DIAGRAMS AS MEANS AND OBJECTS OF MATHEMATICAL REASONING

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According to Peirce a great part of mathematical thinking consists in observing or imagining the outcomes and regularities of manipulations of all sorts of diagrams. This tenet is explained and substantiated by expounding the notion of diagram and by analyzing several specific cases from different parts of mathematics.

INTRODUCTION

Ch. S. Peirce (3.363 in Collected Papers) has made, among others, the following comment on a basic feature of mathematics:

It has long been a puzzle how, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while, on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. There have been various attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation. Deduction thus consists of: constructing an icon or diagram the relationships of whose parts shall present a complete analogy with those of the parts of the object of reasoning; experimenting upon this image in the imagination, and observing the result so as to discover unnoticed and hidden relationships among the parts. The very idea of the art of algebra is that it presents formulae which can be manipulated and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries, which are embodied in general formulae. These are patterns, which we have the right to imitate in our procedure, and are the icons *par excellence* of algebra.

The term diagrammatic reasoning (henceforth d.r.) is commonly used to describe this way of thinking. By analyzing various examples from different mathematical areas the potential of d.r. is investigated, thereby, to some extent, substantiating the Peircean tenet. This, therefore, is an epistemological and theoretical analysis which yet is taken to suggest implications for all kinds of actual mathematical activity. It will then be the purpose of another paper to point out inherent limitations to d.r. I only indicate that there are cases where in principle the mathematical notion is not amenable to diagrammatic methods and one has to stick to a kind of conceptual reasoning based on linguistically or metaphorically prescribed properties.

I have chosen to stick to the term "diagram" as it has been used by Peirce and others though being aware that this term might cause some misunderstandings and arouse inadequate expectations. First of all, the reader should dismiss all narrower geometric connotations. This can be seen already from the above reference to Peirce who includes formulae of all kinds into his notion of diagram (or icon). What is important is the spatial structure of a diagram, the spatial relationships of its parts to one another and the operations and transformations of, and with, diagrams. The constituent parts of a diagram can be any kind of inscription like letters, numerals, special signs or geometric figures. This will be elaborated in more detail below.

Mathematical thinking and reasoning was, and is, often considered to be a purely mental activity which is completely separated from any empirical investigations. The "remoteness from sense experiences" is emphasized by Gödel (1964) or for instance by Poincaré (1905) who speaks about the aesthetic character of mathematics "despite the senses not taking part in it at all ". The symbols and diagrams (in our sense) are considered rather as a crutch or as a means of expressing genuine mathematical ideas and notions. An extreme position in this regard had been taken by Brouwer in his conception of intuitionism. For him, mathematics is completely independent of any representational means and in particular language. These representational means only serve to communicate mathematical content and do not have any influence on the language. Symbols thus play only an auxiliary role and can in principle be dispensed with, at least by the individual mathematician in his/her creative process, see for instance Brouwer (1907, 1912). In a Platonistic framework, diagrams, like geometric figures, are just instances of general and abstract ideas which may be investigated by manipulating the representing diagrams. Yet there are positions which emphasize the role of symbols and diagrams in their own right in a similar way to how it is done here or by Peirce. An example is the so-called formal arithmetic which was developed by E. Heine and J. K. Thomae, cf. Epple (1994). Here the meaning of arithmetic is considered to reside in its operation or calculation rules rather than in its reference to the cardinalities of finite sets. The formalistic stance as developed by Hilbert takes a similar position: mathematics studies symbolic structures based on axiom systems defining the respective terms, operation and inference rules. However it should become clear from the examples given below that the stance taken here is much broader than classical formalism by comprising also what for Hilbert was the essential mathematics, especially finite and discrete mathematics before and outside of any axiomatization. There is also a tradition of considering (part of) mathematics in analogy to rule-based games like chess. This view was then extended by the later Wittgenstein in his concept of "language games" (see the exposition in Epple (1994)). It is interesting that in all these discussions the viewpoint taken by Peirce is not taken into account at all. In any case, the topic treated here was in one way or another always of interest to mathematicians and philosophers though the mainstream position was rather to confer only an auxiliary role to symbols and diagrams. In contrast to that, in this paper, symbolic and diagrammatic structures are genuine objects of mathematical, i.e. diagrammatic reasoning which result in theorems expressing general properties and relationships of those structures, i.e. diagrams in the sense used here. For a related position see Detlefsen and Luker (1980) who stress the empirical character of any kind of computation which as they say (p. 813/814) "is always an experiment … operating on symbols and not on the things for which the symbols may be taken to stand".

