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ON ASPECTS OF DIDACTICALLY SENSITIVE UNDERSTANDING OF MATHEMATICS

Abstract:

„Whoever has understood mathematics can teach it effectively.“ This frequently expressed opinion states an automatism between expertise and quality of teaching. A common antithesis says: „Anyone who had problems with mathematics himself will more easily adapt to the students' problems.“ Both positions do not conceive the coherence between expertise and teaching skills accurately enough. For this reason the present paper will transform these statements into a question: „Which kind of understanding does mathematics require in order to be taught effectively?“

1. The Development of Knowledge

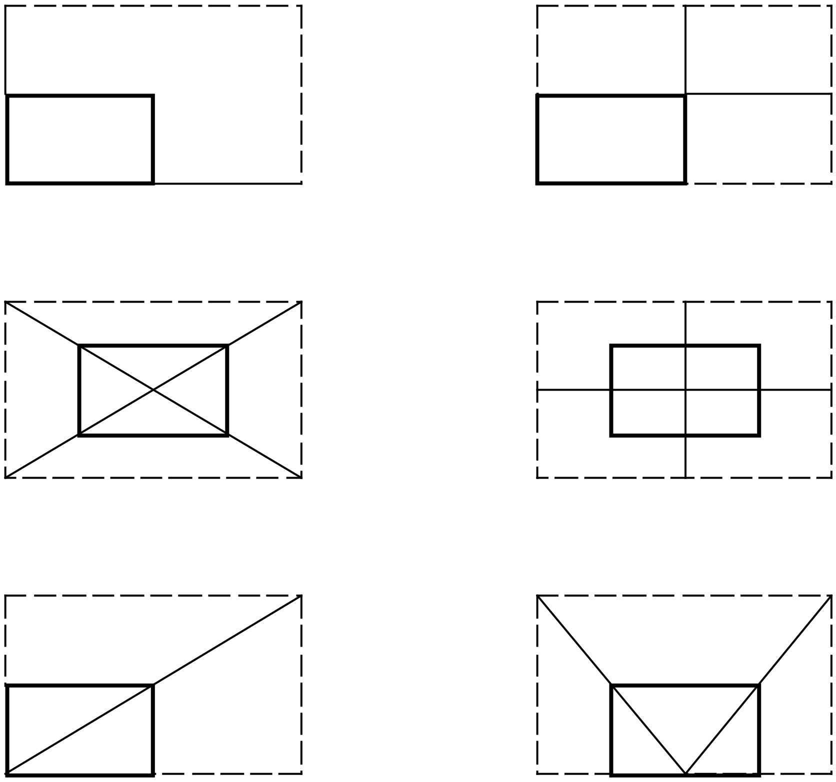
1.1 An Example: Scale-Preserving Drawing and Homothetic Transformation

As a young teacher I had a particular problem with teaching geometry. My problem in plain words was the following: Often it was hard for me to decide what could according to common sense be accepted as intuitively clear and where a more thorough substantiation was required. This problem is related to the subject of natural learning processes and the level of knowledge-development in geometry classes. This can be exemplified considering conformal geometry in secondary school (grades 8 to 10), and in particular the connection between scale-preserving drawings and homothetic transformations.

For a primary understanding scale-preserving enlargement or reduction of a figure means to draw a facsimile of this figure. In doing so the incidence relation and all angles are preserved whereas the length of each line segment is scaled by a fixed factor. The following open ended problem (Becker & Shimada, 1997) is to encourage students to invent and reflect methods of scale-preserving drawing and thus to realize already intuitively existing knowledge.

A rectangle is to be enlarged by doubling the length of each side. Find as many constructions as possible for such an enlarged rectangle and explain your work.

