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HOW GERMAN ELEMENTARY SCHOOL CHILDREN SOLVED ADDITION AND SUBTRACTION PROBLEMS WITH THREE-DIGIT NUMBERS

Abstract:

The present paper describes of the methods (mental, informal written and standard) the success, and the strategies almost 300 primary students used while working on six addition and six subtraction problems with three-digit numbers. These twelve problems were administered by means of a class test given in February (grade 3; nine-year-olds) before the standard methods were introduced, in June after they had been covered and in October at the beginning of grade 4.

1 Background and Research Questions

Considering that mathematics education claims to be the science of mathematics teaching and learning, it is apparent that our discipline provides - generally speaking - surprisingly little knowledge about how children deal with mathematical problems. Certainly some topics have been studied in considerable detail, as for example young children's arithmetical pre-knowledge (see BAROODY 1989, GINSBURG 1977, HUGHES 1982 or STEFFE et al. 1983) or addition and subtraction with numbers up to 20 (see FUSON 1992 or GRAY 1991). However it is equally true that we do not know very much about children's mathematics with respect to various other topics. This deficit applies, for example, to addition and subtraction with three-digit numbers, as there are hardly any research reports concerning this issue – other than that of FUSON et al. (1997).

Thus this present paper aims at analyzing 300 German primary school children's solutions to problems of this kind. It should be borne in mind that a study conducted within the classroom - like this one - does no more, but equally no less, than shed light on the procedural path taken by students inside school. It is well known that these can differ considerably from those which are applied in out-of-school situations (CARRAHER et al. 1987).

Before the design (section 2) and the results of the study are described (sections 3 through 5) and summarized (section 6), relevant aspects of the existing literature are presented. Following the structure of sections 3 through 5, I first want to deal with the different methods of arithmetic (mental, written informal, written standard) in section 1.1, followed by a discussion of relevant issues concerning primary school children's success in solving these kind of problems (1.2). Finally, I come to the main solution strategies which the literature documents (1.3).

1.1 Methods

Traditionally, mathematics education distinguishes three main arithmetical methods: written standard (algorithms), written informal (by putting down equations or non-standardized notes) and mental arithmetic (without making any notes) (KRAUTHAUSEN 1993; PLUNKETT 1979). The written standard methods are characterized by the fact that numbers are split up into digits which are then manipulated by means of the addition and the multiplication table according to explicitly prescribed rules (apart from long division beginning with the smallest place value, from small to big). The standard algorithms can also be labelled digit methods and distinguished from the so-called number methods, a notion that encompasses mental as well as informal written arithmetic. The common characteristic of all number methods is that the children operate with (split) numbers without precise prescriptions (as a rule beginning with the largest place value, from big to small; see SELTER 1999). Whereas it is relatively easy to distinguish between digit and number methods, the border between mental and informal written arithmetic is not a fixed, clearly defined one.

At the moment our discipline is witnessing a worldwide discussion about the relative importance of these three different methods (see for example VERSCHAFFEL & DE CORTE 1996). In particular the role of the written standard algorithms in the arithmetic of the 21st century is being discussed in detail - with different conclusions. In some countries, for example in Denmark, the standard methods are given a marginal position. In other countries mathematics instructors give reasons why it is still justified to include them as a crucial part of the primary school curriculum (see for example Germany: BAUER 1998; SCHIPPER 1998).

1.2 Success

Another important topic of discussion among the mathematics education community, as well as among the public at large, is the evaluation of daily teaching. In this connection, our discipline has repeatedly elaborated that success cannot be paralleled solely with the availability of facts of knowledge and the ability to perform routine procedures. The goals of mathematics teaching comprise much more. However, we should not forget that attempts to determine success and failure of conceptions of mathematics teaching have also to take into account children's performances with respect to these goals. Thus this paper aims at investigating the success of typical German conceptions of dealing with addition and subtraction with three-digit numbers.

