Argumentations in proving discourses in mathematics classrooms

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This paper focuses on argumentations in proving discourses in mathematics classrooms. Examples of different types of argumentations that were observed in a comparative study of French and German lessons on the Pythagorean Theorem are presented. These illustrate different argumentations within two kinds of proving discourse. In one case the argumentation is characterised as intuitive-visual, and in the other as conceptual. It is suggested that there is an epistemological basis for the differences between these discourses and that comparative studies like this one provide a way to investigate these further.

In this article two examples of argumentations are discussed: one in which one argues visually and the other in which one argues conceptually. The argumentations are collective processes in which students and teacher develop the proof together. Such collective proving processes have received little attention in educational research to date, except in the work of Sekiguchi (1991) and Herbst (1998). Herbst examines discursive and socio-didactical conditions that make the public formulation and validating of statements in the class possible. By stressing the public negotiation of knowledge, Herbst makes visible the regulation and the division of labour between teachers and students which characterize these situations (Herbst 2002). This makes it clear that social rules are necessary in collective processes and can at the same time hinder learning processes. In the research reported here epistemological aspects rather than social aspects of collective proving processes are the focus of interest. The central question is: *By what kinds of argumentations are proving discourses created and shaped in mathematics classes?*

Proof and argumentation

Research on proof and proving has focused on different types of reasoning and arguments in proving processes of students (Reiss et al. 2002, Reid 1999 and 1995, Healy & Hoyles 1998, Balacheff 1988). Balacheff distinguishes between *pragmatic* and *intellectual* proofs (Balacheff 1988 and 1999). In pragmatic proofs, statements are validated by concrete or mental actions; proofs of this type include naive empiricism (*empirisme naïf*) or crucial experiments (*expérience cruciale*). On the other hand, arguments in intellectual proofs are based on concepts and language. Intellectual proofs are not necessarily formal, but they are detached from concrete actions. Generic examples (*l'exemple générique*) or mental experiments (*l'expérience mentale*) can be the basis of these kinds of proofs. I observed another kind of proof in classroom practices, in which one argues visually and at the same time mentally, independent of the concrete representation. Here visual proving combines arguments based on generic examples and mental experiments. In the literature, proofs of this kind are called *anschauliche Beweise* (Kautschitsch & Metzler 1989) or *preformal* proofs (Blum & Kirsch 1991). In related research it has not, as yet, been analysed by what kinds of argumentations proofs of this type are created and shaped in the classroom. Analyses of classroom discourses and of collective argumentations are important to come to a better understanding of these kinds of proof and their importance in teaching.

In other research Toulmin's model of arguments (1958) turns out to be a powerful tool for characterising different types of arguments, including formal and informal ones in class (Pedemonte 2002, Krummheuer 1995). Pedemonte uses the model to characterise abductive and deductive types of arguments in proving processes of students and analyses the *cognitive unity* or *break* in those processes (Pedemonte 2002). The functional analysis of arguments exposed in the Toulmin model turns out to be equally fruitful in the research presented here. The analyses of statements in terms of their function within an argument, i.e. as data (D), conclusions (C), warrants (W) and backings (B), helped particularly in singling out distinct arguments in proving discourses in ordinary classroom situations.

Rav reminds us that mathematical proofs involve *sequences of claims* where the passage from one claim to another is generally *not formal*.

> A proof in mainstream mathematics is set forth as a sequence of claims, where the passage from one claim to another is based on drawing consequences on the basis of meanings or through accepted symbol manipulation, not by citing rules of predicate logic. (Rav 1999, p. 13).

This point seems to be particularly relevant in proving processes in learning-teaching contexts (Hanna 1989, Wittmann & Müller 1988). When proving is not formal deductive reasoning it is not evident that passages from one claim to another can be described purely as the recycling of conclusions of one claim as data for the next. As well, proving discourses in class are complex, distinct arguments overlap in these discourses, and the overall structure cannot be described only by single steps. This raises the questions: What kinds of passages from one claim to another are to be found in informal proving discourses? How do these passages and the whole argumentation structure intertwine? In what ways are these passages negotiated in class? This paper attempts to describe different types of argumentations in proving discourses in class by analysing non-formal passages within multi-step argumentations.

