Mechanisms of the Taking Effect of Metacognition in Understanding Processes in Mathematics Teaching

Elmar Cohors-Fresenborg
University of Osnabrück
<cohors@mathematik.uni-Osnabrueck.de>

Christa Kaune
University of Osnabrück
<ckaune@mathematik.uni-Osnabrueck.de>

In the last ten years of international discussions about mathematics education, making reflection and metacognition a central component of mathematics teaching has been increasingly advocated. When analysing the components of metacognition, one usually starts out from situations in which a mathematical problem has to be solved. In this paper we will analyse components of metacognition that are employed during understanding of mathematical concepts and procedures. We will develop a theoretical model and apply it to the analysis of four teaching scenes. These scenes all have in common that, within a mutually shared discursive lesson culture, pupils discuss the interplay of external and internal mental representations of the things being said and those being meant.

In the last ten years of international debate about mathematics education, it has increasingly been proposed to make reflection and metacognition a central component of mathematics teaching. An overview of early approaches to research in mathematics education concerning metacognition can be found in Schoenfeld (1992). In the field of educational psychology, Boekaerts emphasised (1996, 1999) which role metacognition plays regarding self-regulated learning. Whereas there are various examinations of the benefit of metacognition in other fields (comments in Boekaerts, 1996), there are only few studies regarding successful approaches when considering metacognition in mathematics teaching (see De Corte, 1995; De Corte et al., 2000). Occasional reports about the effectiveness of a use of metacognition are available rather than having concepts at one's disposal about how to use metacognition generally in mathematics teaching.

The cognitive science approach of two school experiments in classes 7 to 10 at grammar schools in the years 1987 to 1995 (see Cohors-Fresenborg, 2001; Cohors-Fresenborg et al., 2002), which were scientifically guided by us, led, right from the beginning, to changes in the teaching-learning processes. Here the promotion of metacognitive activities plays an important role (Cohors-Fresenborg & Kaune, 1993; Kaune, 1999). A different teaching culture is therefore necessary (Kaune, 2001a), which uses a different curriculum with different kinds of books (Cohors-Fresenborg, 2001) and problems (Kaune, 2001b; Sjuts, 2001). The current project "Analysis of teaching situations for the training of reflection and metacognition in mathematics teaching in forms 7 to 10 at grammar schools"[1] deals with the development of a typical teaching scenario, the central idea of which is the solving of problems which are in a special way suitable for the promotion of reflective and metacognitive activities. Its objective is to analyse more exactly metacognitive activities by means of video-documented teaching examples and to reveal, in detail, the mechanisms which illustrate their effectiveness.

Schoenfeld's (1992) and De Corte's et al. (2000) analyses of the components of metacognition are based on situations in which a *mathematical problem is to be solved*. Reflection refers to the choice of suitable tools. It can, however, also refer to the choice of a suitable order of use of the tools when a certain set of tools is available. Therefore an analysis, for example of the latest state of what has been achieved, is necessary. A comparison with the

[1] The project is supported by the Deutsche Forschungsgemeinschaft under reference Co96/5-1.

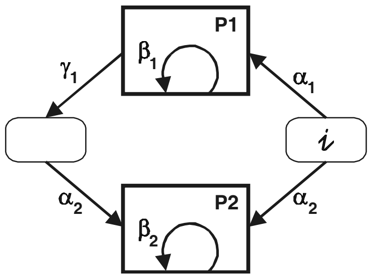
goals set needs to be made and the administration of this comparison is called monitoring or controlling.

Our analysis of components of metacognition is restricted to the aspect of describing thinking processes, which occur when *understanding mathematical concepts*. All the following dialogues, involving pupils from mathematics lessons at grammar school chosen by us for analysis, have in common the fact that the discussion centres on whether a given representation refers to mentioned or supposed ideas. The objective of metacognition is to judge the adequacy of the production of representations from ideas or to carry out the steps backwards from representation to the presumed ideas of classmates.

The thought that the controlling of the choice of tools and the strategic meaningfulness of the combination of tools is the subject matter of metacognition goes along with the aspects of metacognition found in the literature. In contrast to the examples from the field of problem solving, the tools here are "precision" and "analysis of the interplay of different (colloquial or formal) representations of concepts".

