**HANS-GEORG WEIGAND, WÜRZBURG**

# UNDERSTANDING THE FUNCTION CONCEPT WHILE USING A COMPUTER ALGEBRA SYSTEM AN EXPLORATORY STUDY

**Abstract.**

This study is an empirical investigation of 11th graders at a German high school (Gymnasium). Working over a 24-hour period in a computer lab, we investigated students' use of quadratic functions with 'Derive', and trigonometric functions with 'Mathplus'[1] (or 'Theorist' for Macintosh). We were particularly interested in the working styles of students while they solved problems and looked for changes in these styles, as compared to traditional paper and pencil activities. While students worked on the computer, their activities (such as inputs from the keyboard, menu choices or mouse movements) were saved by a special program, which ran in the 'background'. We are interested in the possibilities of developing a research method based on these 'computer protocols'. The study should be seen as an exploratory study for developing hypotheses for further empirical investigations.

## 1. Computers and functions

The possibilities of concept formation in a computer supported environment are very often discussed in connection with the function concept. On the one hand, *new content* was proposed:

– Starting a computer-supported approach to the function-concept with real-life-models: E. g. the CIA-(Computer-Intensive Algebra)-Project of Heid (1996) or the ACT-(Applications, Concepts and Technology)-Project of Mayes et al. (1996).

– Looking for a better basis for the concept development while working with different representations.: E. g. the 'Functional Approach to Algebra' of Kieran a. o. (1996), Demana a. Waits (1997[4]), the Austrian or the French 'Derive-Project' (Heugl a. o.1996, Hirlimann 1996).

– Getting a new approach to functions with two variables: E. g. Weigand a. Flachsmeyer (1998) or Neveling (1996).

On the other hand, classroom experiences in computer-supported environments claim for *new teaching methods*:

– There is a possibility to switch between numerical, graphical and symbolical representations while only pressing a button: Heugl a. o. (1996) speak about the 'Window-Shuttle-Principle' (p. 196ff).

– Working with modules gains importance, because you may see functions as objects or modules on the computer screen: Terms can be substituted (e. g. $T(x) \rightarrow T(x+c)$) or changed, graphs can be transformed, reflected, dilated and functions can be added, multiplied or iterated (See Borba a. Confrey 1996).

– Working experimentally and doing conjectures about solutions through systematic search processes gains prominence (See Heugl a. o. 1996).

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[1] The program was later called "Mathview" and is now sold as "LiveMath". See: http://www2.schroedel.de/detailseiten/gymnasium/mathe/livemath/

- We gain the opportunity to pose more problems with open-ended approaches (See Mayes u. a. 1996).

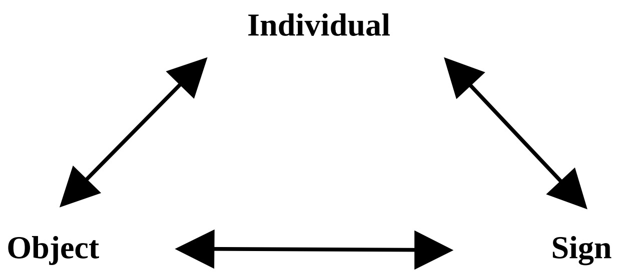
Of course, these proposals and expectations are not new. They have been discussed for many years in mathematics education. But, what is really new is, that nowadays we have a technological tool, which provides hope, that we will be able to fulfill these demands in a better way than in the past.
Looking at classroom activities, there are two important questions: First, do students really use the new possibilities they are afforded by the new technologies, and second, do they get a better understanding of the function concept?
Referring to the expectations concerning the methods of teaching, you may especially ask, whether students are able to do the transfer between different representations, whether they are able to work with functions as modules, whether they are overwhelmed by the variety of possibilities while working experimentally, and whether they do come beyond the 'trial and error'-mode and perceive the necessity for theoretical considerations,
Despite the actuality of these questions and despite the large number of proposals for classroom-activities for working with computers, there are only a few empirical investigations in this area. As we speak about computers, in the following we think of two words Systems (CAS).

