yor [12], or Yor ([24], formula (2.5) for $\delta = 3$):

$$\left(\frac{\lambda t}{\sinh(\lambda t)}\right)^{\frac{3}{2}} \exp\left[-\frac{c^2}{2t}(\lambda t \coth(\lambda t) - 1)\right]$$

Thus, to summarize, we first obtain:

$$Q_a^{(\lambda)}(T_0 \in dt) = |a| \frac{\exp(\lambda a^2/2)}{\sqrt{2\pi}} \exp\left[\frac{\lambda}{2}(t - a^2 \coth(\lambda t))\right] \left(\frac{\lambda}{\sinh(\lambda t)}\right)^{\frac{3}{2}} dt \quad (8)$$

as found in a number of papers, e.g. Yor ([25], Exercise p. 56), Elworthy et al. [3]. Secondly, the knowledge of the density of $Q_a^{(\lambda)}(T_b \in dt)/dt$, for all a and b is equivalent to the knowledge of the joint Laplace transform of $(\int_0^t X_s ds, \int_0^t X_s^2 ds)$ under the BES$^{(3)}$ bridges laws $P_c^{(3)}(\,\cdot\,|X_t = 0)$.

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