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PROOF AND PROVING PROCESSES: TEACHING GEOMETRY IN FRANCE AND GERMANY

Abstract

In this paper, the first results of a comparative study on proof and proving in geometry teaching are presented. Twelve eighth grade classes (approximately 14 year-old students) were observed in France and Germany, in order to analyse the impact of culturally-embedded classroom practices on the teaching and learning of proof. In addition, the differences in forms and functions of proofs based on the observed French and German teaching practices are also presented here. In particular two types of mathematical culture in classroom practice have been singled out.

Towards a Comparative Analysis of Proof Teaching

Previously mathematics had always been seen as a value-free, non-cultural subject and as many still believe, numeration is numeration. However, in contrast to such a concept, Bishop (Bishop 1988) and Gerdes (Gerdes 1996) have asserted that mathematics, as a product of human activity, is based instead on culturally-embedded values (Howson and Wilson 1996; Powell and Frankenstein 1997). Recent results of international studies which compare mathematics teaching, further suggest that cultural diversity in mathematics education goes far beyond the commonplace notion that the settings are affected by traditional instructional methods. This research has outlined qualitative differences between the focus on lesson topics and the actual expectations for the students when studying these specific topics.

While reflecting upon the idea that ethnomathematical questioning of proof is just as necessary as the accepted epistemological inquiry, Balacheff stated that, "any didactic transposition of proof in mathematics takes into account the rationality that is dominant in the society and the culture within which this transposition unfolds" (Balacheff 1999b). Hitherto there has not been much research done in this field.

Although teaching and learning practices have become institutionalised within society, they may not merely reproduce the rationality of this society. Teaching and learning within lessons have been shown to be complex and fragile processes (Cobb and Bauersfeld 1995). Results of international comparisons have put forward the theory that different "mathematical classroom cultures" do exist and that comparing classroom practices in different countries could provide useful insights. The hope is that this will enrich the research in the proof and proving field from an inter-cultural as well as a classroom-cultural perspective.

In my comparative case study on proof and proving in French and German geometry lessons, differences in classroom practice and their implications for the learning and teaching of proof will be analysed. This study poses the question: *How do culturally-embedded classroom practices influence the teaching and learning of proof?* For this I focus on the differences within proof forms and functions from two perspectives: a subject-bound and an argumentative perspective.

Throughout this analytical discourse, I will be looking specifically at proof in both the French and German classroom and will present the first results of my research on the differences in form and function of the proof in these environments. However, before I begin, I will delve briefly into the concept of proof itself and the methodological design of my study.

Proof and its Teaching

Researchers working directly with the concept of proof were particularly interested, among other things, to find out why students, once convinced of a statement's validity, experience so much difficulty understanding the importance of proving the statement. In the following, different approaches to this problem will be presented and discussed briefly.

On the one hand a series of learning situations, including proving exercises, have been created in order to stimulate student processes for formulating both conjectures and arguments which may refute these conjectures. (Balacheff 1987). Therefore problems were designed to lead to controversy among students which should involve them in a process of proving. Arguments for or against conjectures should be induced by the social nature of the situation itself (Balacheff 1991; Boero, Garuti, and Mariotti 1996), and in this approach the social dimension of proof and proving is seen as essential for didactical considerations.

However the teaching practices of proof, and the forms of proofs being taught, have been analysed for their impact on students' difficulties with proving. Hanna and others criticised the overemphasis of formal proofs in everyday teaching (Hanna 1989; Hanna 1996) which do not give students an adequate understanding of the meaning and function of proofs. Several authors emphasise that proofs do have a variety of functions that cannot be reduced to a deductive concept of proof which is typical for formal proofs. Research in this field has made it clear that proofs have a variety of functions (de Villiers 1990), for example as an epistemological function that is to understand why, whereas a systematic or universal function makes visible the place of subjects learned within a larger mathematical structure. Further different forms of proofs as empirical, intuitive and action proofs and their role within the learning of proof have been discussed (Blum and Kirsch 1991; Semadeni 1984; Wittmann and Müller 1990).