1.3 Strategies

In line with various researchers, as for example WITTMANN & MÜLLER (1990, 87f.), KLEIN et al. (1998) or RADATZ et al. (1999, 87ff.) the following main strategies of number arithmetic can be distinguished. It should be noted that table 1 offers no more than a superficial classification, as the students' strategies are often more complex and manifold (SELTER 1998).

Table 1: main strategies of number arithmetic

	addition (example 527+399)	subtraction (example 701–698)
stepwise	527+300; +90; +9	701–600; –90; –8
hundreds, tens, units (htu)	500+300; 20+90; 7+9; 800+110; +16	700–600; 0–90; 1–8; 100–90; –7
stepwise and htu	500+300; +27; +90; +9	700–600; +1; –90; –8
auxiliary task	527+400–1	701–700; +2
simplifying	526+400	700–697
adding up		701–698–>698+__=701

As a rule, the examples in table 1 represent several different strategies. The addition problem 527+399 for example can be solved by using 'stepwise' in several variations: +9; +90; +300 or +70; +3; +300; +20; +6 or +300; +9; +70; +20 or …

There is rarely any disagreement in the existing research reports (see e. g. FUSON et al 1997 or BEISHUIZEN 1997) that the first two strategies in table 1 can be observed quite often whereas fewer children make use of the mixture ('stepwise' and 'htu'). An exception is the study by

THOMPSON & SMITH (1999) where this was the main strategy used for solving addition problems. In addition, the studies show that the other three strategies enumerated in table 1 could be observed relatively rarely. This is at least partly due to the fact that there is a limited field of application for them. 'Auxiliary task', for example, seems to be easy for 527+399, but rather difficult for 365+271.

1.4 Research Questions

Taking into account these remarks, the following research questions can be identified:-

Methods:

- How frequently are the three methods being used for solving addition and subtraction tasks with numbers up to 1000? Is there (possibly specific to the German situation) an inclination to use the standard algorithms?
- Does it a make a difference whether addition or subtraction is required? Does the relationship of the given numbers make a difference? Do children show a sort of stability and tend to always use the same method?

Success:

- How successfully do German primary school children solve addition and subtraction problems with three digit-numbers?
- Does it a make a difference whether addition or subtraction is required? Does the relationship of the given numbers make a difference?

Strategies:

- How often are the different strategies used? Can strategies be observed that were not taught or do the children only make use of those which were dealt with in class?
- Are the two main strategies dominant even while solving those problems where another strategy would seem to be more appropriate (from an adult's point of view)? Do the children switch between different strategies or do they stick to their (personal) main one?

In addition, the study questions to what extent the children's decisions regarding the choice of methods and strategies, and the percentages of correct answers, changed during the project period (from February until October).

2 Design of the Study

2.1 Overview

In 1999 a study was conducted with 298 German primary school children. The problems were posed not once, but three times: in February (grade 3; nine-year-olds) before the standard methods were introduced, in June after they had been covered in class and in October at the beginning of grade 4. All children were given twelve problems by means of a class test. In addition, 36 students were selected to take part in an interview study in which the same problems were posed a second time on one of the following days. This paper concentrates on the findings from the written test. The analysis of the interviews is described in SELTER (in prep.).

2.2 The Problems

Six addition and six subtraction problems were selected. Context problems were deliberately not included, as the children participating were to be given the opportunity to concentrate on arithmetical processes. This also made the data analysis easier.

Table 2: The 12 problems

problem	example for a skilful strategy
527+399	auxiliary task (+400; –1)
199+198	auxiliary task (for example 199+200; –2)
250+279+250	combining (250 and 250)
119+120+121	combining (119+121) or balancing (3·120)
286+437	no specific one
345+634	no specific one
845–399	auxiliary task (–400; +1)
649–347	auxiliary task (49–47; or adding up: 47+__=49)
701–698	adding up (698+__=701) or auxiliary task (–700; +2)
610–590	adding up or auxiliary task
836–567	no specific one
758–515	no specific one

Four problems were each constructed in order to make the use of a strategy of skilful number arithmetic possible; two additional problems were control problems which did not lend themselves to being solved in the same way. One of these was constructed as an easy one requiring no 'carrying', the other one required 'double carrying'.

