On the Nash Program for the Nash Bargaining Solution

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Abstract

The present paper provides three different supporting results for the Nash bargaining solution of n-person bargaining games. First, for any bargaining game there is defined a non-cooperative game in strategic form, whose unique Nash equilibrium induces a payoff vector that coincides with the Nash solution of the bargaining game. Next this game is modified in such a way that the unique Nash equilibrium that supports the Nash solution is even in dominant strategies. After that an n-stage game in extensive form is presented whose unique subgame perfect equilibrium supports the Nash solution of the bargaining game. Finally, the support results are shown to induce implementation results in the sense of mechanism theory.

1 Introduction

The Nash Program, as it was termed by Binmore (1987),[see also Binmore (1997)] is a reasearch agenda trying to support solutions of cooperative NTU-games by equilibria of non-cooperative games in strategic or extensive form. An example of a contribution to the Nash Program is Moulin's (1984) implementation of the Kalai-Smorodinsky solution for two-person bargaining games in subgame perfect equilibrium. Haake (1998) applies

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our present approach to derive an alternative subgame perfect equilibium implementation as well as a dominant strategy equilibrium implementation for the Kalai-Smorodinsky solution. In the present paper we are concerned with the Nash solution. The present state of the Nash Program as far as this solution is concerned is not perfectly satisfactory. The first contribution is due to Nash (1953) who related the Nash solution of two-person bargaining games to the equilibria of what he called a simple demand game. Unfortunately, the Nash solution only coincides with one of an infinity of equilibria of that game. This negative result led Nash to the consideration of a smoothed demand game. Here the original game is considered as the limit of a class of increasingly less smoothly distorted versions of that game. The only limit of all possible sequences of equilibria of the distorted games turns out to be the Nash solution. Another example for an approximate support of the Nash solution is Rubinstein's alternate bargaining game when discounting is increasingly neglected [cf. Binmore, Rubinstein and Wolinsky (1986)]. The unique subgame perfect equilibrium converges to the Nash solution in that case.

The only direct support of the Nash solution is given by Howard (1992). Here the Nash solution coincides with the unique subgame perfect equilibrium payoff vector of an extensive form game. From the puristic point of view of non-cooperative game theory this result is not perfectly satisfactory, however, as there are several subgame perfect equilibria yielding this payoff vector. So there remains a coordination problem which equilibrium should be chosen.

One might be irritated by all the effort made on getting support results for the Nash solution while there is obviously a simple approach providing such a support by a dominant strategy equilibrium of some game in strategic form. Given some two-person bargaining game V just let both players choose any non negative real number. If they jointly happen to choose the Nash solution $N(V) = (N_1(V), N_2(V))$ of the game they receive their respective coordinates as payoffs. Otherwise they get 0 (or what ever the disagreement point dictates). This game provides support in a dominant strategy equilibrium for the Nash solution. It fails, however, to be a "sensitive strategic model" as Osborne and Rubinstein (1998) require it for the Nash program. And it does not supplement the cooperative bargaining game such that, in the words of Nash (1953) "each helps to justify and clarify" the other. In fact, any arbitrary bargaining solution could be supported in the same way. Clearly, there is no formal theory telling us what kind of games are sensitive strategic models. But any game can be judged under the aspect whether it helps to justify or clarify the Nash solution. The matter of what reasonable games are becomes even more delicate when the analysis is extended to implementation proper. Then information and enforcement aspects have to be considered which are important for the design of a mechanism. This problem is extensively discussed in Trockel (1999) [see also Jackson (1992) and Jackson, Palfrey and Srivastava (1994)].

In our present paper we shall prove three different support results for the Nash solution of n-person bargaining games. One result states the support of the Nash solution by the unique Nash equilibrium of a game in strategic form. The second result provides support of the Nash solution by the dominant strategy equilibrium of a game in strategic form which is a modified version of the first game. The third result establishes the support of the Nash solution in an n-person bargaining game by the unique subgame perfect equilibrium of an n-stages game in extensive form.

Our approach to the support of the Nash solution is based on the characterization of the Nash solution of a bargaining game as the unique Walrasian equilibrium of a naturally induced bargaining economy [cf. Trockel (1996)].

The set of feasible utility allocations of the players in a bargaining game is interpreted as a production possibility set describing all technologically possible ways of producing joint utility vectors. All players have equal shares in this technology set and, hence, in any resulting profit from production. The different players' utilities are the commodities. Each player, as an agent of the economy, is only interested in "his" commodity, namely his utility. Endowments are zero for each player to guarantee that the only source of income is the profit earned from production of joint utilities. In this framework the equilibrium price system can be used to evaluate utility allocations and, thereby, to give a specific meaning to the notion of equity in a world of non-comparable cardinal utilities. Equity had, together with efficiency, been the basis for the definition of Shapley's (1969) λ-transfer value (or NTU-value). This value, on the class of bargaining games, is unique and coincides with the Nash solution. From Nash's simple demand game where each efficient utility allocation is supported by one equilibrium we see that mere efficiency is not restrictive enough to yield a unique support result for the Nash solution. As we shall show equity embodied into the payoff rule in a suitable way will enable us to single out a unique equilibrium for our game from the infinity of efficient Nash equilibria of the simple demand game. A modification then allows to make this equilibrium even one in dominant strategies.

In Nash's simple demand game each player's strategy set is the set of utility levels for him which result from feasible points in the bargaining set. If players' choices are consistent in the sense that the resulting strategy profile is an element of the feasible bargaining set then each player receives the utility level he proposed. Otherwise, all of them receive their disagreement payoff zero.