These two approaches came to similar distinctions for proof forms (see Balacheff 1999a) and both focus on the subject and its learning environment, trying to single out conditions for learning and teaching proof successfully.

However, in proof research few empirical studies exist on proofs and proving in everyday classroom situations (see Herbst 1998). This means that we have little evidence showing how much students' difficulties are due to everyday mathematics teaching.

In contrast, proof conceptions of teachers and students as well as students' aptitudes in proving have been examined in several empirical studies (Healy and Hoyles 1998; Reiss and Heinze 2000; Stein 1988). These studies focus on cognitive rather than on social aspects, although proving processes are included. These studies have given interesting insights into the implicit proof conceptions of students, but they do not allow any conclusions to be drawn in so far as these conceptions are due to how proof is taught in mathematics lessons.

Research on argumentation processes gives a clue to the argumentation formats in classroom situations, but is limited however to the primary school level (Krummheuer 1993; Krummheuer 1999; Schwarzkopf 2000). It might be interesting to see how argumentation formats in class might be linked to proving processes although such analyses which have not yet been done.

In my empirical study I focus on proofs and proving processes in ordinary geometry lessons at secondary school level. In this case study, types of proofs and proving in French and German geometry lessons at level 8 / 9 (14 year-olds) are analysed. The results of my research lead me to assume that proof functions in the classroom give a way of understanding how proof embeds itself into the underlying culture.

Methodological Approach

This case study, which is carried out within a framework of a qualitative paradigm, seeks to explore differences in forms and functions of proof as they are found in French and German geometry lessons. Therefore six classes of eighth and ninth grade students (14 year olds) were observed during a two-week period in each country. Of the observed classes, two were from a French-German college while the others were from ordinary college classes in Paris and German Gymnasium and upper level comprehensive school classes in Hamburg.

With regard to proving on the one hand and variation of subjects on the other, six units concerning Pythagoras' Theorem and six further units dealing with similarity and special lines in triangles were chosen. Participant observations of the classes were expanded by recording the lessons, so that analyses of transcripts and analyses of blackboard drawings and writings could be carried out later.

Analyses of the observed proofs are intended to be done from two perspectives which are supposed to be complementary: firstly, subject-bound analyses and secondly, reconstruction of argumentation. This means that forms and functions of proofs are analysed with respect to their mathematical substance and not separate from it. I propose, as does Granger, that form and content of proofs cannot be separated (Granger 1994). Content means, in this context, mathematical substance and not kinds of logical reasoning (Miyazaki 2000).

Further research on Pythagoras' Theorem by Fraedrich and on geometrical frames by Parzysz and Douady is referred to as a theoretical framework for the subject-bound analyses (Douady and Parzysz 1998; Fraedrich 1995). Analyses of the structures of arguments are guided by the theoretical work of Duval and Toulmin (Duval 1995; Toulmin 1958). Duval's theoretical analyses allow for a distinction between argumentation and proof by formal aspects, whereas Toulmins' scheme helps to work out different argumentation structures. Results of the latter analyses will be published elsewhere.

In the following sections I will first present results of the subject-bound analyses. Exemplary analyses of forms and functions of proofs of Pythagoras' Theorem as observed in everyday classroom situations will also be presented and differences singled out.

First results by subject-bound analyses

For each class observed, one of the following pseudonyms (without any ulterior motive) have been chosen: Lüders, Nissen, Grimm are given to three observed classes of the German Gymnasium, one of them a German school abroad. The other three classes also observed during units on Pythagoras' Theorem in France are denoted by Dupont, Pascal and Petit, two classes in an ordinary college, the third one from an elite school.

Characteristics of forms of proofs

Sorting proofs of Pythagoras' Theorem, which have been observed in classroom situations in France and Germany, by a classification of Fraedrich (Fraedrich 1995) showed four different forms of proof.