2.3 The Written Tests

Via permutation of the problems, four different test versions were put together, two of which started with addition and subtraction problems. Six addition and subtraction problems were presented on one page with enough space on the paper below the problems. The children were told that they were free to decide which strategy and which method to use. The only thing they had to do was to write down the result at a specific place on the page. The test was administered by members of the research team. The class teachers were present, but not allowed to intervene. The children were given 45 minutes to solve the problems. Those who needed a little bit more time were allowed to return the test sheet some minutes later, those who finished earlier engaged in some other individual assignments.

2.4 The Context

The six schools taking part were selected by the local school advisory board to present a cross-section of schools across the region. Germany does not have test scores of schools available for use as reference points thus the selection does not claim to be representative, as this was not achievable given the financial and personal support available. All the same, there were almost 300 children participating in the study. This makes it possible to at least identify certain trends. In my opinion quantitative case studies like this one which take a bigger number of children as one case to be analyzed are as important as case studies of individual children and also as large-scale studies with representative samples.

There were two participating classes from each of the six schools. The teaching was mainly orientated to the textbooks used. A careful analysis of the didactic conceptions expressed by these would lead to an extensive description; thus only a few relevant aspects are outlined in the following sections.

Double the time was used for standard written methods of digit arithmetic than for number arithmetic. In the course of grade 3, first mental, then informal written and finally standard written arithmetic was dealt with in the lessons. In February, when the first tests were administered, number arithmetic had more or less been completely dealt with. In June, when the next series of tests took place, the standard written methods had been introduced and been intensively practised. Finally, by October, a short repetition of some written informal and a lot more written standard arithmetic had taken place.

The main strategy of number arithmetic in the textbooks was 'stepwise' (see 1.3). 'Htu' was dealt with to a lesser extent in ten classes, but solely for addition problems, not for subtraction. 'Adding up' and 'auxiliary task' received very little attention (half a page in books with more than 100 pages). The mixture 'stepwise and htu' and 'simplifying' were not mentioned at all. The culture of the classrooms can be described as first introducing a strategy by the teacher and subsequently practising it. The textbooks used do not give much scope for arithmetic via personal methods.

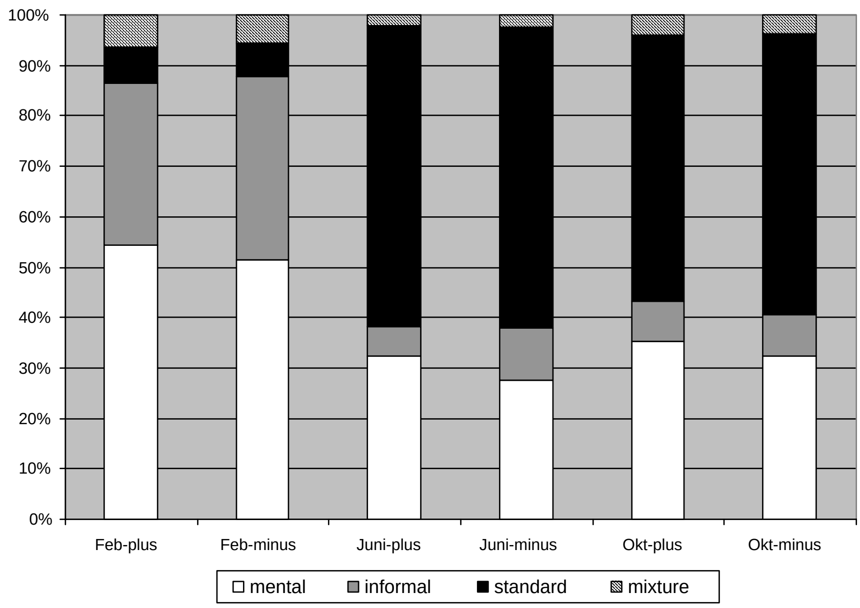
3 Methods

The solutions were categorized according to the methods used. Beside the three modes already mentioned in 1.1 a fourth category was introduced. It was called 'mixtures' and was meant for those cases where children used several methods, for example to add two numbers mentally first and then to add the third one by means of the standard algorithm. Solutions where the children put down equations were labelled informal written methods while those where nothing was put on paper were coded mental. We had another code available for those ways when just some preliminary results or incomplete equations were noted. As solutions of this kind were almost non-existent we decided to label the very few ones which arose as mental also.