DIAGRAMS

Despite the fact that in this paper the stance taken is that mathematical development to a great part consists in the design and intelligent manipulation of diagrams, no general definition of the notion of diagram is given but rather several examples and descriptive features are presented. Generally speaking, diagrams are kinds of inscriptions of some permanence in any kind of medium (paper, sand, screen, etc). These inscriptions are mostly planar but some are 3-dimensional like the models of geometric solids or the manipulatives in school mathematics. Mathematics at all levels abounds with such inscriptions: Number line, Venn diagrams, geometric figures, Cartesian graphs, point-line graphs, arrow diagrams (mappings), arrows in the Gaussian plane or as vectors or commutative diagrams (category theory). However there are also inscriptions with a less geometric flavour: arithmetic or algebraic terms, function terms, fractions, decimal fractions, algebraic formulas, polynomials, matrices, systems of linear equations, continued fractions and many more. There are common features to some of these inscriptions which contribute to their diagrammatic quality as understood here. Nevertheless I emphasize that not all kinds of inscriptions which occur in mathematical reasoning, learning and teaching have a diagrammatic quality. Quite a few of what are taken as visualizations or representations of mathematical notions and ideas do not qualify as diagrams since they lack some of the essential features. Mostly this is the precise operative structure which for genuine (Peircean) diagrams permits and invites their investigation and exploration as mathematical objects. On the other hand, diagrams are of such a wide variety that a generic definition appears both impossible and impractical. Accordingly, the various kinds of diagrams in a Wittgensteinean sense are connected by family resemblances and by the ways we use them. Some widely shared qualities of diagrams are proposed in the following:

- Diagrammatic inscriptions have a structure consisting in a specific spatial arrangement of, and spatial relationships among, their parts and elements. This structure is often of a conventional character.
- Based on this diagrammatic structure there are rule-governed operations on and with the inscriptions by transforming, composing, decomposing and combining them (calculations in arithmetic and algebra, constructions in geometry, derivations in formal logic). These operations and transformations could be called the internal meaning of the respective diagram.
- Another type of conventionalized rule governs the application and interpretation of the diagram within and outside mathematics, i.e. what the diagram can be taken to denote or model. These rules could be termed the external or referential meaning (algebraic terms standing for calculations with numbers, a graph depicting a network or a social structure). The two meanings closely inform, and depend upon, each other.
- Diagrammatic inscriptions (can be viewed to) express relationships by their very structure from which those relationships must be inferred based on the given operation rules. Diagrams are not to be understood in a figurative but in a relational sense (like a circle expressing the relation of its peripheral points to the midpoint).
- Diagrammatic inscriptions have a generic aspect which permits construction of arbitrary instances of the same type of diagram. This leads to, among others, consideration of the totality of all diagrams of a given type (like all triangles, all decimal numbers).
- There is a type-token relationship between the individual and specific material inscription and the diagram of which it is an instance (as between a written letter and the letter as such).
- Operations with diagrammatic inscriptions are based on the perceptive activity of the individual (like pattern recognition) which turns mathematics as d.r. into a perceptive and material activity.
- Diagrammatic reasoning is a rule-based but inventive and constructive manipulation of diagrams to investigate their properties and relationships.
- Diagrammatic reasoning is not mechanistic or purely algorithmic, rather it is imaginative and creative. Analogy: the music by Bach is based on strict rules of counterpoint but yet is highly creative and varied.
- Many steps and arguments of diagrammatic reasoning have no referential meaning nor do they need any.
- In diagrammatic reasoning the focus is on the diagrammatic inscriptions irrespective of what their referential meaning might be. The objects of diagrammatic reasoning are the diagrams themselves and their already established properties.
- Diagrammatic inscriptions arise from many sources and for many purposes: as models of structures and processes, by deliberate design and construction, by idealization and abstraction from experiential reality, etc. Accordingly they are used for many purposes.
- Efficient and successful diagrammatic reasoning presupposes intensive and extensive experience with manipulating diagrams. A widespread "inventory" of diagrams, their