Methodological considerations

The empirical investigation on which this paper is based involved comparative analysis of proving processes in six French and six German classes. The differences in instructional practices across cultures allowed me to identify different proving processes in these classroom contexts. Such a comparative procedure corresponds to work in comparative education research (e.g., Alexander et al. 1999; Broadfoot et al. 2000; Pepin 1999, 1997). In mathematics education the potential of international comparative research on instructional practices is only beginning to be realised, although comparative research in many disciplines, for instance in political science, has shown itself to be extremely fruitful for development of theory.

My analysis of instruction revealed different types of argumentations in classroom proving processes. Based on Max Weber's methodological concept of the *ideal type,* ideal-typical characterisations of discourses were developed by comparing cases of argumentation during proving processes. This involved comparative analyses of all observed episodes "*from an initial interpretation of those episodes to a later theoretical exploration of those episodes*" (Krummheuer & Brandt 2001, 78). Instruction processes were compared both on the level of context analyses and on the level of argumentation analyses with the aim of developing a typology. The formation of an ideal type in Weber's sense can be described in this way:

> An ideal type is formed by the one-sided accentuation of one or more points of view and by the synthesis of a great many diffuse, more or less present and occasionally absent *concrete individual* phenomena, which are arranged according to those one-sidedly emphasized viewpoints into a unified *analytical* construct. (Lassmann 1989, p. 249)

An *ideal type* is therefore an analytic construction for further discussion with the purpose of isolating and clarifying theoretically important characteristics of social actions. The *ideal* *type* does not correspond to some real entity, rather through it an understanding of, and an explanatory model for, reality is formed.

I claim the generality of my results only in this sense. Different types are revealed by argumentations during classroom proving processes which make it possible to analyse classroom proving theoretically. However no claim is made that the typology is complete or that the instruction observed is representative of German or French mathematics instruction in general.

Prototypical cases or *prototypes* form the basis for the construction of ideal-typical characterisations of proving processes in my work. A prototype is a case "*in the sense of a concrete model"* (Zerssen 1973, 53), not an ideal type, i.e. not an ideally formed theoretical construct. Rather it is a case that can apply to a group as representative in the sense that through it special characteristics of a group of cases become clear (Kluge 1999). Descriptive typical characteristics can be worked out by the characterisation of the prototype. Singling out prototypes forms an intermediate step in the process of constructing ideal types. The comparison of prototypes with further cases is also crucial here. In the light of other cases, typical features become clearer in contrast to individual characteristics. The ideal-typical characterisations developed in this way have a heuristic function, because *"the pure type contains a hypothesis of a possible occurrence"* (Gerhardt 1991, 437). The cases discussed below represent prototypes in this sense.

Method and design

The instructional units were selected according to curricular criteria and cover topics in geometry. Analyses and results from six instruction units on the Pythagorean Theorem are presented here. French and German curricula, which I analysed before beginning my data collection, list proofs as an explicit topic in geometry for the first time in grade 8. For this reason instructional observations were done in geometry classes at these levels. The data collection was carried out at six Collèges in the Paris region and three *Gymnasien* and two *Gesamtschulen* in Hamburg. Two of the observed cases in France are classes in a bilingual stream and are highly selective.

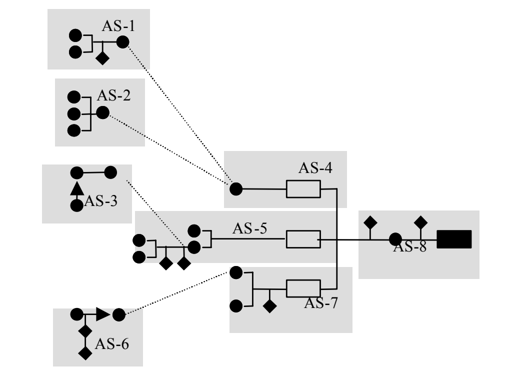
French curricula apply nationally, so that all classes at this level in the Collège are intended to study the same material. Hence I feel that my decision to carry out investigations in different Collèges in the Paris region has not resulted in a special sample of proofs and proving processes at this level. With the exception of the bilingual classes, whose students are selected to a considerable degree according to achievement in ordinary schools, the cohort was near average and heterogeneous.