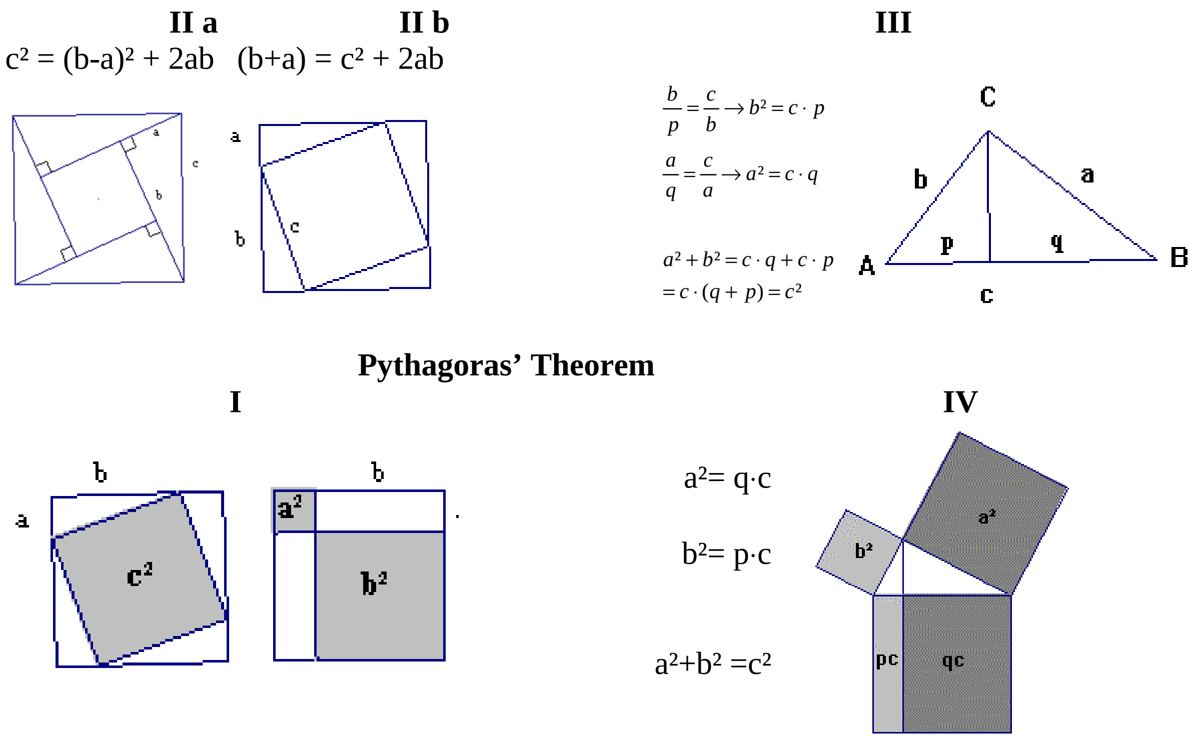


Figure 1

1) Proofs based on comparisons of areas
2) Proofs based on calculations of areas
3) Proofs by applying theorems on similarities
4) Proofs using visualisations of the theorem of Euclid, meaning $a^2 = qc$ and $b^2=pc$.

In the first proof type, two figures - a square with area c^2 and the union of two squares with area $a^2 + b^2$ - are complemented by four equal right triangles to create two equal squares, evidently of the same area. In the proofs of type two, Pythagoras' Theorem is derived by calculating the area of a square in two different ways, i.e. firstly by finding the area of the big square, then by calculating the sum of the area of the small square and the areas of four right triangles. In proof three, proportions - based on the similarity of the three right triangles ABC, ADC and DBC - are considered. Pythagoras' Theorem is deduced by transformation and addition of the generated algebraic equations. In the fourth proof the square built up on the hypotenuse is divided into two rectangles whose areas are known to have the property, which has been demonstrated in former lessons, that $a^2=qc$ and $b^2=pc$. Analysing these proofs shows that in proof one and four a^2, b^2 and c^2 can immediately be interpreted as areas of the three surfaces appearing in the figures which are to be compared. Contrary to this, proofs of type two and three mainly use algebraic operations. Nevertheless, there is a substantial difference between proof two and three since in proof two the generated equations are still interpretable as areas, which is not the case in proof three, where quotients of lengths appear. These proofs make evident two different interpretations of Pythagoras' Theorem.