Our first game will behave exactly like Nash's demand game as long as the profile of chosen strategies is feasible. Yet, for non-feasible strategy profiles, our payoff functions will discriminate between the players. Our payoff functions react in case of inconsistent strategy choices to the "lack of adequateness" of each player's strategy choice. In case of an inconsistent strategy profile we judge, separately for each player, the players' proposed utility claims, i.e. their strategy choices, under the aspect of "Walrasian realizations" of their claims, the equilibrium values of these realizations, and the relation between an equal share of this value and the proposed utility claim.

Any player's, say player 1's, strategy choice x_1 defines the x_1-section of the original bargaining game V, which is again a bargaining game but for players 2, $\ldots$, n. The claim x_1 of player 1 together with the unique Walrasian allocation of utilities in the $(n-1)$-person bargaining economy induced by the choice x_1 defines a utility allocation for V. This can be evaluated by its (after normalization) unique supporting price system. In a thought experiment player 1 gets the n-th part of the resulting value as income, which he can use to buy his utility-commodity in a hypothetical competitive market system.

If his demand can be satisfied with the "available amount", defined by his claim x_1 he receives his demand. Otherwise he receives his claim x_1. The effect of this payoff rule, reflecting a Walrasian evaluation of utility allocations, is an "adequate" claim of each player. A very modest utility claim of player 1 results in a low price of his utility (commodity 1), thus, as he is endowed with the n-th part of the total value, in a high level of his demand for commodity one, which turns out to be in excess to x_1. So he gets only his modest claim. A very high utility claim, on the other hand, results in a high price for his utility, and thus in a low level of his resulting demand. Then he receives only this small demand. For all players together it is strategically optimal to claim "adequate" utility levels, i.e. those which coincide with the derived demands. If all players act accordingly, there is no need for the hypothetical market system, as the resulting utility allocation is feasible in V. The strategic equilibrium of our game coincides then with a no-trade Walrasian equilibrium of the bargaining economy, hence with the Nash solution of the bargaining game. In the modified second game we employ the Walrasian evaluation even for the determination of payoffs in case of feasible strategy profiles. All this is very much in the spirit of Shapley's λ-transfer principle, where the NTU value is realized within the original NTU-game without the need of actual transfers.

2 Background and Concepts

As argued in the introduction one should expect a possibility to support the Nash solution by the unique Nash equilibrium of some game in strategic form only if equity in addition to efficiency is somehow built in. To express equity in a framework of cardinal non-comparable utilities of players requires some method of evaluation of the different players' utilities. For the class of bargaining games such a method is explicitly provided by the Walrasian approach to bargaining games due to Trockel (1996). There, the Nash solution of any bargaining game is shown to coincide with the unique Walrasian equilibrium of a naturally induced economy with production and private ownership. The equilibrium price system evaluates the allocated utilities of players (interpreted as commodities) such that each player gets the same part of the total utility allocation in terms of value. We shall exploit this evaluation point of view to construct for any bargaining game some game in strategic form. The payoff functions of that game are based on an endogenous valuation of efficient payoff profiles by their supporting efficiency prices. Equity is then represented by consistent valuation of all players and equal distribution among them of the resulting value in the equilibrium of the game.

We fix some $n \in \mathbb{N},\ n \geq 2$ for the rest of this paper.

For any $k \in \mathbb{N},\ k \leq n$ we define a k-person-bargaining game V^k as a non-empty compact, strictly convex subset of $\mathbb{R}^k_+$ that is comprehensive with respect to $\mathbb{R}^k_+$
(i.e. $x \in V^k, 0 \leq x' \leq x \Longrightarrow x' \in V^k$).

The symbols $\geq$ and $\gg$ used to compare vectors mean $\geq$ and $>$, respectively, coordinatewise. The sign $>$ means: $\geq$ but not $=$. Moreover we assume the existence of some $d_k \in V^k$ with $d_k \gg 0$. This point d_k is interpreted as status quo or disagreement point.

For convenience we make some additional assumptions for our bargaining games, which are not necessary, however, for the derivation of our results.

We require that for any V^k the map

$$g : \partial V^k \to \mathbb{R}^k{}_{++} : x \mapsto g(x) \text{ with } ||g(x)||_2 = 1$$

is well defined as a continuously differentiable mapping. Here $g(x)$ denotes the normal vector at x to the efficient boundary ∂V^k of V^k.

We also choose the origin of $\mathbb{R}^k$ as the disagreement point d_k. Doing so, we exploit one of the two degrees of freedom in representing a player's preference relation by a cardinal utility which is determined only up to positive affine transformations. In fact, for $k = n$, we also use the remaining degree of freedom and assume $\max\limits_{x \,\in\, V^n} x_i = 1$ for all $i = 1, \ \ldots\ , \ n$.

As we shall look at bargaining games for $k < n$ players resulting from n-person-games by fixing $n - k$ coordinates for such games V^k we typically get $\max\limits_{x \,\in\, V^k} x_i < 1, i = 1, \ \ldots\ , \ k$.