**2. Empirical investigations for the understanding of the function concept**

In a report to the British Schools curriculum and Assesment Authority (SCAA) (1997) Ruthven evaluated the worldwide use of CAS. He found only a few studies (Heid 1988, Palmiter 1991, Mayes 1994, Repo 1994, Smith 1994) that were of a scientific research design (in his sense, it is an investigation with experimental- and control-group and pre- and posttests). The most important results of these investigations are, that the experimental classes had a better problem solving-ability and a deeper "relational understanding" (Skemp[2]) of concepts. He attributed this outcome to less training of routine skills, but noted that experimental classes scored nearly as well as the control classes at the final exam of routine skills. Some other investigations show a similar improvement of the "relational understanding", e. g. O'Callaghan (1998), Alexander (1993) or Müller-Philipp (1994).
But these investigations did not consider questions concerning problems and difficulties students encounter while they are working with the computer or how their working style changes in comparison with the traditional paper and pencil work. Investigations which will give answers to these questions do not have to be focused only on *the result* or *the product* of thinking or acting, but should reveal what happens in the minds of the students while they solve problems. These investigations should include description and analysis in *the process* of thinking and acting.
A first step to the description of processes like these while working with a CAS are the investigations of Heugl u. a. (1996, 209f, they describe difficulties in the structuring of solutions), or Hunter a. o. (1993) or Hillel a. o. (1992), whose findings suggest difficulties while reading graphical and numerical representations. But these investigations can be seen only as a beginning of a systematic evaluation of working methods of students, because the results are only

[2] This means, that the students had "a broader array of appropriate associations" (Heid 1988, S. 15) when explaining concepts.

documented by spontaneous observations of teachers. More detailed technical and conceptual difficulties of students while working with a CAS as suggested by the video-taped lessons and interviews of Warmuth (1995)[3] , vom Hofe (1998)[4] and Krummheuer (1993)[5].
Some years ago Pea (1985) and Dörfler (1991) predicted that the computer could be viewed as an amplifier of our mental abilities. Their basic hypothesis was, that mental objects and their real representations are unseparably connected. This also reflects the hypothesis of the semiotic thinking proposed by Ch. S. Pierce. He expressed the relation between the mental object, the representation (sign) and the individual in the diagram of the semiotics triangle.




Moreover Pierce's pragmatic basis suggests that the meaning of concepts and signs can only be clarified if you see them in relation to possible actions. The computer is a tool, which allows a person to work with representations on the screen in a new way: The computer is a tool with special mathematical notations and special menu- or mouse-driven commands. Indeed one may ask the question about the meaning of the computer in the frame of the semiotic triangle: Will the computer be a help or an obstacle for the development of new concepts?

The following study is about the working styles of students while engaged with a CAS in comparison to the traditional working style with pencil and paper. The study should be seen as an explorative study for getting hypotheses for further empirical investigations.

## 3. Starting questions of the investigation

### 3.1 Structural understanding of the function concept

Understanding the function concept means knowing what functions are used for in real life situations, knowing about properties and representations of functions and seeing their relationships to other concepts (Vollrath 1984, S. 215f). Furthermore, it is necessary to have a *structural understanding* of the function concept. This means looking for functions as objects and operating with functions as a whole, similar to work with procedures or modules in programming languages. The computer can be seen as a tool, which enables one to operate with

[3] The students worked with the CAS 'Theorist', which has 'Drag-and-Drop'-commands and an interactive connection of term and graph. One of his results is: "The technical difficulties are not as small as expected" (S. 154).

[4] He shows 'mental obstacles' in the development of the concept of limits and how the students tried to overcome them with computer visualizations.

[5] He shows the importance of the social student-student- and student-computer-interaction for the development of problem solving strategies..

the concept - or better: with representations of the concept - on the computer screen. This leads us to the first question:

**1. question: How does the computer change students’ working style with the function concept compared with traditional working with pencil and paper?**

### 3.2 The term-graph-interaction and search strategies

It is possible to distinguish among menu-driven CAS like ‘Derive’, symbol-oriented systems like ‘Mathplus’ and command-driven systems like ‘Mathematica’ or ‘Maple’. Especially for the development of concepts, it is important to know advantages and disadvantages of different systems. A very important feature of the program 'Mathplus' - which is not implemented in 'Derive' - is the interactivity between term and graph. If the term is changed, the graph changes automatically, too and this might be an advantage for experimental working, for the development of search strategies. This leads to the second question:.