> properties and relationships supports and occasions the creative and inventive usage of diagrams. Analogy: an expert chess-player has command over a great supply of chess-diagrams, which guide his or her strategic problem solving. Consequence: learning mathematics has to comprise a great variety of diagrammatic knowledge.

USING DIAGRAMS

Another dimension for explaining the notions of diagram or of d.r. is through the uses made of them in mathematics. Firstly, the most widespread usage is to apply the admissible operations and transformations to solve a given task. This comprises calculating a numeric value, solving equations, constructing a proof in geometry, finding a derivation (in formal logic) and many others. Thereby one operates with the inscriptions by exploiting and observing their structure and its changes. Thus this is a material and perceptive activity guided by the diagrammatic inscriptions. It is as in other material actions: to be successful one has to have acquired an intimate experience with the objects one is operating with, which here are the inscriptions. This is crucial in comparison to abstract or conceptual knowledge. There are algorithmic operations (consider the Gauss algorithm) but much of d.r. is highly creative because the appropriate operations with the diagrams have first of all to be devised and deployed.

This first type of use is the only one which I want to subsume as diagrammatic reasoning. It is essential that the diagrammatic inscriptions themselves are the objects of the activity which produces knowledge about, and experience with, the diagrams. Secondly, I will call the other usages of diagrammatic inscriptions representational. The first kind of representational use is when a diagrammatic inscription is taken as a model for some other material or virtual structure from any science including mathematics itself or from any practice. This is captured by terms like application of mathematics or mathematization. It is not the place here to discuss that any further. I only remark that therein lies an important source for the design of diagrams which then become the topic of d.r. within mathematics, see Dörfler (2000). A second type of representational use is widespread in mathematics education: to use diagrams as representations of abstract objects constructed by the learner. The diagrams are taken as a means for mental or cognitive constructions and thus have little interest in themselves. They are then more a kind of methodological scaffold, possibly unavoidable but to be dismissed when successful. This is diametrically opposed to d.r. where the focus is on the diagrams and operations themselves as the objects of study and not on their doubtful mediation with virtual objects. In this representational view mathematics is a predominantly mental activity supported by diagrams whereas mathematics as d.r. essentially is a material and perceptual one. This does not reduce mathematics to meaningless symbol manipulations since the diagrams have meaning through their structure, their operations and transformations and of course via their applications. This holds for all diagrams as considered here in a way completely analogous to how geometric figures can have meaning.

DIAGRAMMATIC REASONING: EXAMPLES

For each of the mathematical topics referred to below, the reader could consult any standard textbook. Browsing through any book on mathematics reveals an abundance of diagrams but it does not show how they are used or exploited. These examples intend to highlight d.r. as a specific perspective on diagrams. This perspective takes the view that diagrams of many different kinds are the objects of mathematical activities. Those activities consist of exploring properties of the diagrams and of various operations with them, see also Dörfler (2001). In the sense of Peirce, an important aspect of this kind of mathematical activity is then the observation of the impact and outcome of empirical activity. The detected regularities give rise to concepts which describe specific properties of the diagrams. Invariant relationships between those properties and the respective concepts are formulated as theorems. It should be mentioned that there is the (mostly realized) possibility of then arguing with the concepts and the already proven theorems without explicit recourse to the underlying diagrams.