Modelling with αβγ–automata processes of understanding

Up to now, there exist great difficulties in decomposing metacognition. In this paper we want to take single components of how persons imagine other people's ideas. As a metaphor (according to Lakoff, 1980), we choose the theory of αβγ–automata, through which Rödding (1977) modelled mechanisms of social behaviour. An individual modelled by an αβγ–automaton receives, with the help of the function α, a piece of information i from its environment in state s, in which the person's knowledge has been coded. The α-transition produces an idea i.e. an inner representation while the β-transition describes inner mental processes in the individual. By means of a γ-transition the individual carries out an action, e.g. passing on a piece of information to the outside or producing a (written or oral) description – in essence presenting an outward representation of their idea. We are now going to look at a network of two αβγ-automata.



An external representation i is given which serves as an input for person 1 as well as for person 2. Person 1 in state s_1 forms an idea of information i with the help of α_1, which leads to a new state, and processes it with β_1. With the help of γ_1, the person passes a representation of $\beta_1(\alpha_1(s_1, i))$ on to the outside. Person 2 perceives this representation in state s_2 and forms an idea of it by means of α_2. Moreover person 2 has – like person 1 – realized the external representation i and has formed an idea $\alpha_2(s_2', i)$ of it with different knowledge s_2'. He/she compares it with $\alpha_2(s_2, \gamma_1(\beta_1(\alpha_1(s_1, i))))$. Let us presume that these two do not fit together and that person 2 assumes that the reasons for that do not lie in his/her own thinking processes α_2. Person 2 can now suppose that person 1 has made a mistake in the representation [mistake regarding γ_1] or a mistake in his/her logic when processing $\alpha_1(s_1, i)$ by β_1 or that he/she has formed a misinterpretation of i [mistake regarding $\alpha_1(s_1, i)$]. The thinking about which of the cases mentioned above is plausible belongs to the field of metacognition.

The process mentioned above describing the analysis of ideas and their representations will be illustrated with a hypothetical example. As an input we choose a subject matter which – speaking in terms of set theory – deals with the situation where two sets are to be joined together. Person 1 gives the representation $A \cap B$. Person 2 can, on one hand, suppose that a representation or writing error [γ_1] has occurred or, on the other hand, that person 1 has got a false idea [a combination of β_1 and α_1] of how the word "together" has to be expressed. Person 2 supposes in both cases that his/her own mental constructions [$\alpha_2(s_2', i)$ and $\alpha_2(s_2, \gamma_1(\beta_1(\alpha_1(s_1, i))))$] have worked correctly. By means of a question from person 2 addressed to person 1, person 2 can exclude the possibility that a representation mistake has been made. Person 2 presumes by means of his/her own knowledge that person 1 has become a victim of the misconception that the mutual contemplation of the sets A and B has to be expressed by the logical composition term "and" (instead of "or").

Analysis of Metacognition in Teaching Situations

We now want to show the explanation value of this model in order to understand teaching situations. We do this with the help of several examples taken from mathematics teaching to classes that have been taught according to the Osnabrück Curriculum. In the first example, a pupil reveals one of his classmates' misconceptions. In the second example some pupils anticipate possible misconceptions of their classmates, which could have been caused by certain representations. The third example shows the difference between things being said and actually meant. In comparison to the previous examples, the fourth example furthermore shows that cognition about metamathematical questions is interwoven with metacognition.

Example 1:

Within a teaching sequence, the reflection of parabolas was to be expressed by the manipulation of the terms describing them as functions with several variables. A grade 9 class had to deal with the following problem:

The graph of a function defined according to the scheme $p(a, b, c, x) = a(x-b)^2 + c$ with $a \neq 0$ is a parabola. Such a parabola is reflected in its origin. State the corresponding scheme for the reflected parabola.

The pupil, Judith, makes comments on her classmate's solution
$p(a, b, c, x) = - a(x + b)^2 - c$ written on the board.