**2. question: What is the meaning of the term-graph-interactivity for the development of search strategies while working with functions?**

### 3.3 Student-computer-mathematics-interaction

Over the last few years there has been a growing interest in looking at student-student-interactions and the communication and the meaning of the spoken and written language in classroom activities. The “Standards” promoted by the NCTM (1989) call for ‘mathematics as communication’ in view of the new technologies.

> „Society’s increasing use of technology requires that students learn both to communicate with computers and to make use of their own individual power as a medium of communication.“ (NCTM, p. 78)

But communication has to be based on mathematics and presumes mathematical knowledge. The third question is about the meaning of mathematics in the frame of the student-computer-interaction.

**3. question: What is the meaning of students' basis-knowledge in mathematics in the frame of the student-computer-interaction?**

### 3.4 Computer protocols

This study shows the possibilities of a research method, which is based on 'computer protocols'. While working on the computer students' activities like inputs per keyboard, menu or mouse were saved by the program 'screencam', which was running in the background of the CAS. The screencam-files can be viewed like a film, they give a real-time-description of students' activities on the computer and these activities can be evaluated. The protocols give answers to different questions: How has the student solved the problems? What time has he/she needed? What and how many representations has he/she used? How often has he/she changed between different representations? Moreover it is especially possible to look for obstacles and misleading strategies of students while they engaged in solving the problems. This leads to the fourth question:

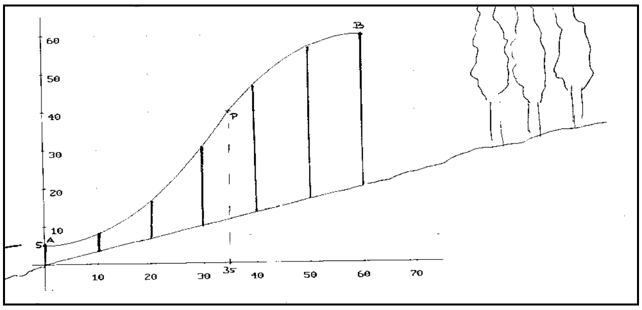
**4. question: What are the possibilities and the boundaries of the production and the evaluation of computer protocols?**

## 4. Empirical methods

We[6] have taught our lessons with 17 11th-grade students. The lessons took place in a computer lab, but we left it to the students to decide whether or not to use the computer for problem solving. During the first part of our lessons (8 hours) we investigated **quadratic functions with Derive**, the second part (also 8 hours) we introduced **trigonometric functions with ‘Mathplus’**. We used multiple methods for getting empirical data: written notes, videotapes, worksheets, computer-protocols and questionnaires. Weigand a. Weller (1996, 1997) give a more detailed description of the actual activities. The following problems are highlights from the lessons.

## 5. Results
### 5.1 Working with functions

During the first part of our lessons (quadratic functions) we gave our students the following problem: A ski-jump consists of two parabolic arcs, combined in point P. What are the equations of the two parabolas and how long are the pillars?




We noticed different working styles of the students:

- 4 students solved the problem (in two groups two by two) only by working with 'paper and pencil' and the hand-held calculator.
- 2 students worked (each on his own) only with the computer.
- A third group (11 students) started with 'paper and pencil' and switched - after 9, 11, 12, 15, 16 minutes - to the computer.
- Some of the third group worked only with the computer.
- Some of the third group sometimes switched back to 'paper and pencil' to solve special problems.

The solution of this problem can be divided into three steps:

**First step: Finding the equations of the parabolas**

If you have found the parameters of the two equations $y = ax^2 + c$ and $y = a\cdot(x-b)^2 + c$ or $y = ax^2 + bx + c$ (by help of the computer or 'only' by hand), the computer graphics serve as a control instrument for the matching of the two parabolas at the intersection point. If a student saw that the arcs didn't match, we noticed two different reactions: Some students had controlled their solution again and had tried to find their fault, others had changed their problem solving strategy and had tried to find the solution by a search process: They hat changed e. g. the parameter a of $f(x) = ax^2 + 5$ as long as the graph matched the point (35, 40). You will find a computer protocol of one of these search processes at the appendix.