In their famous book "Grundlagen der Mathematik", Hilbert and Bernays analyze operations with arrays of strokes (or points) the observation of which leads to much of what is taken to be properties of natural numbers. The natural numbers are interpreted as types of arrays of strokes, two of which are of the same type if they can be matched one by one. Addition and multiplication appear as operations with these arrays which clearly show a diagrammatic character. Properties like evenness and oddness are observable qualities of such diagrams in the form of specific arrangements of the strokes. A good example of d.r. is the statement that the sum of two odd numbers (diagrams) is even. This results from observing the combining of two odd diagrams in an appropriate way. In this kind of d. r. that statement is a way of reporting one's observations (and not a statement about abstract objects):

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********                 *********                *****************
*******       plus      **********      gives     *****************
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Here the generic character of the diagrams is an important feature which provides the generality of the assertions about the diagrams. Similarly d.r., by inspection of the following diagram,

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 ********                  *****                   *************
 ********      plus        ****        gives       ************
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implies the rule "even + odd = odd". In the same manner the corresponding rules for multiplication are obtained by d.r. with rectangular (product) arrangements. That any number either is prime or divisible by a prime number similarly results from d. r. by using rectangular arrangements of the arrays of strokes. Another kind of d.r. is also possible here using algebraic expressions and their properties. For "odd + odd = even" this could be: $(2m+1)+(2n+1)=2m+2n+2=2(m+n+1)$. Again, this d.r. is a manipulation of diagrams and an observation of the outcome by knowing that a diagram of the form $2(\ldots)$ is equivalent to evenness. Here we have the very common phenomenon that diagrams of very different kinds describe or model each other. This is also termed as "isomorphic representations" in the representational view of diagrams. Here this simply means that one can translate between the two kinds of diagrams in such a way that the operations and properties of both uniquely match each other.

In this vein, a Ferrer's graph is a diagram for a partition such as $12=2+2+3+5$ in the form of an array of lines of points:

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**            transposed        ****
**                              ****
***                             **
*****                           *
                                *
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By inspection, various relations can then be observed and formulated as properties of partitions, see Liu (1968). For instance, one can exchange the lines and columns in a Ferrer's graph which in the example corresponds to $12 = 4 + 4 + 2 + 1 + 1$. A case of d. r. is then the observation that to any partition with at most m parts corresponds one with no part greater than m and vice versa. Therefore the number of those two kinds of partitions is the same.

The young Gauss is reported to have found the sum of the first 100 positive integers by thinking of those numbers being written down in the following way:

1	2	3	4	…	49	50
100	99	98	97	…	52	51

and adding the two numbers to get 101 in each of the 50 columns. Thus the sum is 50 times 101 which equals 5050. I consider this as a good example of d.r. since it depends on the structure of the rectangular array which can be used for every even number $2n$ in the form:

1	2	3	4	…	$n-1$	n
$2n$	$2n-1$	$2n-2$	…	…	$n+2$	$n+1$

Here each of the n columns adds up to $2n+1$ and the sum total is $n(2n+1)$. This is gleaned from the diagram by observing certain regularities and relationships. This type of diagram permits, in an analogous way, the summing of the integers from $k+1$ up to $k+2n$:

$k+1$	$k+2$	...	$k+n$
$k+2n$	$k+2n-1$	...	$k+n+1$

Here each column adds up to $2n+1+2k$ and thus the sum is $n(2n+1+2k)$ where $k=0$ corresponds to the original diagram.

The creative act consists in inventing this diagram and, most likely, results from extensive experience with number relations of all kinds. There are other diagrams which can be used in this case and also for any number of integers. Consider the following pattern:

$$\begin{array}{c} 1 \quad + \quad 2 \quad + \quad 3 \ + \ldots + (m-1) + m + \\ + m + (m-1) + (m-2) + \ldots + \ 2 \ + \ 1 \end{array}$$

where each of the m columns adds to $(m+1)$ and thus the sum $1+2+3+\ldots+m = (1/2)m(m+1)$. One could say that the above diagram results from the diagram $1+2+3+\ldots+m$ by an appropriate transformation. This might even become more diagrammatic when imagining the whole process as being carried out with collections of, say, pebbles which for $m=6$ leads to the diagram:

* * ** ** ***

*	*	**	**	***	***
***	**	**	*	*	*
***	***	**	**	*	

Combining column-wise gives a union of 6 sets of 7 elements which has twice the number of elements one is looking for. The main activity is thereby the experimental investigation of diagrams whereby one has to stick to the operational rules (either for finite sets or numerals). The latter diagram can also be organized in a well-known pattern with 6 rows of 7 items each:

○	*	*	*	*	*	*
○	○	*	*	*	*	*
○	○	○	*	*	*	*
○	○	○	○	*	*	*
○	○	○	○	○	*	*
○	○	○	○	○	○	*

The generic and general quality of all those diagrams which permit consideration of the general case in a specific example is very important. Even an inductive argument however will use (algebraic) d.r. of a form like:

$$(1+2+\ldots+n)+(n+1)=(1/2)n(n+1)+(n+1)=(1/2)(n(n+1)+2(n+1))=1/2(n+1)(n+2)$$

What is, in my opinion, not valued appropriately is the role of perception, of the material activity (with the inscriptions) and of pattern recognition. Too much emphasis is laid on the "ideas" and on purely mental activity, but I assert that the ideas emerge from the observation and manipulation of the diagrams, of course always under a specific perspective. This perspective is expressed by the operation and transformation rules which are permitted for manipulating the inscriptions. In addition, I think one should resist the temptation to view that kind of d.r. only as a means of discovering or justifying properties of – in this case – natural numbers. I prefer to say that the discourse of natural numbers expresses the observations made about the diagrams and that it describes properties of, and relationships among, the latter. For this a great many diagrams of different kinds are used and investigated.

Elementary linear algebra offers many striking examples of d.r. Consider the following formula $(A\alpha,\beta)=(\alpha,A'\beta)$ where α,β are (column-)vectors of R^n, A is an $n\times n$ matrix, A' its transpose, and (. , .) is the usual inner product. I present two different kinds of d.r. which demonstrate the formula by observing transformations and patterns of diagrams of linear algebra. The first takes recourse to the components a_i and b_i of α and β and the elements a_{ij} of A. The left side then gives rise to the following diagram:

$$\begin{pmatrix} a_{11}a_1 & + & a_{12}a_2 & + & \ldots & + & a_{1n}a_n \\ a_{21}a_1 & + & a_{22}a_2 & + & \ldots & + & a_{2n}a_n \\ \ldots & & \ldots & & \ldots & & \ldots \\ a_{n1}a_1 & + & a_{n2}a_2 & + & \ldots & + & a_{nn}a_n \end{pmatrix} \text{ inner product with } \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{pmatrix}$$

which according to the rules transforms into:

$$\begin{array}{l}(a_{11}a_1 + a_{12}a_2 + \ldots + a_{1n}a_n)b_1 + \\ (a_{21}a_1 + a_{22}a_2 + \ldots + a_{2n}a_n)b_2 + \\ \ldots \\ (a_{n1}a_1 + a_{n2}a_2 + \ldots + a_{nn}a_n)b_n \quad .\end{array}$$

This can be transformed according to the rules of elementary algebra or by reading the rectangular schema columnwise into the diagram:

$$\begin{array}{l}(a_{11}b_1 + a_{21}b_2 + \ldots + a_{n1}b_n)\cdot a_1 + \\ (a_{12}b_1 + a_{22}b_2 + \ldots + a_{n2}b_n)\cdot a_2 + \\ \ldots \\ (a_{1n}b_1 + a_{2n}b_2 + \ldots + a_{nn}b_n)\cdot a_n\end{array}$$

which is a pattern resulting from

$$\begin{pmatrix} a_{11}b_1 + a_{21}b_2 + \ldots + a_{n1}b_n \\ a_{12}b_1 + a_{22}b_2 + \ldots + a_{n2}b_n \\ \ldots \\ a_{1n}b_1 + a_{2n}b_2 + \ldots + a_{nn}b_n \end{pmatrix} \text{ inner product with } \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

The left side is just $A^{'}\beta$ and thus $(\alpha, A'\beta)$ is the final result.