Judith:	*Well, I agree. Because when it is reflected in its origin, the opening has to be upside down and he has achieved this with the -a. It has been reflected at the axis of argument, and then there has to be +b, as the b has to be in the minus area and then there has to be -c as well, -c ... ?*
	Yes, sure. Yes, it is then reflected at the axis of function values and this has to be the case in the minus area, too, so there has to be a minus-sign.
T.:	*Chris!*
Chris:	*Yes, Judith's solution is actually almost right, apart from her saying that it has to be opened upside down. I would prefer to say "the other way round".*
T.:	*Can you explain why there is a difference if you say "opened upside down" or "the other way round"?*
Chris:	*Because a is not necessarily a positive figure; the variable a can also be a negative one. I think Judith has imagined that a is only a positive figure and if you put a minus sign in front of it, it is negative.*

We analyse the dialogue taking three aspects into consideration. In a first analysis we leave out lines 3 to 5. We take Judith as person 1, Chris as person 2 and the part term "*-a*" from the pupil's solution on the board as information i.

Chris picks up what Judith said in line 1: "*The opening has to be upside down*". He supposes that his mental representation [α_2] works without problems. He furthermore supposes that Judith has correctly expressed her idea of the problem's solutions so that no mistake needs to be looked for in this representation process [γ_1]. This leaves Chris with the idea that, in an abstract way, there has to be a subject matter (i) from which Judith has formed an idea [$\alpha_1(s_1, i)$] and from which, on the other hand, Chris has got his own, differing idea [$\alpha_2(s_2', i)$]. Chris's statement "*I think Judith has imagined that a is only a positive figure and if you put a minus sign in front of it, it is negative*" is aimed at the fact that the idea which Judith has formed of the subject matter [$\alpha_1(s_1, i)$] is a misconception. Starting from this misconception, Judith chooses [β_1] a matching prototype of a parabola in the first quadrant with its opening at the top.

In the second analysis we concentrate on lines 2 to 5. Judith is person 1, the analyst is person 2. Judith forms an idea [α_1] of the "reflection in the origin" [i] demanded by the problem and reduces [β_1] it to a concatenation of reflections in two axes. The analyst also forms an idea [$\alpha_2(s_2', i)$] together with a suitable decomposition [$\beta_2(\alpha_2(s_2', i))$], analyses the discrepancy first of all supposing that the mental processes [$\beta_2(\alpha_2(s_2', i))$ and $\alpha_2(s_2, \gamma_1(\beta_1(\alpha_1(s_1, i))))$] have worked properly. He furthermore assumes that Judith's mental processes [α_1 and β_1] have also worked properly and that only her talking [γ_1] about the two axes contains a mistake, i.e. an exchange of the indication of the two axes.

In the third analysis, we concentrate on lines 3 to 5. This paragraph describes Judith's "thinking aloud" when she analyses her own partial solution "*-c*". Therefore we take Judith not only as person 1, but also as person 2. The first "*-c*" in line 3 corresponds to a concatenation of γ_1 and β_1. The second "*–c?*" in line 3 as well as the following "Yes, sure" in line 4 give clues about a thinking process β_2. This thinking process is metacognitive referred to β_1.

Example 2:

For the analysis of the transcript below the following information is necessary for the reader: The teaching scene is taken from a lesson series *Introduction into the world of computers with registermachines* (textbook: Cohors-Fresenborg, Kaune, & Griep, 1995) at the beginning of grade 7. A *computing network* (mentioned in the following), built with counter-bricks from the didactical material *Dynamische Labyrinthe*[2], can be regarded as a mechanically working flow chart whose correct functioning cannot be proved by concrete examples but only by general considerations.

Jasmin and Elfi have drawn their homework (Cohors-Fresenborg, Kaune, & Griep, 1995, p. 13, exercise 1.10), a construction of a computing network (containing three counter bricks) for the addition of three figures, on a transparency and are presenting their results to the class. The transparency is not like a photograph of the computing network, but an abstract description of the mathematical idea. In the description of the three counter bricks the windows, which in the real bricks show the corresponding storage content, have been kept empty. The pupil Herta asks that some figures are written into the empty windows on the transparency. Then the following discussion begins:

Jasmin: *We didn't put any figures in there deliberately.*
Herta: *But isn't that better, because then you can ….*

[2] The mentioned textbook unfortunately only exists in German. The didactical approach used is also developed in Cohors-Fresenborg (1993).