For any $k \in \mathbb{N}, k \leq n$, we denote by G^k the set of k-person bargaining games as defined above. Let $G := \bigcup_{k=1}^{n} G^k$. For games in G^n we will sometimes drop the superskript n. Using the disjoint union we define a *bargaining solution* on G as a map

$$\varphi : G \to \bigcup_{k=1}^{n} \mathbb{R}^k : V^k \mapsto \varphi(V^k) \in V^k.$$

The *Nash bargaining solution* N on G is defined as the unique maximizer of the Nash product, i.e. by

$$N(V^k) := \underset{x \in V^k}{\arg\max} \quad x_1 \cdot \ldots \cdot x_k.$$

Any bargaining game $V^k \in G$ induces a *bargaining economy* $\mathfrak{E}_{V^k}$ as follows. $\mathfrak{E}_{V^k} := ((\succeq_i, e_i, \vartheta_i)_{i=1,\,\ldots,\,k},\ Y^k)$, where $Y^k := V^k$ is a *production possibility* set producing the players' utilities as commodities, $\vartheta_i := 1/k$ is agent i's *equal share* in Y^k, $e^i = 0 \in \mathbb{R}^k$ is agent i's initial endowment (leaving each agent with his share of the profit of production as his only income), and $\succeq_i$ is agent i's *preference relation* on $\mathbb{R}^k_+$. The agents of this economy are identified on the interpretational level with the players of the bargaining game. This fact is expressed by the assumption that agent i's preference $\succeq_i$ can be represented by the utility function $\mathrm{proj}_i : \mathbb{R}^k_+ \to \mathbb{R}$. That means, every agent is interested only in "his" commodity, i.e. his own utility in the game V^k.

The bargaining economy $\mathfrak{E}_{V^k}$ has a unique Walrasian equilibrium. The equilibrium allocation x_W^k of $\mathfrak{E}_{V^k}$ coincides with the Nash solution $N(V^k)$ of the game V^k and, therefore, with the utility allocation determined by the Shapley λ-transfer value [cf. Shapley (1969)]. The λ of the λ-transfer value is the (up to normalization) unique equilibrium price vector for the economy $\mathfrak{E}_{V^k}$ [cf. Trockel (1996)].

Consider a game $V = V^n$ or, equivalently, the economy $\mathfrak{E}^n_V$. Now take any $x_1 \in [0, 1]$ and consider the set $W^{n-1}_{x_1} := V \cap (\{x_1\} \times \mathbb{R}^{n-1})$. Projecting this set to the subspace $\{0\} \times \mathbb{R}^{n-1}$ and skipping the first coordinate, 0, yields an $(n-1)$-person game $V^{n-1}_{x_1}$ and the induced economy $\mathfrak{E}_{V^{n-1}_{x_1}}$. Let $x_W^{n-1} = x_W^{n-1}(x_1)(= N(V^{n-1}_{x_1}))$ denote the unique Walrasian allocation of $\mathfrak{E}_{V^{n-1}_{x_1}}$. We consider the point $s(x_1) := (x_1, x_W^{n-1}) \in \partial V^n$. Evaluation of $s(x_1)$ by its associated efficiency price vector $g(s(x_1)) \gg 0$ yields $w(x_1) :=< g(s(x_1)), s(x_1) >$. Here $< \cdot, \cdot >$ denotes the standard inner product on $\mathbb{R}^n$.

We can consider now agent 1's demand at the price system $g(s(x_1))$ and his income $b_1 := w(x_1)/n$. Due to the specific structure of $\succeq_1$ it is $d(x_1) = (d_1(x_1), 0)) \in \mathbb{R} \times \mathbb{R}^{n-1}$, i.e. a corner solution.

The following lemma whose proof is given in the appendix is the main tool for deriving our results. Analogous versions hold true if the bargaining game under consideration is not normalized.

Lemma: The function $d_1 : [0,1] \to \mathbb{R}$ is strictly decreasing and continuous.

For $n = 2$ the assertion of the lemma is quite obvious. The larger $x_1 \in [0,1]$, the more to the right is $s(x_1)$ on ∂V^2, the steeper is the negative slope of the tangent at $s(x_1)$ to ∂V^2, the closer to zero does this tangent intersect the x_1-axis, the closer to zero is agent 1's demand of "his" commodity. The reason is that ∂V^2 can be looked at as the graph of a concave decreasing function of x_1 defined on $[0,1]$. It is not, however, generally true for larger n that the tangent hyperplane at some point $x = (x_1, x_2, \ldots, x_n) \in \partial V$ is steeper in the direction of x_1 than that at $x' = (x'_1, x'_2, \ldots, x'_n) \in \partial V$ with $x'_1 > x_1$, even if $(x_2, \ldots, x_n) = (x'_2, \ldots, x'_n)$. The lemma says that it is true when $x = s(x_1), x' = s(x'_1)$ and $x'_1 > x_1$.

Again, it is quite intuitive that the larger x_1 is, the less is left for all the others if they share the rest in an equitable way. Notice, that this statement is not in conflict with the well known lack of monotonicity of the Nash solution.

3 Supporting the Nash solution by a unique Nash equilibrium

In this section we shall construct the game Γ^V whose unique Nash equilibrium supports the Nash solution $N(V)$ of a given $V \in G^n$. As in Nash's demand game [Nash (1953)] the strategy sets of the n players are the images of V under the projections to their respective coordinate axes. In our case, due to the chosen normalization, the strategy sets for the n players are $\Sigma_i = [0,1], i = 1, \ldots, n$.

The next step is the construction of the payoff functions $\pi_i, i = 1, \ldots, n$.

We define the payoff function of player 1 by
$\pi_1 : [0,1]^n \to [0,1] : x = (x_1, \ldots, x_n) \mapsto x_1 \chi_{V^n}(x) + \min(s_1(x_1), d_1(x_1))(1 - \chi_{V^n}(x))$
$= x_1 \chi_{V^n}(x) + \min(x_1, d_1(x_1))(1 - \chi_{V^n}(x))$. Here $\chi_V(\cdot)$ is the indicator function of V. The payoff functions $\pi_i, i = 2, \ldots, n$ are defined analogously.