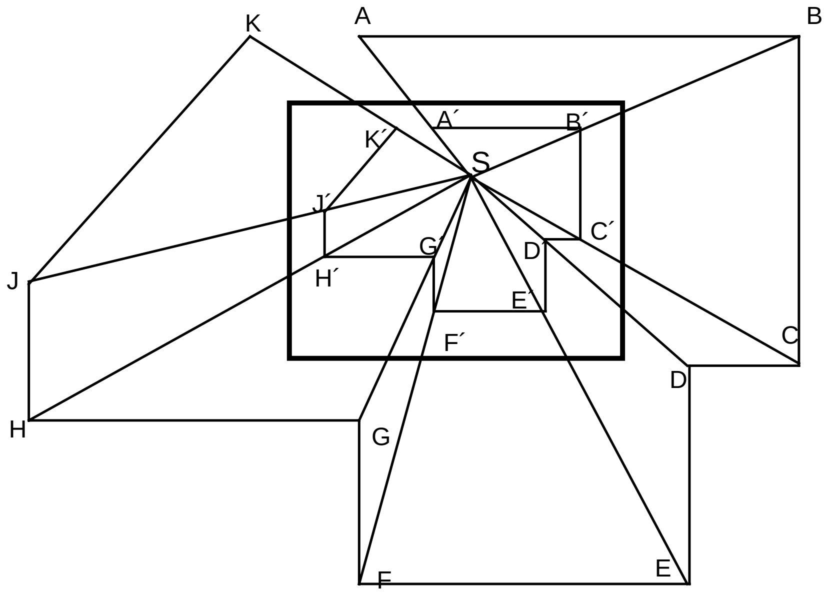
The students found 14 different solutions (l.c.). Here are six of them:



All of these use the intuitive idea that it is sufficient to enlarge a few supporting segments by the given scale and then to complete the figure by connecting points and/or drawing parallel lines. Some of the solutions already imply the idea of a homothetic centre.

A straightened approach for this idea is the following (Ostermann & Steinberg, 1978): Draw a large polygon on the board, pin a letter size paper inside the drawing. With a question like "How can we copy the polygon on this sheet?" ask the students to find an instruction for downsizing the figure.

A suitable approach would look like this: Mark a point S on the paper, which will be called the *centre* of measurement. The distance between S and each vertice of the polygon is measured. In order to reduce the figure we have to shorten each distance by the same factor – like 0.3.



This reduction assignes to each point P of the given polygon exactly one point P' on the paper in the same way the images of the basic points have been found. This correlation can be easily described in mathematical terms: Each vector $\vec{SP}$ is scaled by the same linear factor r. For every point P in the plane there is one and only one point P' satisfying: $\vec{SP'} = r\,\vec{SP}$. This mapping is called homothetic transformation with center S and homothetic ratio r>0 or Z(S;r) for short.

It can be shown that this instruction already contains all basic mathematical concepts for scale-preserving drawing of an arbitrary figure. It only refers to points but all phenomena of similarity transformations are included. For the moment this can be verified with experiments or theoretically proven by following these steps:

1. The images of all points on a line segment (AB) set up the line segment between the images A' and B'. Thus the mapping is preserving line segments (and hence lines). Under this construction a line segment is always parallel to its image; the ratio of their lengths corresponds to the scale of the mapping.
2. All angles are preserved.
3. Hence the resulting figure has indeed the same shape. It does not depend on the center. The size is depending on the scale of the mapping.

So this instruction for drawing homothetic transformations, initially found as a description of a procedure which was discovered experimentally, can serve as a basic definition. By this definition the intuitive knowledge about scale-preserving drawing can be formalized, reproduced, presented, secured and developed on a theoretical level. In this context the proof for line-preservation, which was applied without reflection in the intuitive approaches, obtains the status of a theoretical verification. It verifies that the definition can accomplish what is expected from the phenomenological point of view. This idea can be further developed by introducing negative scaling factors.

The knowlegde acquired on a theoretical level can be used to discover and prove further mathematical topics. The theorem of the nine-point circle is one of the highlights of the theory of conformal geometry in secondary school, it can be proved most elegantly using homothetic transformations.

This outline of ideas about conformal geometry describes the development of knowledge from the level of "intuitive understanding" to the level of "exact deduction" (Wagenschein, 1965), it shows explicit levels using scientific methods:

1. An exercise allows an explorative introduction of the topic. It enables the students to become familiar with the material on an informal level. The goal is to develop and test a sufficient instruction for drawing based on intuition.
2. The method will be put in mathematical terms and secured by further experience (e. g. by measuring and comparing the length of a line segment and its image).
3. The next step is to transform the drawing-instruction from a description to a basic definition which could be the root of a theory. The proof wich states the properties of preservation for the mapping assures that the obtained notion holds in theory what has been suggested or expected in the explorative stage.