3.1 Different Dates

Diagram 1 shows that the standard methods were given clear priority by the children (between 53% and 60%) after their introduction following the first test in February. The informal written methods almost completely disappeared (as a rule not more than 10%), whereas mental arithmetic was still used by approximately 1/3 of the children.

Diagram 1: different methods used in February, June and October

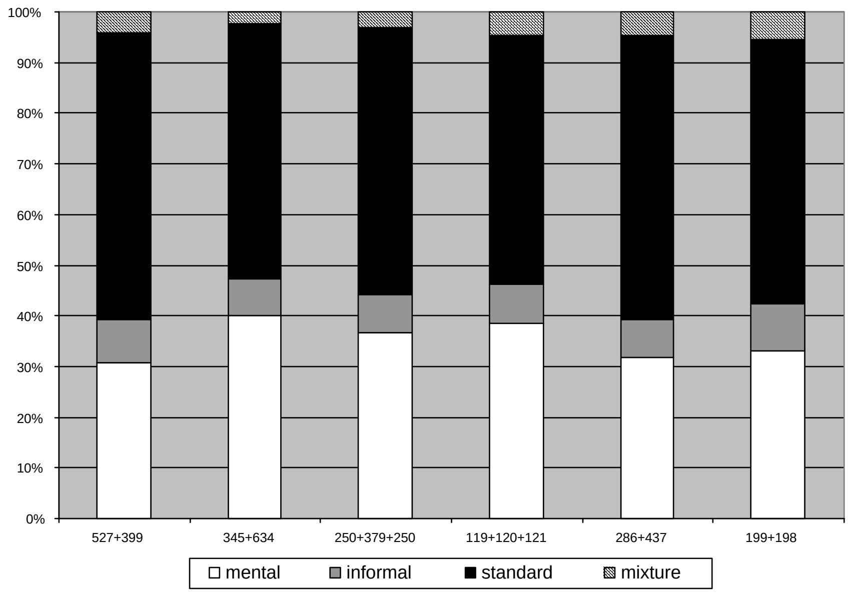


Taking into account the great amount of time given to the standard methods in comparison to informal written and especially to mental arithmetic, it is surprising that so many children used mental arithmetic. It is remarkable that there were no big differences in respect of this if addition and subtraction are compared. However it should be noted that, as a rule, subtraction translated into slightly less mental and slightly more informal (February and June) and then standard written arithmetic (October) respectively.

3.2 Different Problems

The data in diagram 1 are average scores for all addition and all subtraction problems. Thus it is necessary to examine how the various methods are distributed if one looks at single problems. It is of course possible that many children solved 701-698 mentally and many calculated 836-567 by means of the standard algorithm. The analysis of the data however suggests that in all three tests the children showed a stable decision pattern. The October test serves as an example in the following discussion.

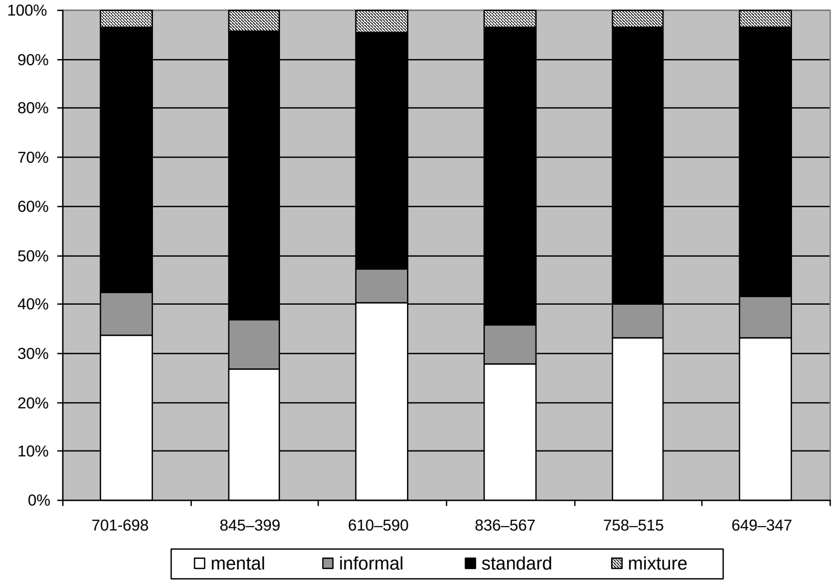
Diagram 2: distribution of methods in October, addition problems