T: *She says she didn't put in any figures deliberately. May I just ask her why not?*
Jasmin: *Yes, because some people always say that machines only run with certain figures. And if we had put in for example one, two, three, everybody could have said, perhaps they only run with these figures.*
Herta: *I think it is better, Elfi, if you can put some figures in there!*

Jasmin and Elfi suppose that putting in figures is a metaphor for concreteness. They are quite aware of the problem that with definite figures one cannot prove the functioning of an algorithm in general. We know from individual examinations that it is significant for pupils who think in a *predicative* way (Schwank, 1993) to associate the question of general workability of a computing network only with the question of whether a computing network works for the given example with specific figures without considering the arbitrary, general case. After having checked several numerical examples, these pupils quickly assume the attitude that a further examination is not necessary and trust that there will not be a counter example. In the end, they do not really understand the general ability of a computing network to function. Pupils thinking *functionally* (Schwank, 1993), however, are capable of analysing the functionality of computing networks in a prototypical way.

The difficulty of modelling the pupil's text lies in the fact that Jasmin's argumentation is indirect. We therefore first analyse the implied, direct (part-) argumentation: Elfi and Jasmin describe "*some*" of their classmates' imaginations which arise when they "*put figures in*" the transparency. Such classmates then associate that the computing network "*perhaps only works with these figures*". In order to avoid this misconception they "*didn't put any figures in there deliberately*". We know from examinations (according to those of Schwank, 2002) that Elfi prefers a functional way of thinking, whereas Herta prefers a predicative way of thinking. Herta's repeated objection reveals her dilemma - without figures she cannot analyse the effectiveness of the computing network, even with the help of a definite example, while with figures she becomes a victim of exactly that misconception mentioned by Jasmin and Elfi.

This direct argumentation shown above can be interpreted as pre-knowledge which led to the statement: "*We didn't put in any figures deliberately*". When modelling, person 1 stands for the prototype which is described as "*some*" by Jasmin. Elfi/Jasmin are person 2 and i stands for a drawn computing network, which has figures in the counters. s_1 describes the disposition for the misconception of person 1, $\alpha_1(s_1, i)$ describes the misconception of person 1 and β_1 represents the misjudgement about the functioning of the computing network, which is mentioned by γ_1, "*perhaps they only run with these figures*". $\alpha_2(s_2, \gamma_1(\beta_1(\alpha_1(s_1, i))))$ describes Elfi's and Jasmin's idea of person 1's misconception and β_2 their knowledge of its reasons and causes. γ_2 describes the subsequent removal of the figures. The complexity of the formula $\beta_2(\alpha_2(s_2, \gamma_1(\beta_1(\alpha_1(s_1, i)))))$ is a metaphor for the complexity of this metacognitive thinking process described by β_2. This shows that modelling with $\alpha\beta\gamma$–automata, which in the first instance seems to be impractical, is able to make the complexity of the thinking processes involved transparent.

Example 3:

This example shows two representations of one pupil with the same subject matter, i.e. something said about the subject matter and something formally represented. In an advanced mathematics course at grade 13 the question is discussed as to how a function can be differentiated which has been defined as the absolute value of another function.

Soren: *That is the question: Do absolute values stay unchanged during differentiation?*
Actually this should be the case, shouldn't it? Because otherwise it would be something else.
T *Soren has asked: Does an absolute value stay unchanged during differentiation? Whatever this might mean to him.*

Soren:	*Well, you could, if you write it down mathematically* $\vert f(x)\vert = \vert f'(x)\vert$ (he dictates) *Question mark. This would basically be what it means. The bar signs of absolute value round the function or within the function are to be kept* (4 seconds).
T:	*So, are there any more questions?*
Benno:	*I think it is a completely different question when we put it down than what Soren is now* (...)
Soren:	*Yes.*
T:	*Yes, he has dictated it himself.*
Benno:	*This is not what he wanted to ask* (laughs).
T:	*A moment please. Do you mean he has dictated something that he didn't want to ask, something that doesn't really interest him?*
Benno:	*Yes.*
Soren:	*Yes.*
T:	*Tell me then, Benno. What does interest Soren? Benno is now going to interpret what Soren is interested in.*
Benno:	(...) *interested in whether, when you differentiate function f, if, hm,* $\vert f\vert$ *is differentiated, if the bar signs of absolute value of the derivative are also kept around it, and here he has, here is the question if* $\vert f(x)\vert$ *is the same as* $\vert f'(x)\vert$.