All in all, the presented idea assumes a learning process that starts on a vivid intuitive level and then consciously passes over to a mathematically advanced view. During this process an accurate reasoning should adopt the primary understanding rather than repressing or replacing it a priori (Wagenschein 1983). In particular, the levels should not be mixed.

1.2 Importance for the organisation of learning processes

The teaching philosophy intruduced by an example in 1.1 was characterized in general by R. Thom (in Howson 1973, p. 200):

> „In good teaching one introduces new concepts, ideas etc. by using them, one exlains their rules of interaction with primitive elements one has assumed to exist, one makes them familiar through handling these rules. It is only later that one will be able to give the abstract definition which allows one to verify the consistency of the theory extended in this way. Mathematics, even in its most elaborate form, has never proceeded otherwise (except perhaps for certain gratuitous generalisations of algebraic theories).“

This way of teaching mathematics, which "is focusing on the natural *cognitive processes of developing and applying mathematics*" (Wittmann, 1981, p. 130), is called *genetic* in didactics. Known representatives, who introduced and developed the genetic method, are F. Klein, O. Toeplitz, H. Freudenthal, A. Wittenberg, M. Wagenschein.

The opinion that mathematics can only be learned and understood profoundly by comprehending its development, but not as a completed unit, also means to give up the belief that the only way to understand mathematics is following the logical and narrow description of the material in a precise mathematical language. Such a belief disregards the fact that the language of mathematics already includes mentally formed experience (Steinbring, 1998).

This experience can not be accessed automatically by observing the symbols, but must be acquired individually.

Modern constructivistic theories of learning (cf. v. Glasersfeld, 1995) generally assume that it is impossible to adopt knowledge and abilities directly from somebody else. In fact, every individual has to reconstruct sense and meaning of the material by himself. The already established network of knowledge as well as the dispute with the environment and social interaction are crucial for this. The induced adaptation effort leads to the fact that the individual reconstruction process mentioned above is not proceeding arbitrarily, but groups of individuals are developing a common knowledge they can share.

But if learning does not mean to take over knowledge directly, then teaching can not be the direct transfer of knowledge and abilities. It can only offer appropiate chances. Hence teaching is – independent of the form of presentation – understood as the creation of a learning environment, where the individual is inspired to cognitive constructional activity and dispute is provoked (Bauersfeld, 1998). This can be managed well or worse.

Genetic teaching requires an *epistemologic view of mathematics, that takes into consideration the development and structure of knowledge and the inner variety of aspects.* Now it is not only possible to present the contents, but to experience how the corresponding development of knowledge works (H.-H., 1997). The following sections consider details of the accompanying supportive activities.

2. The global coherence of sense

2.1. An example: Fractional equations with parameters

After dealing with simple fractional equations in an eighth grade, a teacher introduces fractional equations with parameters. His guideline is generalization.

Example of a fractional equation	
without Parameter	with Parameter
$\frac{2}{x}-1=\frac{1}{2}$ Domain: $Q \setminus \{0\}$ Common denominator: 2x $\frac{2\cdot 2x}{x}-2x=\frac{2x}{2}$ $4-2x=x$ $4=3x$ $x=\frac{4}{3}$ Set of soulutions: $\left\{\frac{4}{3}\right\}$	$\frac{2}{x}-1=\frac{a}{2}$ Domain: $Q \setminus \{0\}$ Common denominator: 2x $\frac{2\cdot 2x}{x}-2x=\frac{a\cdot 2x}{2}$ $4-2x=ax$ $4=(a+2)x$ Ist case: $a+2\neq 0$. $a\neq -2$ $x=\frac{4}{a+2}$ Set of solutions: $\left\{\frac{4}{a+2}\right\}$ 2nd case: $a+2=0$ Set of solutions: $\{\ \}$

He carefully proceeds step by step. First he reviews fractional equations by an example using a common algorithm to solve it. Then he modifies the example by replacing a coefficent with a parameter, and he changes the algorithm accordingly. The strict parallel arrangement of the steps supports the comparison of the case with and the one without parameter. Locally the proceeding is carefully planned so the teacher does not expect any difficulties.

