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STUDENTS' IDEAS ON THE DENSITY OF FRACTIONS

Abstract:
Fractions (on the number line) are dense; in other words, there is an infinite number of further fractions between any two different fractions. This empirical study examines how familiar students are with this aspect of the fraction concept. Subjects were 411 students attending 7th-grade advanced mathematics courses at 8 German comprehensive schools. In spring and summer 1996, they completed a written test and personal interviews on this and other issues. Results showed that only a few students realised that an infinite number of further fractions exist between two different fractions.

1. Introduction

Twenty-eight of these students had to tackle these items a second time during personal interviews approximately 2 months after the written test. These interviews also asked how many fractions there are between two specific different fractions.
Items 45 and 46 require two deeper insights into the concept of fractions: first, the realisation that there is always an infinite number of further fractions between two different fractions; and, second, the knowledge that common fractions and decimal fractions are two different ways of expressing numbers and not two different *kinds* of number.

2. RESULTS OF THE WRITTEN TEST

Results of Item 45 were as follows:
Correct: 41 students (10%)
No answer: 77 (19%)
Wrong: 293 (71%).
Only 41 students managed to produce a correct answer. There were two correct solutions with a frequency of more than one percent: twenty students (5%) answered "0.5" and seven students (2%) gave "0.4". Almost every fifth student did not answer the task, indicating that many of them found it too difficult. There was a total of 21 different wrong answers. Four of these had a frequency of more than one percent:

There isn't one:	172 (42%)
A common fraction was reported:	93 (23%)
1.5:	5 (1%)
0.15:	5 (1%)

Many students believed that there could not be any decimal fraction between 1/3 and 2/3 because the number 2 follows directly on from the number 1 (see, also, findings from the interviews below). Two further potential causes could be that

- the task is too difficult for many students, so they simply guess,
- they consider decimal fractions and common fractions to be two different sorts of number that are unrelated.

There are three possible reasons for giving a common fraction as the solution;

- students do not pay enough attention to the task,
- they misunderstand the technical term "decimal fraction" to mean a common fraction,
- they view decimal fractions and common fractions as being two different kinds of number and present a common fraction as the solution.

Five students answered "1.5", and another five offered "0.15". These students concentrated on the numerators of the specified common fractions. Because $1 < 1.5 < 2$, they wrote down "1.5" as their solution. The false solution "0.15" could also be largely due to the misconception that "fractions are always smaller than 1".

The students solved Item 46 as follows:

Correct: 74 students (18%)
No answer: 118 (29%)
Wrong: 219 (53%).

Almost 30% of the students did not answer this task. This also indicates that it was too difficult for many of them. Five correct and three wrong answers predominate. First, the five correct answers and their frequencies:

4/10:	21 (5%)
1/2:	19 (5%)
5/10:	11 (3%)
2/4:	6 (1%)
3/6:	6 (1%)

It seems that many students know that a common fraction stands for one half when the numerator is exactly half the size of the denominator.

Students gave a total of 31 different wrong solutions to this task. Three of these occurred with a frequency of at least one percent:

There isn't one:	68 (17%)
A decimal fraction was reported:	101 (25%)
1/4:	6 (1%)

Many students believed that no common fraction could exist between 0.3 and 0.6. Possible reasons for this could be that

- the task is too difficult for many students, so they simply guess,
- the students consider decimal fractions and common fractions to be two different sorts of number that are unrelated.

There are at least three possible reasons for giving a decimal fraction as the solution;

- students do not pay enough attention to the task,
- they misunderstand the technical term "common fraction" to mean a decimal fraction,
- they view decimal fractions and common fractions as being two different kinds of number and present a common fraction as the solution.

Almost 1% of the students reported the wrong solution of 1/4. One possible reason for this could be the misconception that 1/ 4 = 0.4 (see, also, the personal interviews).