As input i we have the problem if, when a function which has been defined as the absolute value of another function is differentiated, the bar signs of absolute value in the representation of the result are to be kept round the derivative of the inner function. We have not put down the subject matter precisely, i.e. formally, as this would not correspond to the mutually-shared knowledge in this form at the beginning of the discussion. Soren is person 1, Benno person 2. Soren's idea of the problem [i] is described by $\alpha_1(s_1, i)$. His remarks [lines 1/2] "*Do absolute values stay unchanged during differentiation? Actually this should be the case, shouldn't it? Because otherwise it would be something else.*" are described by $\gamma_1(\alpha_1(s_1, i))$. The teacher's remark [lines 3/4] causes Soren [at that moment he is in state s_1'] to think about a formal representation of his question. These considerations are described by $\beta_1(s_1')$. His formal representation "$\vert f(x)\vert = \vert f'(x)\vert$?" describes $\gamma_1(\beta_1(s_1'))$. Benno has his own idea $[\alpha_2(s_2, \gamma_1(\beta_1(s_1')))]$ of what Soren has said [lines 1/2, $(\alpha_2(s_2'', \gamma_1(\alpha_1(s_1, i)))]$ and compares it with the idea $[\alpha_2(s_2', i)]$, which he produced by himself.

Benno supposes that what Soren said at first refers appropriately to what was meant and also to what his idea of the subject matter corresponds. Neither does he imply any mistakes as regards his idea of the formal representation. His explanation of the discrepancy between $\gamma_1(\alpha_1(s_1, i))$ and $\gamma_1(\beta_1(s_1'))$ [line 12] is that Soren has not put down formally in writing what he wanted to express verbally. Soren's "yes" in line 16 and Benno's remarks in lines 19 to 21 prove that our analysis is correct.

Example 4:

The acceptance of the validity of the equation $0.\bar{9} = 1$, or in words, that both terms are names for the same figure, keeps causing problems to pupils of all age groups (see e. g. Tall, 1977). The following extract from a transcript shows the struggle of a group of pupils to bring their ideas into line with the representation.

Jens:	*... but Peter says that zero point nine is the same as one and that there always has to be a figure in between ... I mean that there has to be at least one figure between two decimal numbers. And in this case, it is different and therefore it is logical that this should actually be correct.*
Mona:	*Well, I think that there is a figure. It may, however, only be zero point infinite zero and then a one, which means sometime or other* (she laughs) ...
T:	*Can you please write it on the board, how you imagine it?*

Mona:	*No, not really, you can't ... as a figure ... you can't write it down. But logically it would be possible. Because if you have a periodic continued zero and then a one this is as if … I mean ...* (she laughs). *You cannot write it down as a figure, but logically you can imagine it.*
T:	*I would like to know if it is at all clear to everybody what Mona wants to say and about what figure she has been talking. She said: "I cannot write it down."*
Mona:	*Well, I meant, hm, the figure that you would need in order to make zero point periodic continued nine a one is the figure I have been talking about. If there are many, many zeroes and then at some time or other a one, but this doesn't really exist.*
Suse:	*This is what I also wanted to say. There is an infinite number of nines behind the zero, hm, periodic continued nine, ... and she thinks that, that there should be a figure which has exactly as many zeroes, which means an infinite number of zeroes, ... well there is a one in the end so that what you add is one. She is looking for that figure. But you cannot write it down because there would have to be an infinite number of zeroes.*
Suse:	*... If you take for example five and two instead of zero point periodic continued nine and one, you know that they are not the same, as there is a three between them. And regarding zero point period continued nine and one there is no figure between them. You know you cannot write a figure down. Mona, however, thinks that there should be this figure "zero point period continued zero one", but you cannot write it down. Thus this figure doesn't really exist. And therefore this could be right, Mona.*
Mona:	*Well, I only meant: the figure doesn't exist, but logically you could imagine* (laughter) *that it could exist. Therefore ...* (murmuring) *.... but this figure doesn't exist. Hm, that is clear. It won't work.*
T:	*Will you please make your comments so that everybody can hear them, Judith, Juli?*
Juli:	*Yes, Judith and I are just trying to imagine the figure zero point periodic continued one, but this is somehow funny.*
Jens:	*I think that there cannot be a further figure behind a periodic continued figure.*
Suse:	*Yes, that is right, yes, that is true, because the zero, hm, because the periodic line is above it, which means, it is the zero that always repeats. Thus there cannot suddenly be a one behind it. Which means this figure doesn't exist. If it did, the periodic line would have to be above both figures, and then it would be continued in that way: zero point zero one zero one zero one. This would not be the figure Mona meant.*