It becomes clear from this that the proof of the formula is a sequence of manipulations of diagrams which relies heavily on perceiving their structural properties. Nowhere is the (external or referential) meaning of the diagrams needed. What is needed however is experience and fluency with manipulating diagrams and the ability to recognize patterns in them. Another kind of d.r. proceeds as follows by exploiting the formula $(\gamma,\delta)=\gamma'\delta$ for (column-)vectors γ,δ where the product on the right is matrix multiplication:

$$(A\alpha,\beta)=(A\alpha)^{'}\beta=(\alpha^{'}A^{'})\beta=\alpha^{'}(A^{'}\beta)=(\alpha,A^{'}\beta)$$

Again, this consists in no more (but also no less) than effective manipulation of diagrams (≡ formulae here). We know that this is difficult for students, possibly not because of a lack of conceptual understanding but because of a lack of experience in manipulating diagrams. This is very much hands-on experience which presupposes concrete and material activity with the diagrams by investigating their properties through "calculations". What also transpires from these observations is that the reliability and security of mathematical reasoning (partly) resides in the perception of structural properties of diagrams.

Many proofs in calculus show a great deal of d.r. The main steps consist in transformations and combinations of inequalities according to the usual ε-δ-definitions of limits, continuity, etc. The premises and the conclusion likewise correspond to diagrams with a specific interpretation and the idea of the proof is how to transform the former into the latter. This is a creative and at least partly perceptual activity. The diagrams which occur in these cases are like the following:

$$|f(x)-f(a)|<e/2 \quad |g(x)-g(a)|<e/2$$

and

$$|(f+g)(x)-(f+g)(a)|=|f(x)-f(a)+g(x)-g(a)|<$$
$$<|f(x)-f(a)|+|g(x)-g(a)|<e/2+e/2=e$$

Here d.r. is the manipulation and combination of what are considered as inequalities and this d. r. is known to be dependent on much (empirical) experience with those diagrams. Thus d.r. is based on a sort of manipulative skill with diagrams (and possibly less on understanding abstract objects). Finding a proof often consists in exploiting diagrammatic relationships which is very similar to geometric proofs using auxiliary elements and various transformations of figures.

There are mathematical theories like graph theory (see Bondy and Murty, 1976) the objects of which are (types of) diagrams. Of course, these theories furnish prominent examples for d.r. Thus the objects of graph theory are diagrams made up between pairs of points (vertices) and lines (edges) like the following: Choose seven points a,b,c,d,e,f,g in the plane and draw edges (connecting lines not necessarily straight) between the following pairs: a and b, a and c, a and e, b and d, c and d, e and f, f and g, g and e, g and a.

The diagrammatic character might be less evident in the case of polynomials (over some ring R). A polynomial like, say, $5x^3-2x^2+8x+1$ can be considered as a diagram with a conventional structure. Indeed, the spatial relations of its parts are crucial for its being a polynomial and all the operations with polynomials depend on these structural and diagrammatic relations within the polynomial as a diagram. The results of Kirshner (1989) indicate that the operations with algebraic terms for many students are based on visual (i.e. diagrammatic) characteristics of these terms.

The proofs of properties of operations with polynomials consist in transformations of the polynomial-diagrams according to agreed rules. For instance, that the degree of the product of two polynomials is the sum of the two respective degrees results from observing the structure of the product polynomial, i.e. from d.r. It is also by skillfully manipulating the polynomial diagrams that one realizes: If $p(a)=0$ then $(x-a)$ divides $p(x)$, as is visible from the standard proof in any textbook. This underlines the material, experiential and observational aspect of doing mathematics.

These rules can be changed, for instance by "calculating" modulo a fixed polynomial p. This results in a collection of specific diagrams (the polynomials of degree less than $\deg p$). These enjoy many diagrammatic properties, among them a specific kind of product which gives rise to a ring structure for these diagrams. If the ring R is finite, say $R=Z_4$, then all these diagrams can be listed and their algebraic behaviour can be observed by inspection. Thus this is not "abstract" algebra but diagrammatic algebra! Take $p(x)=x^2+2$ (over Z_4), then all polynomials modulo p essentially are the constants and all $ax+b$ with a from $\{1,2,3\}$ and b from $\{0,1,2,3\}$, i.e. we have 16 diagrams. For instance as a sum and a product using $x^2=-2=2$ one obtains: $(2x+3)+(x+2)=3x+1$ and $(2x+3)*(x+2)=3x+2$. Again this is merely manipulation of diagrams according to conventional rules and the mathematics of it states regularities and invariants of these operations and manipulations.