Pythagoras' Theorem can be interpreted as a statement about comparing areas as well as an assertion about relations of lengths, which are evidently not the same. Further, the theorem is given meaning by discussing it, as well as by the proof type chosen. It is interesting that in French teaching only the second interpretation of Pythagoras' Theorem has been found in discussions, whereas some of the proof types found favour the first interpretation. However, in German classes the teacher has insisted on both interpretations of the theorem.

Taking into account that every type of proof favours one or other interpretation of the theorem it is surprising that, in our observations, tackling the meaning of Pythagoras' Theorem is not necessarily coherent with the proof done in class. In the case of Dupont for instance, the teacher has chosen proof one but is interpreting the theorem as a relation of lengths. In Lüders' class, he has chosen to interpret the theorem in both ways; however, only one proof type is used.

When comparing French and German teaching, one has to note that no differences concerning inconsistencies of teaching could be found. Whereas there had been differences in the meaning, which have been assigned to Pythagoras' Theorem, both interpretations – about areas and about relations of lengths - have been found in German mathematics lessons, but not in French classes. Further, in German mathematics teaching, all different types of proofs as systemised above could be found, while in French mathematics lessons only proofs of type one and two have been identified. These correspond with proofs in the curricula and in the textbooks used in class.

Functions of proofs

Analysing the meaning and the role of proofs in the teaching context made it obvious that functions of proofs could not be understood without looking at introductory phases and phases of exercises. Whereas introductory phases seem to be essential for the German teaching which was observed, phases of exercises might play an important role in the French teaching of proofs (see figure 2).

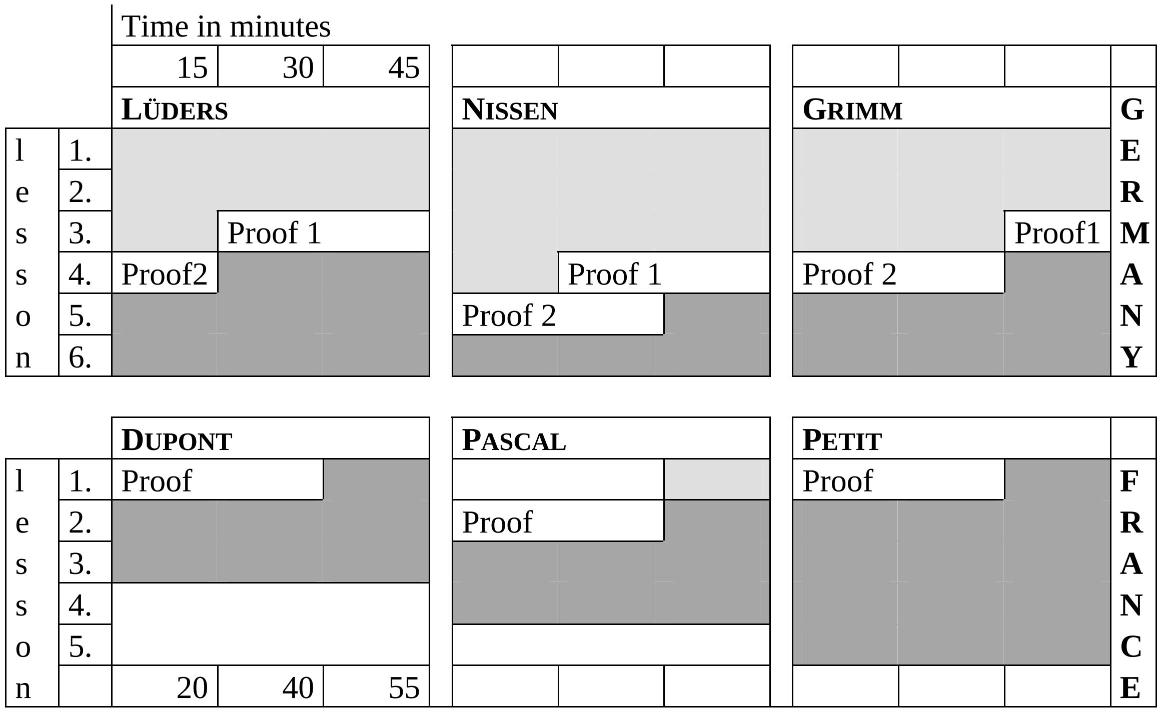
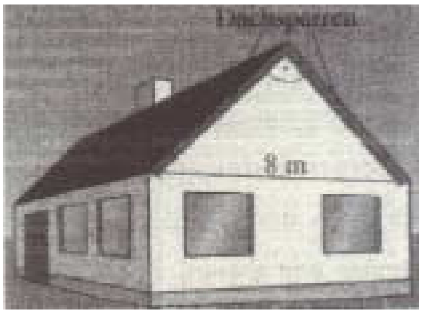