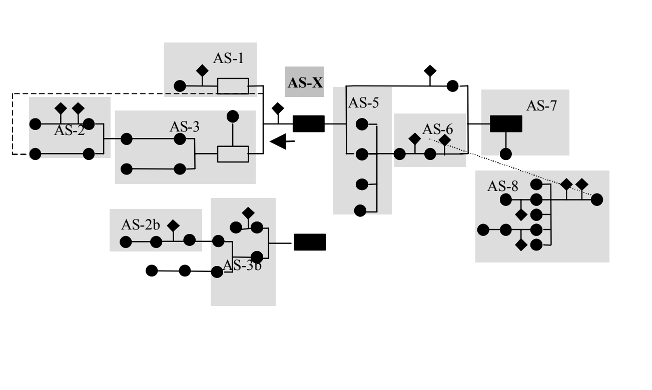
In contrast, substantial curricular differences exist in Germany on a regional level and between different school types. Differences in the topic emphasis of the curricula can be found not only on a regional level, but also, and more so, among the different school types. While a special value is given to proofs nationally in the Gymnasium curricula of these levels, in the curricula of *Gesamtschulen* proofs clearly play an inferior role. In school curricula of *Gesamtschulen* this different valuing of proofs is usually reflected in different targets for courses of the upper and lower sets. The curricula suggested that it would be difficult to observe proofs and proving processes outside the Gymnasium and perhaps also the upper sets in *Gesamtschulen*. Consultations with teachers confirmed this, and so, early in the research, the decision was made to choose German classes selectively. I decided to examine classes in both the Gymnasium and the upper sets in *Gesamtschulen* in case there were differences in the classroom proving processes.

I documented my observations with audio tape recordings and photos of the blackboard. In addition, observations were recorded in the form of process notes which I made after each session. The tape recordings were transcribed to make detailed analysis of the classroom discourses during the proving processes possible, which is necessary for the reconstruction of the argumentations.

Results

The argumentative structures of the proving processes in the observed classes are complex. Individual argumentation strands overlap. For example, different justifications of a desired conclusion are developed in parallel. In the argumentation analyses I took not only different argumentations of individual proof steps into consideration but also analysed the overall structure of the proving processes. Such an analysis cannot make use of the Toulmin model that I used with the analyses of individual argumentative steps therefore I developed a schematic representation for the analysis of the overall argumentative structure of the proving processes. These analyses result in two types of overall structures, which I call *source*-structure and *reservoir*-structure (see Knipping 2002a, 2002b).





● ***Statements (Data (D) or conclusions (C))***

■ ***Target-conclusion (C)***

◆ ***Warrant (W) or backing (B)***

□ ***Statements (D / C) that mark the transition from the first to the second part of the proving discourse***

▲ ***Objection to a conjecture or conclusion (O)***

Figure 1 *Source*-structure

Figure 2 *Reservoir*-structure

In proving discourses with a source structure, arguments and ideas arise from a variety of origins, like water welling up from many springs. This is illustrated by figure 1. Conjectures and different arguments are discussed in public. False conjectures are eventually disproved, but they are valued as fruitful in the meantime. More than one justification of a statement is appreciated and encouraged by the teacher's open or vague questions. The diversity of justifications characterizes an argumentation structure with parallel streams and meandering lines. Not only the target conclusion, but also intermediate statements are justified in various ways. The teacher encourages the students to formulate conjectures which are examined together in class. Students' conjectures are appreciated even when they become publicly contested and refuted.