In this way we associate with any bargaining game $V \in G^n$ a game $\Gamma^V := ([0,1]^n, \pi)$ in strategic form. Notice that this game coincides with Nash's simple demand game, whenever the chosen strategy profile is feasible. Only for inconsistent strategy choices our game has a more sophisticated payoff allocation than just the point of disagreement. Also our game allows it to realize as a payoff vector any feasible utility allocation of the bargaining game. This latter aspect vaguely classifies it as a "reasonable game" from the point of view of mechanism design.

Proposition 1:

For any $V \in G^n$ the game Γ^V has a unique Nash equilibrium x^*. The equilibrium payoff vector $\pi(x^*) = x^*$ coincides with the Nash solution $N(V)$ of V.

Notice, that as with the above lemma this proposition remains true for non-normalized bargaining games.

Proof:

Obviously, any non-efficient point x can be improved upon by some player's unilateral deviation towards the efficient boundary. For any efficient point $x \neq N(V)$ at least one coordinate x_i satisfies $x_i < N_i(V)$. Unilateral deviation of player i to x^n_{Wi} yields him, according to the definition of π_i, the payoff x^n_{Wi} and thus improves him. So it remains to show that, for any $x \notin V$ the fact that $x_1 \neq x^n_{W1}$ implies that $\pi(x) < \pi_1(x^n_{W1}, x_2, \ \dots \ , x_n)$. (Analogous statements are then obviously true for every $i \in N$).

As $g(x) \gg 0$ for all $x \in \partial V$and V is strictly convex we have $d_1(0) > 1/n > 0$. We also have $d_1(1) = 1/n < 1$. As d_1 is strictly decreasing and continuous by the lemma, there is a unique fixed point of d_1. From the Walrasian characterization of the Nash solution we know that the Walrasian allocation x^n_W of $\mathfrak{E}_{V^n}$ satisfies: $x^n_{W1} = s_1(x^n_{W1}) = d_1(x^n_{W1})$. Therefore x^n_{W1} is the unique fixed point of d_1. Obviously, it maximizes uniquely the payoff function π_1 when combined with arbitrary strategies of the other $n-1$ players. Analogous statements hold true for $x^n_{Wi}, i = 2, \ \dots \ , n$. As $x^n_W = N(V)$, and $\pi(x^n_W) = x^n_W$ we have shown that $x^* := x^n_W$ has the claimed properties. This proves Proposition 1.

■

4 Supporting the Nash solution by a dominant strategy equilibrium

In this section we shall modify the game Γ^V from Section 3 to achieve even support of the Nash solution in a dominant strategy equilibrium. We modify Γ^V in the following way. We define the payoffs for the players via demands derived from their claims even for the case of consistent strategy choices that describe feasible utility allocations. This means that the hypothetical market system determines the payoffs of the players now even if their claims are compatible.

So the strategy sets for the n-players are the same as for the game Γ^V, namely $\Sigma_i := [0, 1], i = 1, \ \dots \ , n$. The payoff functions $\tilde{\pi}_i, i = 1, \ \dots \ , n$ are defined as follows:

$\tilde{\pi}_1 : [0,1]^n \to [0,1] : x = (x_1, \ \ldots \ , \ x_n) \mapsto \min(s_1(x_1), d_1(x_1)),$
$\tilde{\pi}_i, i = 2, \ \ldots \ , \ n$ analogously.

Our new game is $\tilde{\Gamma}^V := (\Sigma_1, \ \ldots \ , \ \Sigma_n; \tilde{\pi}_1, \ \ldots \ , \ \tilde{\pi}_n)$. For this game we get an analogue of Proposition 1, namely:

> Proposition 2:
>
> For any $V \in G^n$ the game $\tilde{\Gamma}^V$ has a unique Nash equilibrium $\tilde{x}$. This equilibrium is even in dominant strategies. The equilibrium payoff vector $\tilde{\pi}(\tilde{x}) = \tilde{x}$ coincides with the Nash solution $N(V)$ of V.

Again, the truth of this proposition does not depend on our chosen normalization.

Proof:

The argument for $x \notin V$ in the proof of Proposition 1 can be used now for any $x \in [0,1]^n$. So $x_W^n = N(V)$ is the unique Nash equilibrium. As, independently of the other players' strategy choices each player's optimal strategy choice is his coordinate of $x_W^n = N(V)$, the equilibrium is even in dominant strategies.

It is obvious that the fact that $N(V)$ is the dominant strategy equilibrium payoff vector prevents any payoff vector where some player would get more than his coordinate of $N(V)$. Therefore our game cannot have a full range. Rather the range of utility allocations that can be realized in $\tilde{\Gamma}^V$ via strategic actions is the set $\{x \in [0,1]^n | 0 \leq x \leq N(V)\}$.

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5 Supporting the Nash solution by a unique subgame perfect equilibrium

The basic idea of embodying the Walrasian approach to bargaining into the payoff structure of the supporting game can be used also in the framework of an extensive form game. In the 2-person case the second player takes the role of the payoff function for player 1 in Proposition 2, namely that of pushing the first player to the minimum of what he claimed and the induced demand.