But at the moment when the parameter is introduced resistance is emerging in the class. After reviewing the example the goal is posted like: "We will now introduce a parameter a. Using this parameter the equation obtains a more general form. We proceed with the same algorithm and treat the 'a' like a normal number, but perhaps we must consider a few more things."

The postulation of the goal is founded in the teacher's professional systematic survey of the topic. It turns out that the students do not share this perspective, since they don't have experienced the use of generalization in mathematics yet and know almost nothing about the possible applications of fractional equations with parameters. They are dominated by the impression of an unusual experience: A new kind of variable in an unfamiliar role appears. Hence a general motivation problem follows that is articulated in helpless and concerned questions like:

- What is this good for?
- How can we distinguish between the parameter and the variable?
- Will we have to do all these steps in the next exam?

The loss of orientation leads to a situation where local problems gain additional weight:

- Shall we write the condition a $\neq$ -2 in the domain? (The domain stands for everything that must be excluded.)

- Why does the term $\frac{4}{a+2}$ suddenly appear in the solution set? So far there have been numbers only!
- How do the restrictions of x correspond to those of a?

One observes that the connection to sense and meaning is lost. Even though the lesson is absolutely correct, there is no coherent leading idea for the students. An alternative approach could start like that: "Replace the numerator of $\frac{1}{2}$ in the initial equation by a number of your choice and solve the equation. I will tell you from your solution which number you have chosen." The teacher will be able to keep his promise easily. To explain this trick one has to trace the path of the chosen number through the complete solution algorithm. Hence it makes sense to consider it as a parameter. So the parameter equation results as a description of an equation type in the sense of a construction plan (Vollrath 1994), and the solution formula indicates how the parameter acts like a block during the solution. Behind this example stands a basic concept in algebra: to interprete the solution formula of an equation as the relation between the coefficients and the solutions of an equation and thereby as a possibility to explore one by the other.

2.2 Significance for the motivation of learning processes

If teachers intend to help students to experience the development of mathematical knowledge they must be able to locate and retrieve basic ideas of the subject in the primary context and hence to lead school mathematics away from triviality.

To sentence math lessons as "trivial" threatens their liveliness in two ways:
1. Teachers often regard the material they are teaching as trivial, because they look for assertions which are more than familiar to themselves, but not for the conceptional ideas.
2. Teachers sometimes trivialise the material by reducing it to formulas, probably they want to simplify the material or perhaps they do not recognize the required qualtity of theoretical thinking (cf. Steinbring, 1993).

In both cases it becomes difficult to maintain an intellectual challenge of a topic and to get it across. The following questions are to provide exemplary suggestions on how to give a basic mathematical idea of the number system for three different school levels:
1. How many names can you remember? How many numbers can you remember?
2. Even Pascal had the opinion that if you take four off of zero then zero remains. Why didn't this idea become effective when the calculating rules for negative numbers were formalised?
3. In which sense can the computer calculate limits of sequences?

The first question is appropriate for students in an arithmetic lesson. In an early stage they become aware of the efficiency of number systems. The trick is to "make the unlimited set of numbers accessible and understandable in a way the human memory is not a limiting factor" (Krämer, 1988, p. 10). The second question is to clarify the evolution from intuitive to intellectual constructs at the stage of introducing the negative numbers and explaining the expedience of calculating rules for rational numbers (cf. H.-H. , 1991). The third question is appropriate for pointing out the elementary difference between a computed approximation and the limit of a sequence as an ideal object that can only be understood exactly by using theoretical argumentation (cf. Knoche & Wippermann, 1986; vom Hofe, 1998).

During their education teachers should have learned to ask these questions and should have acquired a sense for basic notions and mathematical concepts in order to be able to draw these connections in class.