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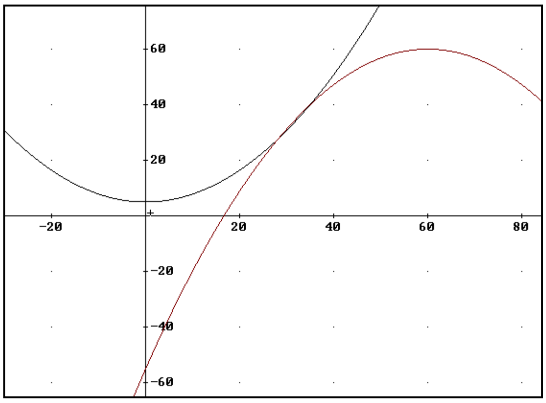
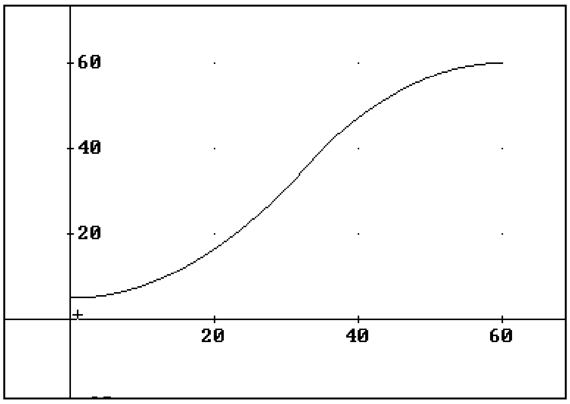
[6] The lessons were taught together with Dr. Hubert Weller (Wetzlar).

**Second step: The graphing of the parabolas**




The difference between working with computer and working with paper and pencils is, that a computer draws the two arcs of the parabolas over the whole window (left picture), while the traditional worker only draws the two necessary 'ski-jump-arcs' within the intervals [0;35] and [35;60]. In 'Derive' you have to use the IF-command to get the above picture on the right side. Moreover is it easier, e. g. for graphing and getting the length of the ‘pillars’, to define the stepwise defined function as ONE function (f1 and f2 are the two parabolas):

$$\text{SKIJUMP}(x)=\begin{cases} f1(x) & \text{for } 0 \le x \le 35 \\ f2(x) & \text{for } 35 < x \le 60 \end{cases}$$

With the computer it is now possible - in the sense of the modular principle -, to work with the function SKIJUMP as one object

**Third step: The drawing of the 'hill' and the 'pillars'**

To solve this problem with Derive, it is necessary to work with a number of different functions. If you draw the ‘ski-jump’, the ‘hill’ and the ‘pillars’ on the screen, you first have to declare functions. Especially it was very strange to some students to see the 'hill' as a function, they first tried to draw the 'hill' - like they would have done it with paper and pencil - by connecting two points. All in all the meaning of functional thinking increased. The drawing of all pillars with one command was a challenging problem.

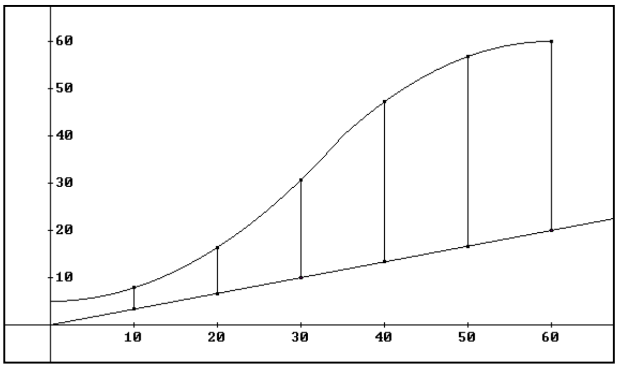
$$F1(x) := \frac{1}{35}\cdot x^2 + 5$$

$$F2(x) := -\frac{4}{125}\cdot(x-60)^2 + 60$$

$$\text{SKIJUMP}(x) := \text{IF}(x \ge 0 \wedge x \le 60,\ \text{IF}(x < 35,\ F1(x),\ F2(x)))$$

$$\text{HILL}(x) := \text{IF}\left(x \ge 0,\ \frac{1}{3}\cdot x\right)$$

$$\text{PILLAR}(x) := \text{VECTOR}\left(\begin{bmatrix} x & \text{SKIJUMP}(x) \\ x & \text{HILL}(x) \end{bmatrix},\ x,\ 10,\ 61,\ 10\right)$$




**Summary of section 5.1**

There are some differences between the computer-supported solution and the traditional paper and pencil solution of this problem.