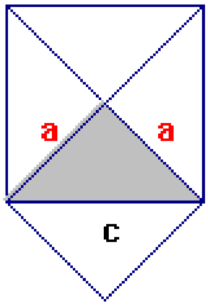
Figure 2 introductory phases phases of exercises

All observed German units start with two or three lessons where students and teachers are engaged in a process of discovery of the theorem before proving it. In contrast to this, in French teaching, Pythagoras' Theorem is introduced directly to the students in the very first lesson of the unit and proved with the guidance of the teacher. Only in the case of Pascal does the teacher reduce the complexity of the proof when she is working on the binomial in the preceding lesson [$(a+b)^2 = a^2 + 2ba + b^2$]. Here the formula is interpreted in a visual way. In order to avoid difficulties on the part of the students, the teacher might have decided to divide the proof into two parts with the more technical part preceding the central idea of the proof, which is to calculate the area of a square in two different ways. Having proven the theorem in French lessons, complex and sophisticated problems have to be solved by students at home as well as in class. These exercises require application of different theorems and concepts, which have been studied in former units, including Pythagoras' Theorem. In German lessons however, all exercises have been analysed as typical routine tasks which request simple applications of Pythagoras' Theorem. Two typical cases of German and French teaching have been chosen and shall be presented below in order to demonstrate the different teaching patterns. Finally, I will discuss how far proofs go in reaching functions by these teaching styles.

The role of discovery of theorems - the case of Nissen

Nissen starts her unit with a calculation problem: the length of the rafter of a rectangular saddle roof is to be computed, given the width of the house (see figure 3). At this time Pythagoras' Theorem has not yet been introduced, so that the students have to find another way of solving the problem. Completing the figure by squares on the sides of the triangle and calculating the areas of these squares leads to a relation between the sides of the triangle: $2a^2 = c^2$.





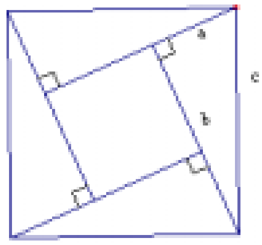


Figure 3 $2a^2 = c^2$ $c^2 = (b-a)^2 + 2ab$

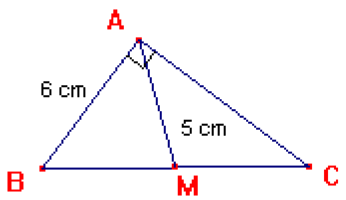
The study of the special case of a right-angled isosceles triangle is used to introduce the central idea of Pythagoras' Theorem in two lessons. This approach allows for consideration of the theorem from two different points of view: the aspect of areas and the relation of length of the triangle's sides. At the same time one of the most important applications of Pythagoras' Theorem, the calculation of length, is already seen. Throughout the course of the unit, in contrast to the idea of proof in the special case, this idea has not been emphasised.