For two persons we define a two-stages game in extensive form as follows. One of the two players, say player 1, chooses in the first stage an $x_1 \in [0,1]$ as in the one-shot game of Section 3. Using the Walrasian economy this choice x_1 results in his demand $d_1(x_1)$. Both points, x_1 and $d_1(x_1)$, define uniquely points in the efficient boundary ∂V of V, say $x = (x_1, x_2)$ and $z = (d_1(x_1), z_2)$, respectively.

In the second stage player two has the choice between "left" and "right". If he chooses "left" the payoff vector for the two players is x, if he chooses "right" it is z. As player 2 as a rational player chooses in such a way that his second coordinate becomes as large as possible he makes player 1's payoff as small as possible. This fact forces player 1 to choose x_1 in such a way that min $(x_1, d_1(x_1))$ becomes maximal. This is the case for the first coordinate of the Nash solution of V, i.e. $N_1(V)$. Though strictly speaking player 2 has two options of choice even if player 1 chooses $N_1(V)$, the outcome $N(V)$ is independent of his choice. For player 1 the choice $N_1(V)$ is the unique equilibrium choice. Therefore, our game has what we call a quasi-unique equilibrium with a unique equilibrium outcome. Therefore there does not remain any coordination problem.

An important property of the two-players bargaining game is the fact that for any two different efficient points one is preferred by player 1 if and only if the other one is better for player 2. An analogous monotonicity property does not hold for bargaining games with more than two players. If in a three person game player 1 prefers one of two efficient points player 2 may prefer the same point. We have to take this fact into consideration when we extend our procedure to the case of general n-person bargaining games.

For an n-person bargaining game $V = V^n$ we define an n-stages game in extensive form, Ex^V, as follows.

In stage 1 player 1 chooses $\bar{x}_1 \in \Sigma_1 = [0,1]$ and, thereby, determines the $(n-1)$-dimensional bargaining game $V^{n-1}_{\bar{x}_1}$. Player 2, in the second stage, chooses $\bar{x}_2 \in \Sigma_2 = \text{proj}_2(V^{n-1}_{\bar{x}_1})$ and, thereby, determines the $(n-2)$-dimensional bargaining game $V^{n-2}_{\bar{x}_1\bar{x}_2}$. This defines the strategy set $\Sigma_3 = \text{proj}_3 V^{n-2}_{\bar{x}_1\bar{x}_2}$ for player 3, and so on. In stage $n-1$ player $n-1$ has to make the first choice in the two-person bargaining game $V^2_{\bar{x}_1\ldots\bar{x}_{n-2}}$. His choice $\bar{x}_{n-1}$ and the induced demand $d_{n-1}^{V^2_{\bar{x}_1,\ldots,\bar{x}_{n-2}}}(\bar{x}_{n-1})$ define the efficient points $\bar{x}^{n-1} := (\bar{x}_{n-1}, \bar{x}_n)$ and $\bar{z}^{n-1} := (d_{n-1}^{V^2_{\bar{x}_1,\ldots,\bar{x}_{n-2}}}(\bar{x}_{n-1}), \bar{z}_n)$ as in the two person case above. In stage n player n chooses between "left" and "right" and determines accordingly $\bar{x}^{n-1}$ or $\bar{z}^{n-1}$, respectively.

Next we have to define the payoff functions. The resulting payoff vector $y(\bar{x}_1, \ldots, \bar{x}_{n-1}, \text{left}) = y = (y_1, \ldots, y_n)$ is defined as follows (for player n's choice "right" analogously).

$y_n =: \bar{x}_n (\bar{z}_n$ in case of "right")

$$y_{n-1} := \min(\bar{x}_{n-1}, d_{n-1}^{V^2_{\bar{x}_1,\ldots,\bar{x}_{n-2}}}(\bar{x}_{n-1}))$$
$$\vdots$$
$$y_2 := \min(\bar{x}_2, d_2^{V^{n-1}_{\bar{x}_1}}(\bar{x}_2))$$
$$y_1 := \min(\bar{x}_1, d_1^{V^n}(\bar{x}_1)).$$

Backward induction describes the subgame perfect equilibria. For any $\bar{x}_{n-2}$ the two-person game between the players $n-1$ and n results in the unique subgame perfect equilibrium which is the Nash solution or, equivalently, the Walrasian allocation of the induced bargaining economy. This fact forces player $(n-2)$ to choose his Nash coordinate in the three-person game between players $(n-2), (n-1)$ and n. So player $(n-3)$ is again confronted with the Walrasian choices of his coplayers and has to choose Walrasian, too. At the end the Walrasian utility allocation, i.e. the Nash solution, is established as the unique subgame perfect equilibrium of the n-stage game Ex^V. (Strictly speaking again we get quasi-uniqueness only). We formulate this result as

Proposition 3:

The n-person n-stages-game Ex^V in extensive form derived from the n-person bargaining game V has a quasi-unique subgame perfect equilibrium with the unique equilibrium payoff vector $N(V)$.

6 From support to implementation

The idea of supporting cooperative solutions by equilibria of non-cooperative games which characterizes the Nash program is different formally and in its intention from implementation of social choice rules in equilibria by a mechanism designer. For extensive discussions of this point see Serrano (1997), Dagan and Serrano (1998) and Trockel (1999).

In the literature, unfortunately, both programs or agendas are frequently mixed up and the term "implementation" is used also in a purely game theoretic context. Some of the support results are in fact even implementation results as they are derived in models with a given outcome space. This is in particular always the case when subgame perfect support is established. By removing just the payoff vectors from the game one is left with the game tree, which in the extensive form framework is a game form (or mechanism). The extension from support results to implementation results becomes problematic, however, in a purely welfaristic context, where only the game in strategic form is given without an underlying model providing an outcome space. In such a situation are we with the Nash demand game and with our modified games in Sections 3 and 4. While Proposition 3 can immediately be interpreted as a result on implementation in subgame perfect equilibrium this is not the case with Propositions 1 and 2.