The existence of a general trend does of course not translate into the non-existence of performance differences. For example, 345+634 (no carrying required) was solved by 40% mentally, whereas for 537+399 ('double carrying' necessary) the percentage was 31.7%. Slightly bigger differences could be observed for the subtraction problems, the most extreme between 845-399 (26.7% mentally) and 610-590 (40.5% mentally).

Nevertheless, as a whole the data for the problems did not disperse as much around the average as expected. It seems as if the children did not make a task-specific decision as to which method to use. This means for example that problems like 701-698, 845-399, 527+399 or 250+379+250 were tackled by more than 50% of the children by means of the standard algorithm. None of the four addition or subtraction problems, to which it was possible to apply a skilful strategy (see 2.2), were solved by more than 40% mentally and by more than 10% by means of an informal written strategy.

Diagram 3: distribution of methods in October, subtraction problems



4 Success

4.1 Different Dates

Table 3 shows the percentages of correct solutions in February, June and October. For addition as well as for subtraction problems an increase of about 20% can be established. Depending on one's own expectations one can interpret this as success as well as failure.

Table 3: percentages of correct solutions

	February	June	October
addition	61.6%	78.3%	82.8%
subtraction	41.2%	57.5%	64.4%

A relatively stable difference of 20% can also be observed if one compares addition and subtraction. It should also be noted that generally speaking about 3/4 of the addition and subtraction problems were solved correctly at the beginning of grade 4. It is surely not satisfactory that roughly only 60% of the addition and 40% of the subtraction problems were solved correctly in February, although mental and informal written number arithmetic had been dealt with in the lessons by that time.

4.2 Different Problems

This deficit becomes even more obvious in diagrams 4 and 5, and the fact that some problems caused big difficulties in February cannot be overlooked. Only about half of the children solved addition problems requiring 'double carrying' correctly. Something similar applies even more to subtraction. Only 16.1% (26.2%) correct solutions for 845-399 (836-567) is simply not an acceptable result. In June, an overall increase of correct solutions can be observed. For 119+120+121, almost 20% and for 199+198 around 27.5%, more correct results were noted. For subtraction an increase is visible as well, ranging from 35.1% (845-399) down to 4.6% (610-590).

An analysis of the October results reveals that there was no regression, but also no significant progress after the holidays. Besides, it should be noted that the percentage of correct solutions for those problems which could be solved by means of skilful arithmetic was quite low: only 48% for 701-698, just 54% for 845-399 and not more than 60% for 610-590. Something similar can also be observed for the respective addition problems. Finally, it is worthwhile mentioning that the percentages did not differ that much for addition (between 76% and 91%) whereas bigger differences could be observed for subtraction (from 48% to 84%).