In the teaching, the first two or three lessons do have a special role, as they lead students to discover the theorem itself and to understand the theorem in two ways, i.e. as a theorem about areas and as a theorem on the relation of lengths of the triangle's sides (see above). Furthermore, because the central idea of its proof is so simple, the special case makes it possible for students to figure out a proof on their own, or with little help from the teacher.

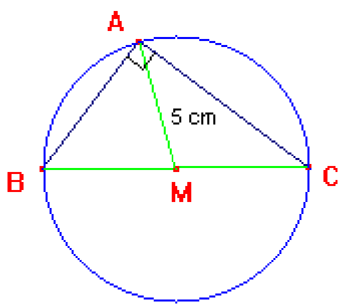
Proofs and problems initiating justifications - the case of Pascal

The problems given to the students in Pascal's class can be divided into two types of problems: routine versus complex exercises. Whereas in the routine tasks Pythagoras' Theorem only has to be applied simply for calculating length, for example the length of a triangle's side, the complex problems cannot be solved without analysing geometrical configurations and using other theorems and concepts, such as similarity or properties of the circumscribed circle (see figure 4).

Calculate the length BC et AC.



(Hachette, Cinq sur Cinq 1998, p. 161, no. 29)



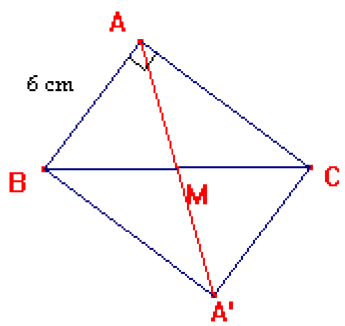


Figure 4 Solution 4a Solution 4b

Subject-bound analyses of these problems have shown that different forms of solutions are possible, some of which are very elementary. These are solutions which are based on concepts and properties that have already been introduced at grade 6 (12 year olds). Reflection as well as the properties of parallelograms and rectangles, for example, offer two different ways to solve problem 29 (see figure 4a and 4b). In class the solution using the theorem of the circumcircle was favoured by the teacher, who had just studied this idea with the students in the last unit. The reasoning in the latter solution is much less complex and briefer than the justification of the former two solutions and can therefore be considered to be more easily retained.

The teacher insists on a complete justification, i.e. why a student chose the way they worked out a solution and whether their use of theorems and concepts is legitimate. The proof of Pythagoras' Theorem, which has been produced in collaboration between students and the teacher, offers a model of how justifications should be structured and sets the level of rigour which is expected by the teacher. Problems and proofs of theorems are functioning as an amalgam where the responsibility for truth switches from teacher to students and vice versa. Everyone is asked to justify the validity of the mathematical statements they have claimed.

Comparing functions of proofs

The role of proofs in teaching can be very different. Different patterns of teaching (see figure 5) produce different functions of proofs.