The starting point of the transcribed discussion mentioned above is Jens' statement that there is no figure between $0.\overline{9}$ and 1. In our analysis, this is taken as input i. Mona is person 1, who picks up Peter's and Jens' idea [$\alpha_1(s_1, i)$] with the help of her pre-knowledge s_1, and thinks about: *"the figure that you would need in order to make zero point periodic continued nine a one"* [She only explains this in lines 13/14]. Presumably, she thinks this figure is bigger than zero and infers that there is another figure between $0.\overline{9}$ and 1. These ideas are described by $\beta_1(\alpha_1(s_1, i))$. She then says: "*It may, however, only be zero point infinite zero and then a one."* [γ_1 in lines 4/5]. Our representation suggests, as well as the sequence of Mona's remarks, that the given figure is supposed to be the figure which should exist. This may, however, not be the case. Even Suse does not note any difference [lines 24/25]. Her first remark [a proof would be the first laughter in line 5] or the teacher's request [line 6] causes Mona [here we take her also as person 2, but in state s_2'], to once more sort out in her mind what she said [$\beta_2(\alpha_2(s_2', \gamma_1(\beta_1(\alpha_1(s_1, i)))))$]. Her laughter in line 5 and her comments in lines 7 and 10 respectively are interpreted as an expression of having found a mistake in her long process of thinking. β_2 describes a metacognitive thinking process referred to her own cognition. In the formula this is expressed by β_1.

The teacher [line 11] causes Mona to repeat her remarks once more. In this new situation (represented by the state s_1') Mona replies with a different formulation [$\gamma_1(s_1')$] "... *If there are many, many zeroes and then at some time or other a one* ..." [line 15]. The process of the discussion, however, shows that she has changed her formulation, but not her opinion. We therefore take this change as a mistake in representation, ["*many, many*" is different to "*an* *infinite number*"] or as a variation in the representation ["many, many" with the meaning of "unlimited", i.e. "an infinite number"]. The reactions of her classmates [Suse, Juli] show that they also take Mona's statement as a variation in representation.

For further analysis, Suse is person 3. In lines 21 to 26, she again refers to what Mona said initially [$\gamma_1(\beta_1(\alpha_1(s_1, i)))$ in lines 4/5]. This forms, together with the pre-knowledge, state s_3, in which Suse notices [$\gamma_3(s_3)$ in lines 23 to 25] that this number does not really exist as you cannot put it down in writing since there is no formal representation. If there is no formal representation of the things having been said, it cannot be of any meaning, i.e. you cannot talk about something existing.

The pupil Juli refers to what Mona has said and gives a formal representation "$0.\bar{0}1$" and tries to imagine the figure represented in that way. The pupil Jens gets into the discussion on this level of representation and criticises that this way of representation is not allowed as there cannot be another figure after a periodic number. This means Jens picks up the form of representation, checks its syntactic correctness and finds a syntactic error. Suse picks up Jens' idea using the semantics of the representation: The periodic line means that there is an infinite number of zeroes, and there cannot suddenly be a one.

The complete dialogue repeatedly deals with representations and ideas, with the presumptions of classmates and with those ideas which other classmates might have [in the case of Mona it is herself] when they have offered a representation. Then the classmates compare them with their own ideas. The pupils have a feeling for the fact that talking, as long as no gradual meaning can be related to the verbal constructions used, sounds meaningful, but does not really have a meaning

The question as to what extent verbal constructions can constitute meaning plays a role when terms [as name replacements] are introduced by denomination operators[3]. Mona introduces the figure that she means by a denomination term. *"Well, I meant, hm, the figure that you would need in order to make zero point periodic continued nine a one"* [lines 13/14]. Now the question arises as to whether the use of a denomination operator is allowed. If the things said were actually the things meant, the figure would be unambiguously defined, i.e. it would be the figure zero. Everything would be in order and the use of the denomination term would be a name replacement of the figure zero. That is, however, not what is meant, as mentioned above. Mona also talks about the past in lines 13/14 before she understands *"…but this doesn't really exist."* When she says in lines 27/28: *"but logically you could imagine that it could exist"*, she presumably means "… it could verbally be formulated in that way". As the figure does not exist, this verbal construction is not allowed *"…that is clear. It won't work"*. [line 28].