Argumentations with a *reservoir structure* flow towards intermediate target conclusions that structure the whole argumentation into parts that are distinct and self-contained. The parts that make up the argumentation are like reservoirs that hold and purify water before allowing it to flow on to the next stage. What distinguishes the reservoir structure from a simple chain of deductive arguments is that abduction allows for moving backwards in a logical structure and then moving forward again by deduction. In argumentations with a reservoir structure initial deductions lead to desired conclusions that demand further support by data. This need is made explicit by an abduction (see dotted line in figure 2). Abductions allow reasoning backward from a desired conclusion to establish data on which further deductions can be based. Once these data are confirmed further deductions lead reliably to the desired conclusion. This characterises a self-contained argumentation-reservoir that flows forward towards, and backwards from, a target conclusion.

Kinds of argumentations

Different kinds of argumentations can be reconstructed in the classroom proofs of the Pythagorean Theorem which I observed (see figure 3). Many of the proofs found in class could not be described appropriately in terms of the categories discussed in the research literature, for two reasons. Firstly, proving processes did not reveal argumentations of only one category, e.g. pragmatic argumentation. Therefore it is more appropriate to speak of argumentations in proving processes instead of a single kind of proof. Secondly, some argumentations occurred that did not fit into the categories suggested in the literature. For example, the differences between *intuitive-visual* and *conceptual* argumentations discussed in this paper could not be made clear using the terminology of Balacheff and others. Instead, a distinction between *semantic* and *deductive/abductive* argumentations was more appropriate for describing the argumentations observed in class.

Pragmatic	***Semantic***	***Deductive***	***Abductive***
Constructive	*Intuitive-visual*	*Conceptual*	
Metric	*Computational*		
	Metaphorical		
	Analogical		

Figure 3 Overview of the reconstructed argumentations

Pragmatic, *semantic* and *deductive / abductive* argumentations are all present in the proving discourses I analysed. *Pragmatic* argumentations, i.e. argumentations that are based on actions, will not be discussed in this article (see Knipping 2002a). Through two examples, of an *intuitive-visual* argumentation and a *conceptual* argumentation, typical characteristics of semantic and deductive argumentations will be identified here. *Intuitive-visual* argumentations, which are characterized by changes of different representation levels and aspects, differ fundamentally from *conceptual* argumentations, in which one argues logically on the basis of mathematical concepts and relations. In the following the difference between *intuitive-visual* and *conceptual* argumentations will be clarified by analysing prototypical examples of each kind.

Intuitive-visual argumentations – the hermeneutic style

In *intuitive-visual* argumentations making reference to figures is, to a certain extent, part of the argumentation. Therefore warrants (i.e. justifications) frequently remain implicit in *intuitive-visual* argumentations. Individual statements are developed not as individual steps in an argumentation chain, but brought out simultaneously on the basis of figures. In addition the conclusions are not constituted as results of individual steps in the argumentation, but are found in and from the figure. The following episode (N5-7 <82-98>), which is an excerpt from a German lesson, makes this clear. In this scene an *intuitive-visual* argumentation is developed by students and the teacher together, and the argumentation occurs on two different levels, i.e. concretely and mentally. Reference to the figure on the board makes such a way of argumentation possible. The figure also permits changes between a geometrical and an algebraic viewpoint.