We shall apply a general procedure due to Trockel (1999) to transform the support results of Propositions 1 and 2 into implementation results. To do this we have to recall first some few concepts of mechanism theory.

Let A be some non-empty set called *outcome space*, U a set of profiles of *utility functions* $u = (u_1, \ldots, u_n)$ with $u_i : A \to \mathbb{R}, i = 1, \ldots, n$. A correspondence $f : U \Rightarrow A$ is a *social choice rule*. If $\Sigma_i, i = 1, \ldots, n$ are strategy sets for n players then a function $g : \Sigma := \Sigma_1 \times \ldots \times \Sigma_n \to A$ is called *outcome function* and the tuple $(\Sigma_1, \ldots, \Sigma_n; g)$ is called *mechanism* or *game form*. Such a game form together with a utility profile u defines a game in strategic form via composition, namely $\Gamma_{g,u} := (\Sigma_1, \ldots, \Sigma_n; u_1 \circ g, \ldots, u_n \circ g)$. For any equilibrium concept E [for instance, dominant strategy equilibrium (DSE), or Nash equilibrium] we say that the mechanism $(\Sigma_1, \ldots, \Sigma_n; g)$ *E-implements* the social choice rule f on U if and only if for all $u \in U$ we have $g(E(\Gamma_{g,u})) \subset f(u)$. (Here $E(\Gamma)$ is the set of E-equilibria of Γ).

For convenience we normalize our n-person bargaining game V^n such that $\Sigma_i = \mathrm{proj}_i(V^n) = [0,1], i = 1, \ldots, n$.

We define the outcome space A by

$$A := \{L \in ([0,1]^n)^{G^n} | \forall V \in G^n : L(V) \in V\}.$$

So A is the set of all bargaining solutions. Now every bargaining game $V \in G^n$ induces a utility profile $u^V = (u_1^V, \ldots, u_n^V)$ on A as follows:

$$\forall_i = 1, \ldots, n : u_i^V(L) := L_i(V).$$

The utility functions are evaluation functions: Rather than evaluating a game V by a solution concept L here the solution L is evaluated by the game V. Notice that the domain U is not a product and that the utility functions in a profile u^V are interdependent. The reason is that the shape of the efficient boundary of V cannot be separated in n parts each of which would describe the preference of one player. Rather V contains the aggregate inseparable information about the players' utilities. So our domain U is restricted and interdependent. The definition of the utility functions says in particular that players in V compare solutions only on the basis of what these would allocate to them in this game V. Two solutions, however differently they may behave for other V's, are considered as equally good by a player if they give him the same utility in game V.

Now we define the Nash social choice rule. For any $V \in G^n$ define V-equivalence on A by: $L \sim_V L' :\Leftrightarrow L(V) = L'(V)$.

The set $[L]_V := \{L' \in A | L' \sim_V L\}$ is the V-equivalence class generated by L.

Now, the Nash social choice rule is the correspondence

$$\mathcal{N} : G^n \Longrightarrow A : V \Longmapsto [N]_V$$

Notice that $\mathcal{N}$ is indeed a social choice rule as G^n is identified with a set of utility profiles.

It remains to define the mechanism for the implementation of $\mathcal{N}$.

The outcome function $g : \Sigma \to A : x = (x_1, \ \dots \ , \ x_n) \mapsto g(x)$ maps any strategy profile x to a solution $g(x)$. This, as a mapping from G^n to Σ will be defined pointwise on G^n. For any $V \in G^n$ we define $g(x)(V) := (\min(x_1, d_1^V(x_1)), \ \dots \ , \ \min(x_n, d_n^V(x_n)))$.

For the game $\Gamma^V := (\Sigma, u_1^V \circ g, \ \dots \ , \ u_n^V \circ g)$ we get

$$g(N(V))(V) = N(V).$$

This follows immediately when we insert

$$N(V) \text{ for } x \text{ in } g(x)(V).$$

This equation asserts that the solutions $g(N(V))$ and N are V-equivalent,
i.e. $g(N(V)) \in [N]_V = \mathcal{N}(V)$.

As by Proposition 1 we have $\{N(V)\} = DSE(V)$ we get

$$g(DSE(V)) = \{g(x)|x \in DSE(V)\} = \{g(N(V))\} \subset [N]_V = \mathcal{N}(V).$$

This establishes the implementation of $\mathcal{N}$ by $(\Sigma_1, \ \dots \ , \ \Sigma_n; g)$.

The resulting payoffs in the game Γ^V are given by

$$u_i^V \circ g(N(V)) = u_i^V(g(N(V))) = u_i^V(N) = N_i(V), \ i = 1, \ \dots \ , \ n.$$

In exactly the same way one can proceed with the game $\tilde{\Gamma}^V$. We have derived a unique Nash and a DSE-implementation of the Nash social choice rule $\mathcal{N}$ to the effect that for any $V \in G^n$ the Nash- resp. DSE- payoff vector in Γ^V resp.$\tilde{\Gamma}^V$coincides with the utility allocation of V prescribed by the Nash solution.

7 Concluding Remarks

In the present paper we provided the first support results for the Nash solution of n-person bargaining games by the unique Nash equilibrium of some game in strategic form. In fact, even support in dominant strategies was established. This extends the result in Trockel (1998) to general n-person bargaining games.