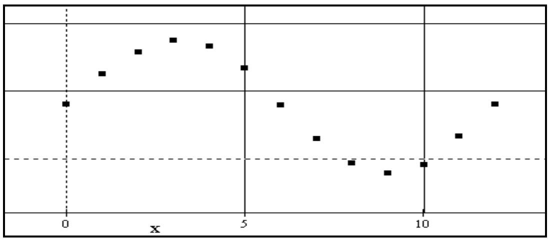
- With the computer, working with functions dominates the actions because you first have to declare a function if you draw the ‘ski-jump’, the ‘hill’ and the ‘pillars’ on the screen.
- To avoid confusion it is getting more important to choose meaningful names of functions or terms like 'skijump(x)', 'hill(x)', pillar(x), instead of f1(x), f2(x), g(x), ....This is important, because the computer screen shows only a part of the already written down solution.
- If you take ONE name for a piecewise defined function, it will be possible to operate with functions (this means with names of functions) like objects or modules on the computer screen
- Producing computer graphics can be a challenging and motivating problem[7]

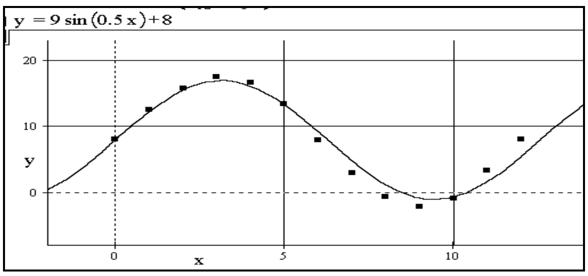
### 5.2 Term-graph-relation and search processes

During the second part of our lessons (trigonometric functions) one of our real-life-situation was the following:
Example: The air-temperature changes daily. If we determine the average temperature per month, we get the following values for Munich (Schmidt 1984, S. 74):

| month | April | May | June | July | Aug. | Sept. | Oct. | Nov. | Dec. | Jan. | Feb. | March | April |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| temp. | 8,0 | 12,5 | 15.8 | 17.5 | 16.6 | 13.4 | 7.9 | 3.0 | -0.7 | -2.1 | -0.9 | 3.3 | 8.0 |

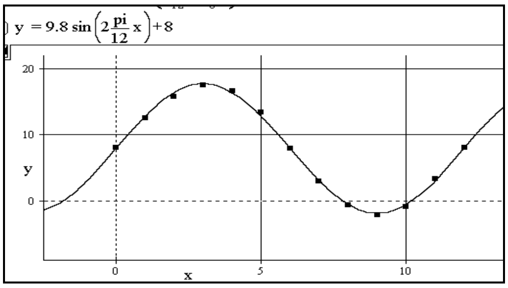
The air-temperature is time-dependent and is approximately described by the sine-function y = a sin(bt) + c. Determine a, b and c.




The left picture shows the graphical representation of the sequence, while the right one shows an approximated real function. A better approximated real function can be achieved in different ways. First you can vary the parameters a, b and c of y = a sin(bx) + c. This search process requires a good knowledge about the meaning of this parameter, because a random choice is unlikely to provide the solution. Most of the students had great difficulties utilizing this problem solving strategy. Many students were overly challenged by this problem. Another strategy to get the values of the parameters from the given table involves theoretical considerations. For example, the amplitude of the sine-function can be determined from the minimum and the maximum of the given values, the parameter b is determined with the 2π-period of the sine-function. This process gives the following graph.

[7] This is also pointed out by e. g. Aldon (1996), Heid (1996) or Goldenberg (1988).