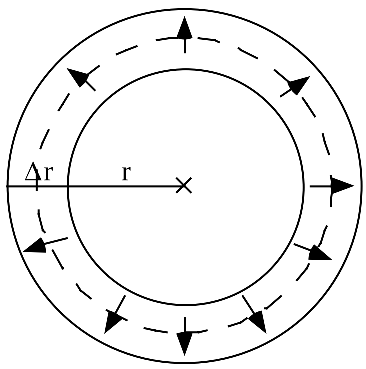
3. Substantial problems

3.1 An example: Differential calculus of a circle

The following problem refers to the center of differential calculus, especially if it can be explained with and without using formulas:

> The derivative of the area of a circle $\pi \cdot r^2$ with respect to the radius r gives the perimeter $2\pi r$.

As a simple application of differential calculus, the relation $\frac{d}{dr}(\pi r^2) = 2\pi r$ is obvious even it may be unexpected. However a qualitative analysis becomes a test for the comprehension of the fundamental ideas of the derivative (R. Fischer, 1976). For this purpose, imagine the circle beeing inflated to a circular ring (Blum & Kirsch, 1996):



The difference $\pi(r + \Delta r)^2 - \pi r^2$

describes the rate of *change* for the area in the interval $[r, r + \Delta r]$, thus the area of the circular ring. The difference quotient

$$\frac{\pi(r+\Delta r)^2 - \pi r^2}{\Delta r}$$

can then be interpreted as the *average rate of change* in the area per unit of length of the radius in this interval. It can be evaluated to $2\pi(r+\frac{\Delta r}{2})$ and then indicates the circumference of the inner circle of the ring; hence it can be represented geometrically in an obvious way. The difference quotient as the limit

$$\lim_{\Delta r \to 0} \frac{\pi(r+\Delta r)^2 - \pi r^2}{\Delta r}$$

finally describes the *instantaneous rate of change* of the area as a function of the radius. In the geometric approach, the boundaries of the circular ring collapse to one circular line.

These considerations use the general terms: rate of change, average rate of change, instantaneous rate of change, which are appropriate to describe the basic ideas of the derivative independent of the context (R. Fischer, 1976) and they can fill these terms with substance in a particular context. The result is a profound comprehension of terms (Vollrath, 1984) and at the same time it can exemplarily experienced how a known situation (here how to measure a circle) can be studied from a higher point of view (here the differential calculus) while both levels gain new aspects.

3.2 Relevance for selection of material

In mathematics there are significant problems and examples which reflect basic ideas of a subject or point out relations between different subjects (cf. Freudenthal, 1973). Furthermore, there are instruments to make a notion precise and to complete a theory. The example with the circle for the calculus lesson belongs into the first category, whereas the subordination of the notion of a function under the notion of a relation belongs into the second category. At the first visit of a topic in mathematics it is more important to pose a problem lively than presenting complete notions.

"The gain of precision at the basics and in the process is payed with an enormous loss of substance. ... (The) status of knowledge about important mathematical problems, however, risks to decrease more and more. ... It has turned out that even in calculus the relation to reality can be disguised by tedious introductions of all sorts of numbers and exaggerative discussions about the notion of a function" (G. Fischer, 1980, p. 155 ff).

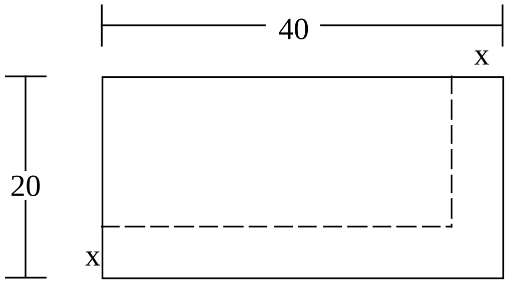
It is part of a didactically sensitive understanding of mathematics to know and to choose such substantial problems.

4. The local variety of aspects

4.1 An example: Reducing a rectangle

The following kind of problem can be found as an application of quadratic equations and inequalities:

"From a 20 cm by 40 cm rectangle we cut off two stripes of the same width at two adjacent sides. The area of the rectangle shall be reduced by at most $\frac{1}{8}$. What is the maximal width of the stripes?"



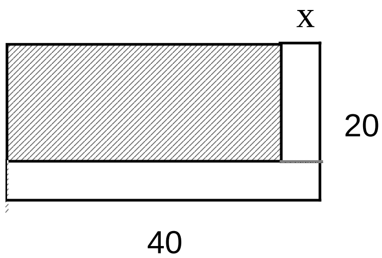
In a flexible preparation of the lesson at least the following aspects of this problem and its solution should be present.