Diagram 4: percentage of correct solutions for addition problems

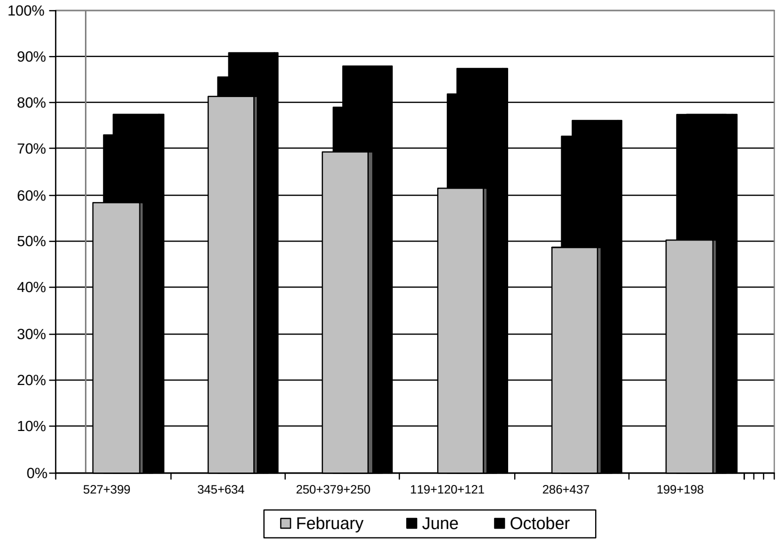
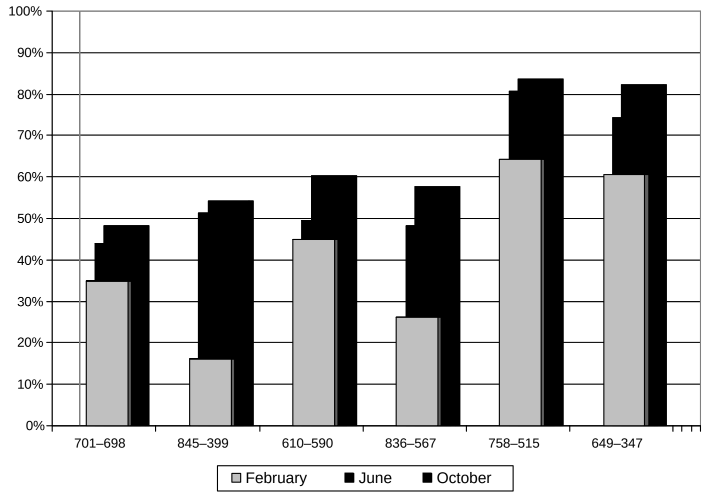


Diagram 5: percentage of correct solutions for subtraction problems



5 Strategies

The number arithmetic strategies to be observed were coded according to the guidelines listed in table 4. It should be kept in mind that the data presented in the following are, in most cases, related to those solutions where informal written arithmetic was shown. It is obvious that we cannot say anything about the mental strategies that were used by almost 30% of the children. In relation to this latter aspect, the interviews are an interesting source of information (SELTER, in prep.). Consequently this means that we are speaking about 1/3 (February) or not more than 10% (June and October) respectively, of the solutions. The data differ slightly if compared to those in subsection 3.1 as very few mental arithmetic solutions were analyzed here when the students clearly indicated the use of a certain strategy (e. g. if a student put down solely 445, while solving 845-399).

Table 4: Coding guidelines for the strategies

strategy	comment
htu, version 1	unit per unit, e. g. 345+634->300+600; 40+30; 5+4; 900+70; +9
htu, version 2	tens and units or hundreds and tens simultaneously, e. g. 300+600; 45+34; 900+79
stepwise, version 1	units separately, e. g. 345+600; +30; +4
stepwise, version 2	tens and units or hundreds and tens simultaneously, e. g. 345+600; +34
stepwise and htu	e. g. 300+600; +45; +34
auxiliary task	e. g. 845–399 –>845–400; +1
adding up	e. g. 845–399 –> 399+ ? =845
simplifying	e. g. 845–399 –> 846–400
combining	e. g. 250+379+250 –> 500+379

The strategies 'stepwise' and 'htu' were classified according to the two versions listed above. It is of course possible that students indicated the use of the shorter version 2 and solved the problem by using version 1. Here we encounter the general and well-known problem that an analysis of written notes cannot translate into more than a justified hypothesis about the student's thinking.

5.1 Different Dates

Table 5 displays the information about the occurrence of the different strategies in percentages (for the three main strategies) and in absolute numbers. With respect to 'stepwise' and 'htu' it should be noted that the first and the second terms of the sum represent version 1 and version 2 respectively.