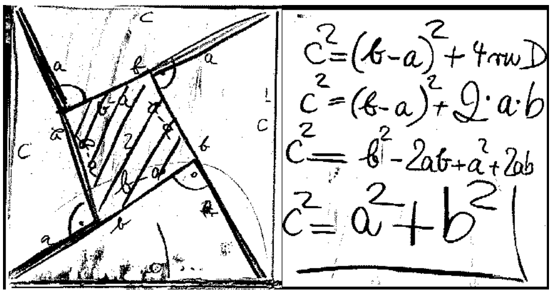


Figure 4 Prototype Nissen - arithmetic proof

Teacher: What was Adam's insight? I think, we should examine this
in detail. What should this be here? What does he mean, probably? Jan,
what does this mean?
Jan: Um, that, so that [*incomprehensibly*], I don't think it's entirely correct, if I
...
Teacher: Look, this is a term, which we know from the second binomial formula [i.e. *(a-b)², C.K.*],
okay, but how does he come to palm that off as part of c squared? Jens.
Jens: So that is the length of the side of this middle square there. But despite #
Teacher: # Thus, Adam says, here I always have to write b minus a, or something like that.
He says these four sides are all b minus a long. How does he know that? b minus a long.
Srike.
Srike: We have the length of one side of the triangle b, uh the leg. And then a,
the leg of the other triangle, is taken off from it and then that,
which is left, is b minus a.
Teacher: Agreed? The yellow part of the green segment is always cut off and what is left
of b is only a remainder. b minus a. Minus a means the yellow is cut off. So b minus
a, aha, and this squared. Great.

With the help of the Toulmin pattern the argumentation can be reconstructed as follows:

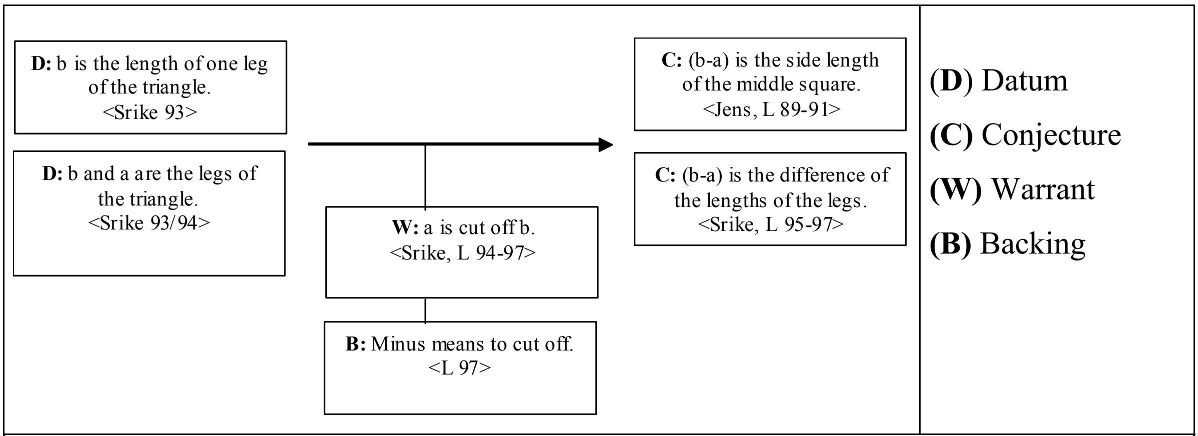


Figure 5 Argumentation N5-7

The students and the teacher argue *intuitive-visually* in the argumentation they develop together. Side lengths and the legs of the triangle are the starting point of the argumentation, and are interpreted and discussed on the basis of the figure. The conclusion is also interpreted on the basis of the figure. Jens and Srike refer in different ways to the geometrical properties within it. While Jens still describes these concretely and representationally, Srike develops a perspective that moves away from such a concrete representation. She speaks of the legs (*Katheten*) and the difference of the sides, not of a specific distance that she sees in the figure. In the argument developed collectively, both the warrant (W) and the backing (B) of the argument can be understood in two ways; pragmatically, (Jens' aspect) and conceptually, (Srike's perspective). So in this *intuitive-visual* argumentation the students argue on two levels at the same time.

The leitmotif of the proving discourse, of which this argumentation is part, is sight. Earlier in the episode (N5 43-49) this theme is clear.