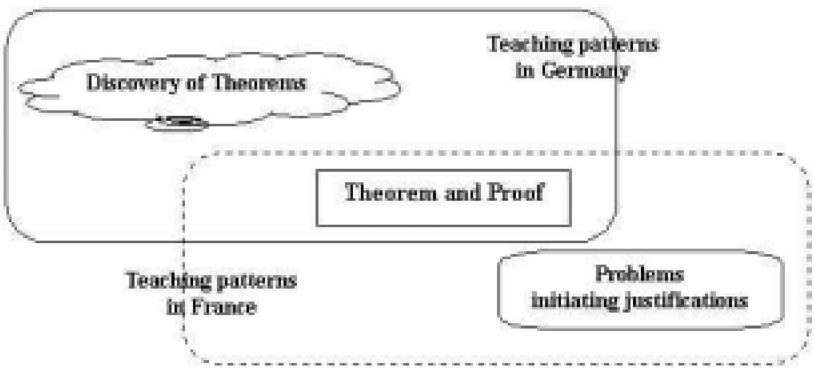


Figure 5

It seems typical in the observed German lessons that the discovery of theorems is based on special cases, where proof and understanding of the theorem merge into one another. With such processes, proofs gain the function of expanding knowledge. This explains why no problems occur when the central idea of proof, used in the special case, is replaced by the general proof. In this pattern of teaching it is more important to make sure that students understand which mathematical ideas are used and why. Whereas understanding and meaning seem to be essential for German ways of teaching proof, successful defence of claims of validity of mathematical assertions can be described as typical for the observed French teaching. Proving is seen here as an activity which characterises the whole teaching and not just phases when theorems such as Pythagoras' Theorem are proven. Every exercise, even those where students are not explicitly asked to prove something, has to be edited so that the given solution is justified. It has to be made clear what are considered as assumptions and which theorems, concepts and properties are applied. The proof of Pythagoras' Theorem, which has been done in collaboration with, but strongly guided by, the teacher, serves as an exemplary scheme on how to organise one's thoughts. This pattern of reasoning, which is acquired through edited justifications for solutions of problems as well as through proofs of theorems, gives proofs the function of applying knowledge. The responsibility of justifying which of the theorems and concepts already studied in class are to be used, and why, shifts from teacher to students and vice versa.

Conclusion

In conclusion I will discuss aspects of the results outlined earlier in order to reflect on a "cultural" questioning of teaching and learning proof.

Comparing the meaning and the role of proof in German and French teaching contexts has raised awareness of the different teaching patterns of proof. These patterns cannot merely be interpreted as different levels of proofs i.e. as more formal and less formal proof types. This distinction is too often linked to normative conceptions of how proof teaching should be undertaken rather than based on empirical evidence. However it is the function of proofs which is different. In the observed German teaching the function of proofs is to "understand why", whereas in French teaching it is important to "defend why".

It seems to me very interesting to see to what extent these differences have an impact on the structures of argumentation in class. We may assume that, in general, assumptions are made explicit in French discourses, whereas in German classes there might be more lenience for assumptions that are implicit in the reasoning. Formats of argumentation might remain implicit in German practices because sharing of meaning is more important than argumentation types. Because argumentation analyses have not been completed, no results can yet be given.

Differences in proof functions might also explain why a plurality of proof forms could be found in German teaching. Different types of proofs can be used beneficially for working out distinct interpretations of theorems' meanings. This might be regarded as a waste of time when proofs' functions are to give a model as to how results should be verified. In mathematics instruction, where processes of "defending why" are typical for teaching and learning in general, it is more important to allow time for students' own reasoning and proving activity. We may presume that this characterises distinct relations to knowledge and rationality as ingrained in "the culture within which this transposition unfolds".

Further, I will argue that comparative studies might help to see "old problems" from new perspectives, because my comparison of French and German geometry teaching has shown that differences in content are important when trying to describe different forms of proofs. It might have not been explicitly said, but the theoretical assumption that the forms and content of proofs cannot be separated is borne out by the very first results of my comparative analyses. Therefore different proof types, which have been singled out, are embedded within different mathematical frameworks. It seems to me very interesting to see how far these differences have an impact on the structures of argumentation in class. These analyses have not been completed yet, so results cannot be given at the moment.

Different mathematical frameworks obviously interrelate with curricula. However impacts of proof types on interpretations of theorems, and inconsistencies induced as a consequence, can rarely be explained just by curricula. This is due to didactical transposition of proof in a broader sense and has not been much focused on, as far as I see. It might be interesting to analyse the construction process of meaning in classes' discussions and to confront the outcome of meaning by this process with meaning induced by proof types.

Analysing "mathematics classroom cultures" from a comparative perspective gives a way to single out different didactical transpositions of proof. These differences might help in understanding student's difficulties with proof from new perspectives. Do different mathematical frameworks cause more problems in the learning of proof than others? What effects do classroom discourses have on the learning of proof? Which functions do proofs gain through teaching and what impact does this have on students' conceptions of proof and their aptitudes for proving?

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