Table 5: occurrence of strategies

	February		June		October	
total	1168		324		284	
htu	277+286 = 563	48.2%	60+110 = 170	52.5%	88+89= 177	62.3%
stepwise	129+281 = 410	35.1%	16+65 = 81	25.0%	25+44 = 69	24.3%
stepwise and htu	147	12.6%	38	11.7%	11	3.9%
auxiliary task	23		27		22	
adding up	6		2		2	
simplifying	4		2		3	
combining	15		4		0	

Table 5 shows that 'stepwise' and 'htu' were used by the vast majority of students. It should not be forgotten that for obvious reasons more than three times as much data were produced in February than in June or October. The increase in the occurrence of 'htu' and, in return, the decrease of 'stepwise' can possibly be explained by more children using 'stepwise' as a mental strategy (instead of informal written). However, at the present stage this is no more than a hypothesis to be confirmed or rejected by the interview study. It is interesting that both strategies were shown more often in version 2 than in version 1.

This means that the children used a strategy version which they were not taught, as only version 1 was dealt with in class. The same was true for the mixed strategy 'stepwise and htu' that did not occur in teaching. Once again, we witness the phenomenon of children developing their own methods. Very few students used the strategy 'auxiliary task', especially in June and October. However, they did not do this in the expected way (845-399-> 845-400; +1), as most of them solved a different task which appeared to be easier, before they did the required balancing as for example: 845-399 -> 800-399; +45.

The other three strategies listed in table 5 were used by less than 1% of the students. In other words the strategies of skilful arithmetic, according to which four of the six problems were constructed, could barely be detected in the informal written methods. It is an interesting question whether this is also true for the mental arithmetic solutions.

5.2 Addition and Subtraction Problems

The general trends appearing in subsection 5.1 can also be detected if one takes into account single problems (no table). Some examples shall be outlined in order to illustrate this observation: '327+599' or '701-698' were, almost without exception, solved by means of the three main strategies of informal written arithmetic. On no more than four occasions could the use of 'auxiliary task' be observed in the expected way in all three tests. For obvious reasons not a single child documented that they used the 'adding up' strategy for solving the problem '701-698' in their writings.

Table 6: occurrence of the main strategies for addition and subtraction problems

	Feb. plus	Feb. min.	June plus	June min.	Oct. plus	Oct. min
htu	316 (c: 76%)	247 (c: 34%)	80 (c: 80%)	90 (c: 32%)	100 (c: 83%)	77 (c: 22%)
stepwise	129 (c: 58%)	281 (c: 60%)	20 (c: 60%)	61 (c: 53%)	25 (c: 64%)	44 (c: 48%)
stepwise & htu	82	65	26	12	9	2
sum	527	593	126	163	134	123

It is striking that 'htu' was used quite often for subtraction problems although it had never been dealt with during lessons (table 6). The percentage in brackets represents the relative occurrence of correct answers. Of course, one has to take into account that the absolute numbers differ quite considerably. In addition, assumptions about all children cannot be drawn from these data as they were produced by those children who indicated the use of an informal written method. However the discrepancies should be noted with respect to the success rates (1) between 'htu' for addition (roughly speaking 80% correct) and subtraction (about 30% correct) as well as (2) between 'htu' (about 80% correct) and 'stepwise' (60% correct).

6 Concluding Remarks

The remark 'Further research is needed' certainly applies to this study as well. The large number of participating students allows us to not only think in terms of individual cases and not only to demonstrate certain phenomena, but to also derive some information about the frequency of their appearance. On the other hand, several questions cannot be answered by a quantitative study. As 36 selected students were interviewed, qualitative data also exist that might contribute to enlightening at least some of the unresolved issues.

In conclusion, the results of the present study do not, in my opinion, contradict the results of other studies which clearly demonstrate children's capabilities of going their own mathematical ways creatively and productively, if they are continuously given the opportunity to do so. However, conceptions about mathematics teaching and learning - expressed in textbooks, teacher education or in didactic handbooks - can obviously contribute decisively to the development of a certain inflexibility in thinking.

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