3. FINDINGS FROM THE PERSONAL INTERVIEWS

During the interviews, 28 students were given Items 45 and 46 for a second time. Results are summarised in the following table:

	None	**Exactly one**	**Exactly two**	**A few[a]**	**Many**	**Infinitely many**	**Task too difficult**
Number for Item 45	**8**	**2**	**2**	**4**	**1**	**4**	**7**
Percentage	**29**	**7**	**7**	**14**	**4**	**14**	**25**
Number for Item 46	**6**	**0**	**7**	**5**	**2**	**2**	**6**
Percentage	**21**	**0**	**25**	**18**	**7**	**7**	**21**

[a]a few = 3-10.

It can be seen that only a few students solved these tasks correctly, and only two of them got both tasks right. Quite a lot of students believed that there could be no decimal fraction between 1/3 and 2/3. This could be due to three misconceptions among students:

- 2/3 follows on directly from 1/3 because 2 follows on directly from 1;
- common fractions and decimal fractions are different sorts of number;

- students are unable to convert a common fraction into a decimal fraction and vice versa.

The following interview transcript shows that Student A 242 believed that 2/3 was the direct successor to 1/3 because the number 2 follows on directly from the number 1.

I (Interviewer):	(presents the task in writing and reads it out loud) First of all, here's the question again: Is there one at all there, or several, or an infinite number?
S (Student):	(spontaneously) No, there isn't one!
I:	There isn't one? Why isn't there one then?
S:	Well, because there isn't any number between 1 and 2.

Many students also believed that there could not be a common fraction between 0.3 and 0.6. Possible reasons for this could be that

- they consider that common fractions and decimal fractions are different sorts of number,
- they are unable to convert a common fraction into a decimal fraction and vice versa,
- 0.3 = 0/3 = 0 and 0.6 = 0/6 = 0, and there is no number between zero and zero.

Here is an example from the interviews (A 288) illustrating the last-mentioned reason.

I:	(presents the task in writing and reads it out loud) First of all, here's the question: Is there one there at all?
S	(spontaneously) No, there isn't one!
I:	Why isn't there one then?
S:	Well, you have to [work out] zero point – well you have to have the fraction zero, then the fraction line, and, underneath that, the number.
I:	That would be zero thirds.
S:	Yes!
I:	And zero point six is zero sixths?
S:	(nods) Yes . . . . and in between [between zero and zero] there isn't one.

Only two students believed that there was exactly one decimal fraction between 1/3 and 2/3. One student offered 0.5 as a solution because 0.5 lies exactly in the middle between 1/3and 2/3. None of the interviewed students believed that there was exactly one common fraction between the decimal fractions 0.3 and 0.6.

In contrast, seven students believe that there are exactly two common fractions between 0.3 and 0.6. The reason for this is obvious: between 0.3 and 0.6, there are exactly two decimal fractions with exactly one decimal place, namely, 0.4 and 0.5. Here is an interview with Student B 330 showing how this idea is revealed only after persistent questioning.

I:	(presents the task in writing and reads it out loud) First of all, I've got a

	question: Is there one at all or several?
S	I reckon there are some.
I:	How many do you reckon there are then?
S:	(spontaneously) Two!
I:	How do you arrive at two?
S:	I don't know.
I:	You could just as well have said three.
S:	No, there are two!
I:	Could you name both of them, or one of them?
S:	One quarter and one fifth.
I:	How did you arrive at those two numbers, one quarter and one fifth?
S:	Because they lie in between.
I:	Yes, but you must have worked that out some how or other. You could just as well have taken one seventh. Why don't you choose that?
S:	(spontaneously) But that doesn't lie between the fractions in the question!
I:	Where does it lie then? Is one seventh larger than 0.6 or smaller?
S:	(spontaneously) Larger!
I:	And if I were to take one half now, is that smaller than 0.6 or larger?
S:	(thinks about this and then answers spontaneously) Smaller!
I:	Then smaller than 0.3 as well?
S:	(coughs and nods)
I:	What have you worked out in your head now? A quarter. How big is that then as a decimal fraction?
S:	(spontaneously) 0.4.
I:	A quarter equals 0.4. And one fifth equals?
S:	(spontaneously) 0.5.
I:	Oh, I see, and both of them are in between [between 0.3 and 0.6]. Now I understand.