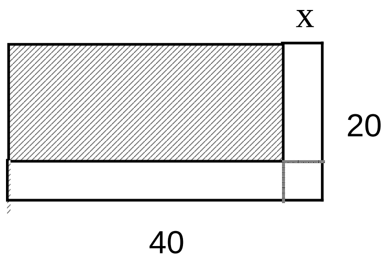
(1) *Structure of the problem*
A rectangular area of size G (initial situation) has to be transformed into a reduced rectangle of size R (final situation) by cutting off an angular strip. The rectangle may be reduced by at most $\frac{1}{8}$ of its initial size. The question of the problem centers the range of the possible width of the removed strip.

(2) *Solving strategy*
In the present context an algebraic solution is expected. The following steps show the necessary translation of the geometric context into an algebraic form: introduction of a variable for the width of the strip, setting up an equation (inequality), derivation and interpretation of the solution set.

(3) *The method for solving this problem in detail*
There are different approaches for the algebraic acquisition of the given facts; they depend on the particular cognitive presentation of the issue. *The presentation of the reduced rectangle of size R* can be done

- in one step as a rectangle with reduced sides: R = (40 – x) (60 – x)
- or as a difference of areas: $R = G - S$. Thereby, S can again be represented in different ways, depending on the arrangement of the removed angular strip:





$S = 40 \cdot x + (20\text{-}x) \cdot x$ $\qquad$ $S(40\text{-}x) \cdot x + (20\text{-}x) \cdot x + x^2$

Finally, this yields

$$
\begin{aligned}
R &= (40-x)(20-x) \quad = \quad 800 - 60x + x^2 \\
S &= 60x - x^2
\end{aligned}
$$

The condition "reduce by at most $\frac{1}{8}$" is difficult to put into an inequality, since here a reduction is bounded from above, and hence two opposite directions of thinking must be reconciled. This only works with a qualified analysis. Among others, the following methods are possible:

- $S \le \frac{1}{8}G$: The removed strip, i.e. the reduction, may be at most $\frac{1}{8}$ of the entire area.
- $R \ge \frac{7}{8}G$:At most $\frac{1}{8}$ of the area may be cut off, i.e. at least $\frac{7}{8}$ of the area must remain.

The first approach translates the condition of the problem immediately after realizing that the reduction is represented by the removed angular strip. The second attempt concentrates on the remaining surface and reformulates the condition of the problem using the duality of the specifications "at most" and "at least".

The solution of the problem: Combining the approaches for the solution, one obtains quadratic inequalities that all lead to the same normal form

$$
\begin{aligned}
& x^2 - 60x + 100 \ge 0 \\
\Leftrightarrow \quad & (x-30)^2 - 800 \ge 0
\end{aligned}
$$

Despite of their mathematical equivalence, the students might experience these statements as different and therefore as unequally evident, since the understanding of isomorphy depends on the particular familiarity. The corresponding equation has the solutions $x_1 = 30 - 20\sqrt{2}$ and $x_2 = 30 + 20\sqrt{2}$. This yields the solution interval $[x_1, x_2]$ for the inequality; for the given geometric problem only the subinterval $[0, x_2]$ makes sense as a set of solutions. The width of the removed strip may therefore take a rounded value of at most 1.72 cm.

4.2 A multifaceted view as an aid to guide the learning process

What we understand here as a high-resolution and multifaceted view of a mathematical topic was described in the introductory example. It regards the structure of the given problem, the specific mathematical strategies used for the solution, as well as individual approaches to describe a problem using mathematical terms. Therefore it is necessary to be conscious *of the way mathematics developes and to be sensitive for the perception for individually different ways of learning and solving mathematical problems*. Both can be practiced by the analysis of mathematical learning processes.

Such a disposition supports the clarifying guidance of learning processes. It makes it easier

- to give orientation about the intellectual development and
- to pick up comments from the students and to support their articulation.

This holds for the discussion in the traditional mathematics lesson which focuses on the teacher and develops by asking questions (Maier & Voigt, 1991) as well as to the guidance of students in an open learning environment.

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