A further result of the present paper was the quasi-unique subgame-perfect equilibrium support for the Nash solution in extensive games with the number of stages being equal to the number of players. A crucial feature of this result is the fact that not only the equilibrium payoff is unique but, up to an irrelevant degree of freedom in the choice of the last player there is also a unique equilibrium strategy profile. So this approach does not create a coordination problem. Finally, we transformed the support results to implementation results by factorizing the games into mechanisms and utility profiles of a canonical outcome space, the space of solutions of the games under consideration.

The implementation results hold for a restricted interdependent domain of profiles of utility functions, which reflects the fact that in a purely welfaristic context as represented by a cooperative NTU-game the characteristic function provides only aggregate information on the players' utilities, which does not allow to elicit full information about the single players preferences. The choice of the outcome space as the set of solutions for the class of games under consideration, i.e. bargaining games in the present paper, appears to be the only way to speak about desirable outcomes under the aspect of a solution concept without having a population of players available whose accordingly chosen payoffs would represent that solution concept.

Our specific choice of the outcome space as the set of bargaining solutions and our definition of the social choice rule to be implemented as intimately related to but different from the Nash solution itself allowed us to establish implementations in Nash and in dominant strategy equilibria that are in contrast to impossibility claims in the literature [(cf. Bergin and Duggan (1996), Dagan and Serrano (1998) and Serrano (1997)]. There is, however, no contradiction between our results and their results as those are in a non-welfaristic framework with an underlying set of physical outcomes and lotteries over these. Also there the Nash solution itself is the social choice rule which is shown to fail to be Maskin-monotonic and therefore cannot be Nash implementable [(cf. Maskin (1999)]. It is the restricted domain in our approach which prevents our Proposition 2 from being in conflict with the Gibbard-Satterthwaite theorem, the famous impossibility result for dominant strategy implementation.

The most appealing aspect of our main result (Proposition 1) is the fact that it is just a modification of Nash's simple demand game that brings about the support result. This modified game is different from Nash's game only in case of inconsistent claims of the players. Here the Walrasian approach to bargaining games allows to estimate in terms of value to what extent the different players are responsible for the inconsistency and to punish them by the payoffs accordingly. Following Shapley's (1969) idea and insisting in equity in addition to efficiency resulted in that modification of Nash's simple demand game which reduced the infinity of equilibria (all efficient points) to just one (the Nash solution).

It is interesting to compare the first two Propositions. Both provide support of the Nash solution by a unique Nash equilibrium. There is a trade off, however, between dominant strategies and full domain reflected by the two results. The punishment of undue behavior by the payoff functions can be made independent of the other players' behavior. Then Walrasian behavior determines a dominant strategy for every player. This is the situation of Proposition 2.

Alternatively, the payoff functions can be structured in such a way that undue behavior is punished only if there are players whose expressed interests are in conflict with that behavior. If covetous behavior of some player is compensated by sufficient modesty of the others he will not be punished and therefore can profitably deviate from Walrasian behavior. This is the situation of Proposition 1. Still it would always be possible for the modest players by unilateral deviation to their Walrasian (or Nash solution) coordinate to improve and thereby to push the greedy player below his Walrasian payoff level. Therefore the Nash solution $N(V)$ is still the only Nash equilibrium.

In all these considerations there is nothing special about the symmetric Nash solution as opposed to the asymmetric ones as far as our results are concerned. Just replace equity, represented by equal shares in the bargaining economy, by a specific inequity represented by shares $\alpha, 1-\alpha \neq 1/2$. Then our results in the obvious modified way continue to be true for the respective asymmetric Nash solution.

Appendix

In this appendix we shall proof the Lemma, which for quick reference we state here again.

Lemma:

The function $d_1 = [0,1] \rightarrow \mathbb{R}$ is strictly decreasing and continuous.

For our proof we apply a part of the Maximum Theorem under Convexity as stated in Sundaram (1996), Theorem 9.17, pp. 237, 238.

Maximum Theorem under Strict Convexity:

> Let $\hat{X}$ and θ be subsets of $\mathbb{R}^m$ and $\mathbb{R}^l$, respectively.
>
> Let $f : \theta \times \hat{X} \rightarrow \mathbb{R}$ be a continuous concave function and $\Phi : \theta \Rightarrow \mathbb{R}^m$ a compact valued continuous correpondence with a strictly convex graph.
>
> For any $\vartheta \in \theta$ let $f^*(\vartheta) := \max\{f(\vartheta, \hat{x}) | \hat{x} \in \Phi(\vartheta)\}$ and
> $\Phi^*(\vartheta) := \{\hat{x} \in \Phi(\vartheta) | f(\vartheta, \hat{x}) = f^*(\vartheta)\} = \arg\max\{f(\vartheta, \hat{x}) | \hat{x} \in \Phi(\vartheta)\}$
>
> define the maximum-value function f^* and the maximizer-correspondence Φ^*, respectively. Then f^* is a strictly concave function.

To apply this theorem we identify the mappings and sets with those in our model.

We have then:

$m := n-1, l := 1, \theta := [0,1], \hat{X} := [0,1]^{n-1}, \vartheta = x_1, \hat{x} = (x_2, \ \ldots \ , \ x_n), f(\hat{x}) = x_2^{1/n-1} \cdot \ldots \cdot x_n^{1/n-1}$ and $\Phi := V^{n-1}$, where V^{n-1} is interpreted as a constant correspondence.