This student converted common fractions into decimal fractions by applying the false strategy of "1/n = 0.n". The misconception of "fraction line equals decimal point" may also have been involved here. On Item 45, only two students believed that there were exactly two decimal fractions between 1/3 and 2/3.

Students who report that "there are a few" or "there are a lot" when answering Items 45 or 46 are on the right path toward recognising that there is always a further fraction between any two given fractions and thus an infinite number of fractions. Frequently they reported that there must also be some kinds of quarter, fifth and other fractions between 1/3 and 2/3, but they were unable to name them. Many students expanded the decimal fractions 0.3 and 0.6 to 0.30 and 0.60, and then reported that there were a lot of common factors between 0.3 and 0.6, but they were not able to extend the decimal fractions 0.3 and 0.6 an infinite number of times and hence recognise that there has to be an endless number of common factors between 0.3 and 0.6.
Finally, a sample interview with Student A 172 who realised in the middle of the interview that there

is an infinite number of decimal fractions between 1/3 and 2/3. Student A 172 was a little bit shy.

I: (presents Item 45 to the student and reads the task out loud) Is there one at all, and, if so, could you name one?
S: (thinks for a long time) Yes, there's one.
I: How many do you reckon there are? One, or are there several, or even endlessly many?
S: (thinks)
I: Or only a few?
S: Many.
I: But not endlessly many, or?
S: (no answer)
I: The problem is to find them [decimal fractions]. What would you do if you had to determine a decimal fraction? Do you know a way to do that?
S: I would divide the one by three and then two by three.
I: Do you know how big one divided by three is?
S: Zero point three three three and so on.
I: Yes, roughly 0.3. And what is two divided by three?
S: (no answer)
I: It's roughly 0.6. Can you now give me a decimal fraction that lies in between?
S: 0.4
I: Another one?
S: 0.5
I: Right. Is there another one?
S: There's an endless number of them! Because there's also 0.42 and so on.

4. SUMMARY OF THE RESULTS

4.1 Typical Student Errors

These findings show that students typically make the following errors:

- a false transfer from natural numbers to the set of fractional numbers (e.g., there is no further fraction between 0.3 and 0.4, or 3/8 and 4/8, because 4 follows on directly from 3);
- they view common fractions and decimal fractions as two different kinds of number that do not relate to each other;
- they cannot distinguish between decimal fractions and common fractions;
- they consider either only the denominator or only the nominator and ignore the other;
- they interpret the fraction line as a decimal point and vice versa (e.g., 0.3 = 0/3 = 0);
- they treat 1/*n* is as being equal to 0.*n*.

4.2 Hierarchies

There is always an infinite number of further fractions between two different fractions.

There are a lot of further fractions between two different fractions, but not an infinite number of them.

There are only a few further fractions between two different fractions.

False transfer from natural numbers to the set of fractional numbers.

No knowledge about common fractions and/or decimal fractions is available.

5. SOME INTERVENTIONS

The following interventions could help to counter typical student errors or encourage a correct understanding of the density of fractions:

- compile a list of errors for the class;
- discuss the errors with all the students;
- gather approximate ideas on the size of the fractions used;
- explain the meaning of "there are endlessly many";
- work out what the set of natural numbers and the set of fractions have in common and in which ways they differ;
- show how common fractions and decimal fractions are merely two different ways of writing down one and the same numbers;
- practice reducing and cancelling; in particular, that one can reduce a fraction infinitely;
- provide iconic illustrations and justifications.

Present special tasks for practising this aspect of the concept of fractional numbers, for example:

- start off with an opinion survey and then get students to discuss and justify their points of view;
- ask how many fractions there are between 0.6 and 0.7;
- ask them to name as many fractions as possible between 0.6 and 0.7. Who can name the most?;
- ask for a number that is as close as possible to, for example, j. Who can give a number that is even closer to j ?;
- ask why is there an endless number of fractions between 0.4 and 0.5. Ask for reasons;
- give students false statements that they should correct.

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