Most of the students 'only' used search strategies without theoretical considerations. Some of the results found by the students:

*y = 9.87 sin(0.522 x) + 7.88*

$y = 10 \sin(\frac{2\pi}{12}\ x) + 7.6$

*y = 10.2 sin(0.523 x) + 7.6*

*y = 10 sin(0.5 x) + 7.8*

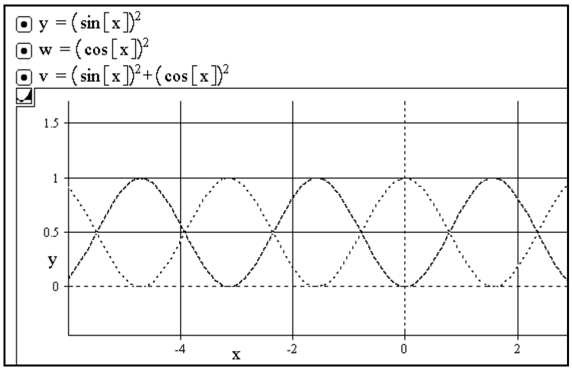
Now it can be discussed, whether this graph is the ‘optimal’ graph (and what ‘optimal’ means), especially because you get a ‘visually’ better fitting graph if you vary the parameters again.

**5.3 The student - computer - interaction**

We noticed again and again, that the concentration on the technical computer handling and the high speed of the computer processes lead to thoughtless actions, thoughtless button-pressing activities, which camouflages even simple mathematical consideration and reflective mathematical thinking. Without a basic mathematical knowledge and without the ability to apply this knowledge while regarding representations on the computer screen, working with a computer may lead to blind actions. Here are two examples:

**1. Example**: The students drew the graphs of ‘$y = \sin^2(x)$’ and ‘$w = \cos^2(x)$’. Then they had to plot the graph of the sum of the two functions.

We often heard comments such as: „It isn’t drawing anything.“ Even if the students knew the formula ‘$\sin^2(x) + \cos^2(x) = 1$’, they didn’t see the relationship to the sine-**function** and cosine-**function**. The students weren’t able to interpret this equation in a functional sense.




**2. Example:** The graphics window in “Derive” is - in the standard modus - a fixed window ($-4 \leq x, y \leq 4$). In 'Mathplus' the section is chosen in convenience to the represented function. Both methods cause difficulties. In 'Derive' the graph of the function with $y = x^2 + 5$ is not drawn in the standard window, in 'Mathplus' y = sin(x) will be drawn in the window with $-1 \leq y \leq 1$, but an additional graph e. g. z = sin(x) + 3 isn't drawn then.

Students' reactions to surprising results are quite different including:

a) After a short period of reflection, the teacher is called.

b) There is a thoughtless movement to the next problem.
c) A series of technical actions starts which lead to a repeated pressing of buttons or a thoughtless input of numerical data.
d) There is a break of reflection without computer inputs.
e) The work is continued by working with pencil and paper.

We noticed that the last two activities were used only by 'good' students (in the sense of good marks in the past). The reason for the activities in a) - c) is often not only the lack of mathematical knowledge, but a lack of concentration on the computer handling, which camouflages their actions with regard to the mathematical contents.

### 5.4 Computer protocols

The evaluation of the computer protocols show possibilities and difficulties associated with this research method. Since this is not the place to discuss the problem in detail, we only list a few aims and problems.

**Aims** of the computer protocols.

- The computer activities of the students can be evaluated in a quantitative way: according to how many inputs they have made while solving a problem, and which and how many representations they used. It is possible to do statistical evaluations.
- The computer protocol gives a real-time description of the computer activities (what appendix).
- The problem solving strategies and the misleading strategies can be classified.
- It is possible to compare problem solving strategies of *one* student while working on *different* problems or of *different* students while working *on the same* problem.

Compared with other methods of empirical investigations like video-tapes, interviews or written tests, computer protocols have some **special characteristics**.