As another example consider the theorem that any polynomial $p(x)$ irreducible over a given field K has a root in a field $K_1 \supset K$ (see Cigler, 1995). This sounds as being about abstract objects but in fact it is about the possibility of designing diagrams of a specific sort. This design runs as follows: $K[x]$ designates all diagrams which are polynomials over K. Calculating in $K[x]$ modulo $p(x)$ amounts to reducing any polynomial h in $K[x]$ by (polynomial) multiples of p (for instance by dividing h by p to get the remainder as the polynomial equivalent to h modulo p - again a diagrammatic operation). When calculating in $K[x]$ modulo $p(x)$ the polynomials of degree less than $\deg p$ enjoy all the properties of a field which we denote by K_1. It contains K in the form of polynomials of degree 0. According to the rules for K_1 we have, for the diagram, x as an element of K_1 such that $p(x)=0$ (p is simultaneously a polynomial over K_1). The crucial point is not that there is a symbol x with $p(x)=0$ but that this x belongs to a collection of diagrams with which we can calculate according to the rules laid down for a field. The "existence" of a root is the design of a diagram with specific properties and as a member of a collection of diagrams (all the polynomial diagrams of degree less than $\deg p$) for which addition and multiplication are realized as diagrammatic manipulations and which show all the properties of a field such as the real numbers.

A special case of this construction is of course the complex numbers. Here a more direct construction of diagrams is common by introducing the "diagram" i which satisfies $i^2+1=0$, i.e. this i is the x of the general design with $p(x)=x^2+1$ and $K=R$ (real numbers). Here $K[x]$ modulo (x^2+1) are just the diagrams (polynomials) $a+ib$ $(a,b \in R)$ with the usual well-known operations which are the operations modulo x^2+1, i.e. exploiting $i^2+1=0$. Thus the essential point is the design of diagrams $a+ib$ and operations with them which satisfy a set of rules, namely mathematics as diagrammatic design (and not as the invention of abstract objects). This design can be carried out also if a and b are taken from any unit ring. In the case of, say, Z_4 then all the designs $a+ib$ can be listed and their properties inspected visually. The resulting diagrams (i.e. the complex numbers over Z_4) are the following ones: $0,1,2,3,i,2i,3i,1+i,1+2i,1+3i,2+i,2+2i,2+3i,3+i,3+2i,3+3i$. Using $i^2+1=0$ or $i^2=-1$ or $i^2=3$, one can easily "calculate" which is a form of d.r. For instance: $(1+i)*(2+3i)=2+2i+3i+3i^2=2+5i+9=3+i$. This because we follow the diagrammatic rules for Z_4 and the generally accepted rules from elementary algebra. This resonates completely with the statement by Peirce.

CONCLUSION

Based on these examples a tentative response to Peirce's statement could be that by d.r. mathematicians (and learners of mathematics as well) investigate diagrams designed (in the form of inscriptions) as empirical and material objects. This leads to the detection of (sometimes even surprising) properties, relations, regularities and invariants. Some of those properties and relations (the axioms) are distinguished as characterizing the respective class of diagrams and taken as the basis for deductive reasoning. The latter, in this view, is another way of talking about the diagrams by using concepts incorporating their various properties and relations.

Much more could and should be said about d.r. For instance,

1. How diagrams are designed to describe certain relations of, or operations with, other kinds of diagrams which results in a layered system of d.r.
2. How mathematics develops a conceptual language to talk about diagrams and how reasoning then occurs in this language, and
3. Which should or could be the consequences for learning and understanding mathematics. Does the idea of d.r. point to the necessity of the acquisition of manipulative skills in operating with diagrams, i.e. a kind of material and observational experience as a prerequisite for doing mathematics? Will this constitute a new but fundamental role for "calculations" as a basis for d.r.?

All this will be the topic of further research.

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