The intervention of the teacher in lines 6 and 11/12 has to be understood in such a way that she foresees that the request to create a formal representation, where syntax and semantics are clearly defined, causes a gradual construction of meaning and mere talking becomes obvious. Suse's statements in lines 33 to 37 prove her to be right.

Summary and Consequences

The four teaching scenes analysed by us have understanding processes of mathematical concepts and procedures in common. Persons create an idea of these and give written or oral representations of these ideas which in turn give rise to new ideas. The discovery of

[3] The concept "denomination operator" has been introduced by Whitehead & Russel (1910, pp. 173-186) together with an analysis of its ambiguity. As the pupils have been taught according to the Osnabrück Curriculum they are familiar with the thoughts about denomination operators ("definite article") (Cohors-Fresenborg, Griep, & Kaune, 1992, pp. 39-40).

discrepancies gives cause for analyzing the different processes of representation. This leads to the assessment of a difference between things being said or meant. It can be mistakes in representation (examples 1 and 3) or misconceptions (examples 1 and 2). An interesting aspect was discussed in example 4, in which the idea was analyzed as to whether something said can have a meaning. Moreover examples 3 and 4 show that formalization under the obligation of precision can considerably help to uncover mistakes in the corresponding representation processes.

In our analysis with a theoretical model, the formal representation by $\alpha\beta\gamma$–automata helped with the clarification. It enabled, in particular, working out more easily that the mistakes in the four examples lie at different places in the representation processes. Furthermore, the modelling elucidates the complexity of the cognitive and metacognitive processes involved.

The mutually-shared discursive teaching culture (Cohors-Fresenborg & Kaune, 2003) of the teachers and pupils is evident in each of these four scenes (which originated from lessons given by three teachers). The metaphor (in the sense of Lakoff, 1980) "To what extent do the things said express the things meant?" could be taken as significant for this thinking culture. Metacognition (as shown here with respect to the stated aspects) and discursiveness (as shown here in the analysed teaching situations) are mutually dependent and support each other.

The consequences for the structuring of mathematics lessons are clear. Apart from the previous objectives of making mathematical concepts and procedures a topic in the thinking, learning and teaching processes, and starting from a cognitive-theoretically orientated point of view, cognition of pupils about transformations of representation forms should also be the subject of mathematics lessons. Sjuts (2002) reports on the benefit of implementing different aspects of metacognition into mathematics lessons. An objective of such an enlargement by metacognitive activities is the analysis of thinking processes, ideas and misconceptions in the lessons by classmates. This makes training in discursiveness and flexibility possible, which subsequently offers scope for enhancing the individual thinking processes of the learner.

References

Boekaerts, M. (1996): Teaching Students Self-Regulated Learning: A Major Success in Applied Research. In J. Georgas et al. (Eds.), *Contemporary Psychology in Europe* (pp. 245-259). Seattle: Hogrefe & Huber.

Boekaerts, M. (1999): Self-Regulated Learning: where we are today. *International Journal of Educational Research, 31*, 445-457.

Cohors-Fresenborg, E. (1993): Registermachine as a Mental Model for Understanding Computer Programming. In E. Lemut, B. Du Boulay, & G. Dettori (Eds.): *Cognitive Models and Intelligent Environments for Learning Programming. NATO ASI Series F. Volume 111*, (pp. 235-248). Berlin: Springer.

Cohors-Fresenborg, E. (2001): Mathematik als Werkzeug zur Wissensrepräsentation: das Osnabrücker Curriculum. *Der Mathematikunterricht*, 1, 5-13.