> Teacher: Mmh. We do not know yet exactly what to write in the centre. But, you know, what I like about your answer is that you *look* for squares here, which have somehow a square measure. But of the internal square, we do not know the edge length of the inner square exactly yet *b* square would be *a* square, which would be here on top somehow, no?
>
> Maren: Umhmm.
>
> Teacher: ... that does not work very well. Perhaps you can find something different, Sarah do not write, do not write, only think, only *look*. Writing it, we can do that later. Jan

In the analysed episodes *sight* is understood in a double sense, which is also apparent in the transcript presented here. On the one hand an *empirical-visual* interpretation of statements is intended. Facts are illustrated visually; mathematical properties and relations are bound to concrete figures and are discussed as knowledge accessible via the senses. On the other hand statements are mentally "held in sight" i.e. taken as *intuitive-visual.*

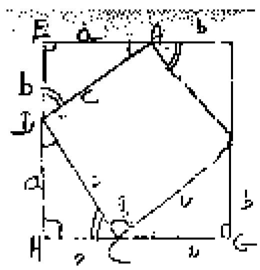
Empirical-*visual* interpretations of statements are not inferior to their reasons, but are treated as supplementing them. The proof figure comes at the beginning of the proving discourse. From this figure a general relationship is to be interpreted and justified and is to be seen mentally. The *intuitive-visual* suggestion of a connection is the starting point of this reasoning. The individual expressions are discerned from the whole; the proving strategy is developed from there.

Reasoning of this type can be interpreted as an outcome of a hermeneutic tradition that has influenced education and in particular mathematics education since Humboldt's reforms in 1810 in Prussia. These turned the ideas of enlightenment i.e. defending Cartesian principles in reasoning, into another approach (Jahnke 1990). The hermeneutic tradition emphasizes reasoning based on holistic interpretation instead of reductionistic reasoning. Meaning and understanding has to be actively constructed each time from concrete givens that convey the general. As Humboldt puts it: *'to recognize the invisible in the visible'* (Jahnke 1990, 29). Humboldt propagated this approach not only for science but for education as well. This meant that teachers should offer learning environments rich in problems that offered opportunities for learning from the concrete to the abstract, and that this process could not be taught but had to be carried out by the students themselves.

Taking the influence of the hermeneutic tradition into account much of the teaching observed in the prototype Nissen becomes comprehensible. The teacher does not want to break the proof down into deductive parts, but wants the students to come to their own understanding starting from the figure given. She wants them to *see* the general statement, including its justification, from the given proof figure. In the complete proof discourse she encourages the students to make up their own arguments. This means their conjectures and different arguments are valued as fruitful and discussed in public. In contrast to this *intuitive-visual* kind of argumentation, the prototype Pascal, which is discussed in the following section, illustrates another type of proving process in which *conceptual* argumentations are dominant.

Conceptual argumentations – the discursive style

In conceptual argumentations, which are important in proving processes of the discursive style, conclusions are understood not visually or mentally, but deduced from concepts. Data can be illustrated and formulated on the basis of figures, but the conclusions drawn from them are *arrived at conceptually*, not *intuitive-visually.* The giving of reasons, i.e. warrants or backings, is central in this kind of argumentation. These lead to conclusions that can be recycled, i.e. used in the next argumentation step as data. Conceptual argumentations of the deductive type are logically correct in structure, although the expression of this reasoning in the analysed episodes is not in a logically correct form. Concepts and mathematical relations, which are the basis of these conclusions, are formulated colloquially in everyday language. This is clear in the following episode (P3 <69-94>).



Since ABCD has four sides of the same length, it is a rhombus. The acute angles of a right-angled triangle are complementary.

The right-angled triangles DHC and BGC are congruent, the corresponding angles are congruent. From this it can be concluded that the angles BCG and DCH are complementary
HCG = 180°, from which follows
BCD = 180 − 90 = 90
ABCD is a square.
F (ABCD) = C x C = c^2
F (ABCD) = $(a+b)^2 - 2ab$
surface EFGH
− surface of the 4 triangles
(surface of the 2 rectangles)
$(a+b)^2 = a^2 + 2ab + b^2$
thus F (ABCD) = $a^2 + 2ba + b^2 - 2ab$
$\mathbf{c^2 = a^2 + b^2}$

Figure 6 Prototype Pascal - arithmetic proof

69 Teacher: Who has an idea of how to find this angle? Marie.
70 Marie : Well, I saw… Because in fact we know that, in all right triangles the angles
71 are …
72 T: the angles…
73 Marie : the angles make up 180
74 Teacher: degrees.
75 Marie : And that, as it has a right angle, that means that the two angles are, uh, ….
76 Teacher: The acute angles in a right triangle are complementary. Very good.