It is immediate that the assumptions of our model imply the assumptions of the above version of the Maximum Theorem. Therefore we get:

$f^*(x_1) = x_{W2}^{n-1}(x_1) \cdot \ldots \cdot x_{Wn}^{n-1}(x_1)$ and $\theta^*(x_1) = x_W^{n-1}(x_1)$.

In fact Sundaram's version of the Maximum Theorem assumes only a convex graph and assures only concavity of f^*. The concavity yields for two points $(x_1, \Phi^*(x_1))$ and $(\bar{x}_1, \Phi^*(\bar{x}_1))$ that for any $\lambda \in]0,1[$

$f^*(\lambda x_1 + (1-\lambda)\bar{x}_1) = f(\lambda(x_1, \Phi^*(x_1)) + (1-\lambda)(\bar{x}_1, \Phi^*(\bar{x}_1))) \geq \lambda f^*(x_1) + (1-\lambda) f^*(\bar{x}_1) = \lambda f(x_1, \Phi^*(x_1)) + (1-\lambda) f(\bar{x}_1, \Phi^*(\bar{x}_1))$.

As both points, $(x_1, \Phi^*(x_1))$ and $(\bar{x}_1, \Phi^*(\bar{x}_1))$, are efficient in V^n the strict convexity implies strictness of the above inequality.

Next define $h : [0,1]^{n-1} \to \mathbb{R} : \hat{x} = (x_2, \ \ldots, \ x_n) \mapsto x_2 \cdot \ldots \cdot \ x_n$.

For any $x_1 \in [0,1]$ the vector grad $h|_{x_W^{n-1}(x_1)}$ is normal to $\partial V_{x_1}^{n-1}$ at its unique Walrasian point $x_W^{n-1}(x_1)$.

If we supplement this vector with the "right" first coordinate, say $p_1(x_1)$, we get $(p_1(x_1), \text{grad } h|_{x_W^{n-1}(x_1)})$ as a normal vector to ∂V^n at $s(x_1) = (x_1, x_W^{n-1}(x_1)) = (x_1, \Phi^*(x_1))$. The last $n-1$ coordinates of $p(x_1)$ are given by $\hat{p}(x_1) := \text{grad } h|_{\Phi^*(x_1)}$
$= (x_{W3}^{n-1}(x_1) \cdot \ldots \cdot x_{Wn}^{n-1}(x_1), \cdot \ldots \cdot, x_{W2}^{n-1}(x_1) \cdot \ldots \cdot x_{W(n-1)}^{n-1}(x_1))$.

In particular, $\hat{p}(x_1)$ is at every x_1 orthogonal to the Walrasian path in ∂V^n at $s(x_1)$ i.e.

$$< p(x_1), (1, \Phi_2^{*'}(x_1), \ldots, \Phi_n^{*'}(x_1)) > \quad = 0$$

or, equivalently

$$p_1(x_1) = -\sum_{i=2}^{n} \Phi_i^{*'}(x_1) \Pi_{j \neq i} \Phi_j^*(x_1) = -(h \circ \Phi^*)'(x_1).$$

Denote the function $h \circ \Phi^*$ by H.

As by assumption $g(s(x_1) \gg \ 0$ for all x_1, we have $p_1(x_1) > 0$ and therefore, $H'(x_1) < 0$.

Next define $\bar{w}(x_1) :=< p(x_1), s(x_1) >= p_1(x_1)x_1 + (n-1)\Pi_{i=2}^n x_{Wi}^{n-1}(x_1)$
$= -H'(x_1) \cdot x_1 + (n-1)H(x_1)$.

We get $n \cdot d_1(x_1) = \frac{w(x_1)}{g_1(s(x_1))} = \frac{\bar{w}(x_1)}{p_1(x_1)} = \frac{-x_1 H'(x_1) + (n-1)H(x_1)}{-H'(x_1)} = x_1 - (n-1)\frac{H(x_1)}{H'(x_1)}$.

This yields $n \cdot d_1'(x_1) = 1 - (n-1)\left[\frac{H'(x_1)^2 - H(x_1)H''(x_1)}{(H'(x_1))^2}\right]$.

To prove our lemma it suffices to show that

$n \cdot d_1'(x_1) < 0$ or, equivalently,

$H'(x_1)^2 - (n-1)H'(x_1)^2 + (n-1)H(x_1)H''(x_1) < 0 \Leftrightarrow$
$(n-1)H(x_1)H''(x_1) < (n-2)H'(x_1)^2$.

Notice, that $H(x_1) = (f^*(x_1))^{n-1}$ and, therefore,

$H'(x_1) = (n-1)(f^*(x_1))^{n-2} \cdot f^{*'}(x_1)$ and

$H''(x_1) = (n-1)[(n-2)(f^*(x_1))^{n-3} \cdot (f^{*'}(x_1))^2 + (f^*(x_1))^{n-2} \cdot f^{*''}(x_1)]$.

It remains to show, therefore, that

$$(n-1)^2(f^*(x_1))^{n-1}[(n-2)(f^*(x_1))^{n-3} \cdot (f^{*'}(x_1))^2 + (f^*(x_1))^{n-2} \cdot f^{*''}(x_1)]$$
$$< (n-2)(n-1)^2(f^*(x_1))^{2n-4} \cdot (f^{*'}(x_1))^2$$
$$\Leftrightarrow (n-1)^2(f^*(x_1))^{2n-3} \cdot f^{*''}(x_1) < 0$$

But this follows from the strict concavity of f^*, as $(n-1)^2$ and $f^*(x_1)^{2n-3}$ are positive.

■

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