- Compared with interviews or written tests, computer protocols are produced during the problem solving situation, they are at the origin of the process of understanding.
- Compared with video tapes and interviews, with computer protocols it is possible to view a bigger group of students simultaneously.
- The received data can be saved in a special file and can be statistically evaluated (This isn't possible with program 'Screencam', you need a special additional program).
- If there is a microphone in the computer, 'screencam' can save the conversation of the students while solving problems.

But there are also some **problems** with the method of computer protocols.

- We had no microphone inside the computer and we weren't able to save the conversation of the students.
- For the evaluation of the computer protocols it is necessary to do a transcription. You have to develop criteria for these transcriptions. This is easily accomplished if you notice the pressed buttons, but it is difficult if the student chose menu-commands like 'zooming'

or if he/she used mouse-clicks. Moreover the production of these transcriptions is very time-consuming.

- It is a problem to gather information specific to the activities of the students, if there are no computer inputs at all. The student may have worked with pencil and paper, they may have talked with neighbours or may have been doing something unrelated to the mathematical problem. For this reason, it is necessary to video-tape the session, too.

## 6. Final remarks

This study showed some changes in the working style while working with a CAS. Our results suggest hypothesis for further investigations concerning the relation between working style and concept understanding.

1. If one uses a CAS, the structural understanding of the function concept is getting more important (compared with pencil-and-paper working): Using meaningful names of variables and functions, seeing piecewise defined functions as one object, knowing about the consequences of symbolic term manipulations in graphical and numerical representations.
2. To develop successful problem solving strategies while working experimentally it is necessary to have a comprehensive basic knowledge. This is expressed in having the ability to recognize "prototypes" (Dörfler 1991) of functions and families of functions in different representations and to be able to do a transfer from one representation to another.
3. With respect to the development of mathematical thinking it is important to bring students from a level of an experimental heuristic working to theoretical reflections about the problem and their activities.
4. Computer protocols are a useful research method to show working styles of students while solving problems and to categorize problem solving strategies.
5. Due to the speed of the produced computer representations it is necessary to 'slow down' the speed of the learning and working process. The learner has to have time for reading and interpreting the viewed representations. Otherwise blind and thoughtless button-pressing activities increase and prevent mathematical understanding. The question of how to integrate 'epistemological obstacles' (Hefendehl-Hebeker 1989 and Sierpinska 1992) into the learning process had to be posed in a new way if one works with a CAS.

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Prof. Dr. H.-G. Weigand
Lehrstuhl für Didaktik der Mathematik
Universität Würzburg
Am Hubland
97074 Würzburg

Appendix: The diagram shows a transcription of a computer protocol. The dark cells show, which windows the student viewed at the given time.

| No. | Time in Min. a. Sec. | Algebra window | Graphics window |
|---|---|---|---|
| 1 | 0.00 | $F(x) := 0.5x^2+5$ | |
| 2 | 1.45 | | |
| 3 | 4.00 | $F(x):=-0.25x^2 + 5$ | |
| 4 | 4.35 | | |
| 5 | 5.10 | $F(x) := 0.01x^2+5$ | |
| 6 | 5.35 | | |
| 7 | 5.50 | $F(x) := 0.08x^2+5$ | |
| 8 | 6.10 | | |
| 9 | 6.50 | | |
| 10 | 8.00 | | |
| 11 | 8.10 | $F(x) :=35$ | |
| 12 | 8.40 | | see graphics |
| 13 | 9.30 | Deleting all expressions except expression 7 | |
| 14 | 9.45 | | Deleting all graphs except graph to the expression 7 |
| 15 | 11.00 | $F(x):=0.06x^2+5$ | |
| 16 | 11.45 | | |
| 17 | 12.10 | $F(x):=0.03x^2 + 5$ | |
| 18 | 12.30 | | Zooming |
| 19 | 14.00 | $F(x):=0.0288x^2 + 5$ | |
| 20 | 14.20 | | |
| 21 | 15.50 | $F(x) :=-0.5(x-60)^2 + 60$ | |
| 22 | 16.50 | | see graphics |
| 23 | 18.50 | $F(x):=-0.03(x-60)^2+60$ | |
| 24 | 19.15 | | |
| 25 | 20.15 | $F(x):=0.032(x-60)^2+60$ | |
| 26 | 20.45 | | |
| 27 | ..... | ... | |