Cohors-Fresenborg, E., Griep, M., & Kaune, C. (1992): Sätze aus dem Wüstensand und ihre Interpretationen. Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Cohors-Fresenborg, E. & Kaune, C. (1993): Zur Konzeption eines gymnasialen mathematischen Anfangsunterrichts unter kognitionstheoretischem Aspekt. *Der Mathematikunterricht*, *3*, 4-11.

Cohors-Fresenborg, E. & Kaune, C. (2003): Unterrichtsqualität: Die Rolle von Diskursivität für "guten" gymnasialen Mathematikunterricht. In *Beiträge zum Mathematikunterricht 2003* (pp. 173-180). Hildesheim: Franzbecker.

Cohors-Fresenborg, E., Kaune, C., & Griep, M. (1995): *Einführung in die Computerwelt mit Registermaschinen. Textbuch für Schüler und Handbuch für Lehrer*. Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Cohors-Fresenborg, E., Schwippert, K., & Klieme, E. (2002): The Osnabrück Curriculum: Mathematics as a Tool for the Representation of Knowledge – An Evaluation Study on the Basis of TIMSS-Instruments. In H.G. Weigand, N. Neill, A. Peter-Koop, K. Reiss, G. Törner, & B. Wollring (Eds.), *Developments in Mathematics Education in German-speaking Countries, Selected Papers from the Annual Conference on Didatics of Mathematics, Potsdam 2000* (pp. 44-55). Hildesheim: Franzbecker.

De Corte, E. (1995): Fostering Cognitive Growth: A Perspective From Research on Mathematics Learning and Instruction. *Educational Psychologist*, 30 (1), 37-46.

De Corte, E., Verschaffel, L., & Op't Eynde, P. (2000): Self-regulation: A characteristic and a goal of mathematics education. In M. Boekaerts, P. R. Pintrich, & M. Zeidner, *Handbook of self-regulation* (pp. 687-726). San Diego: Academic Press.

Griep, M. (2001): Berichtigung von Lernzielkontrollen als Anlass für metakognitive Aktivitäten. In *Beiträge zum Mathematikunterricht 2001* (pp. 229-232). Hildesheim: Franzbecker.

Kaune, C. (1999): Förderung metakognitiver Aktivitäten durch geeignete Aufgabenstellungen. In *Beiträge zum Mathematikunterricht 1999* (pp. 281-284). Hildesheim: Franzbecker.

Kaune, C. (2001a): Merkmale eines konstruktivistischen Unterrichtsskripts und eine Analyse dazugehöriger Lehr- und Lernprozesse. *Der Mathematikunterricht*, *1*, 14-34.

Kaune, C. (2001b): Weiterentwicklung des Mathematikunterrichts: Die kognitionsorientierte Aufgabe ist mehr als die "etwas andere Aufgabe". *Der Mathematikunterricht*, *1*, 35-46.

Lakoff, G. (1980): *Metaphors we Live By*. Chicago: The University of Chicago Press.

Rödding, W. (1977): On the Aggregation of Preferences. *Naval research Logistics quarterly*, *25/1*, 55-79.

Schoenfeld, A. H. (1992): Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Schwank, I. (1993): On the Analysis of Cognitive Structures in Algorithmic Thinking. *Journal of Mathematical Behavior, 12, 2*, 209-231.

Schwank, I. (2002): Analysis of eye-movements during functional versus predicative problem solving. In J. Novotna (Ed.), *European Research in Mathematics Education II. Selected papers from the 2nd Conference of the European Society for Research in Mathematics Education* (pp. 489-498). Prag: Charles University.

Sjuts, J. (2001): Metakognition beim Mathematiklernen: das Denken über das Denken als Hilfe zur Selbsthilfe. *Der Mathematikunterricht*, *1*, 61-68.

Sjuts, J. (2002): Metacognition in Mathematics Lessons. In H.G. Weigand, N. Neill, A. Peter-Koop, K. Reiss, G. Törner, & B. Wollring (Eds.), *Developments in Mathematics Education in German-speaking Countries, Selected Papers from the Annual Conference on Didatics of Mathematics, Bern 1999* (pp. 77-88). Hildesheim: Franzbecker.

Whitehead, A. N. & Russell, B. (1910): Principia Mathematica, Vol. I. Cambridge: Cambridge University Press.