77 Marie : So, because all the angles are equal in the four triangles…
78 Teacher: Yes, as the triangles …
79 Several Students: Are the same, are …
80 Teacher: Are identical, the angles have the same measure
81 Marie : and that, and we know that on a straight angle HCG ´
82 Teacher: Yes
83 Marie : And that, well, HCD and, BCG are, finally they're not the same corners, that's the …
84 Teacher: They are …
85 Marie : … they are opposite, in the end.
86 Teacher: Com-ple....
87 Luc : Complementary.
88 Teacher: The green and the red are ….
89 Marie : They make 90 degrees
90 Teacher: 90 degrees. As the total of the vertices
91 Marie : # That's 180 degrees
92 Teacher: The vertices along angle HCG add up to 180 … What's left in the middle?
93 Marie : 90 degrees
94 Teacher: 90 degrees

The argumentation can be reconstructed as follows:

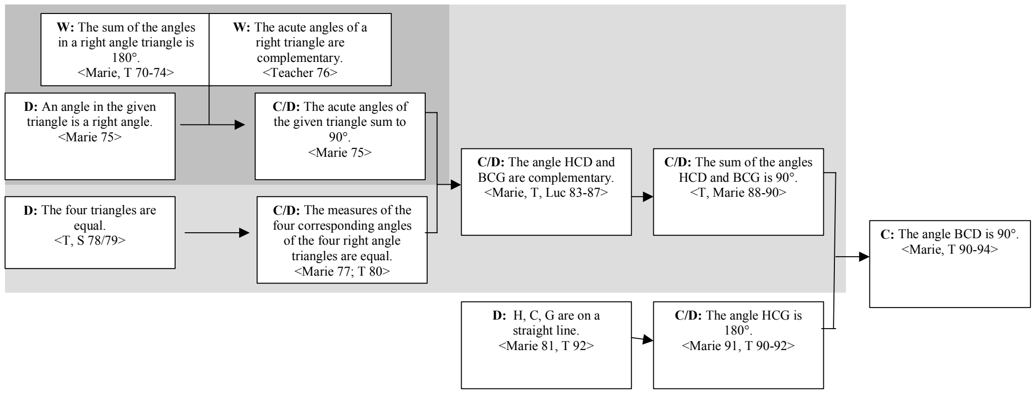


Figure 7 Argumentation scene P3-4

The argumentation developed in this episode can be understood as conceptual and deductive. The warrants of the argumentation are mathematical concepts or mathematical relations between concepts. Thus the basis of the argumentation is conceptual. The way one comes to conclusions in this conceptual argumentation can be described as deductive. From data, that have the status of assumptions, one reaches conclusions conceptually and step by step. Abductive argumentations, in which one also argues on the basis of a concept or a conceptual relationship, have a special role within the total structure of this proving discourse, which I discuss elsewhere (Knipping 2003).

The different status of statements is indicated in everyday language. Data are called reasons and linguistically marked by conjunctions such as "as" ("comme"). Conclusions are not usually marked by specific expressions, but they are represented as results or also as causal consequences. Their status is implicitly assigned to them. Warrants are usually introduced explicitly into conceptual argumentations, and sometimes their status as warrants is made explicit. Thus Marie says: "*Because one knows that*" (70), indicating the generally accepted status of the conceptual warrant.

The argumentation is complex and multi-step, which characterizes the reservoir structure of the global argumentation. The individual strands of the argumentation are developed sequentially. A conclusion is reached from data, and is then reused as data in the next argumentation step. In this complex, multi-step argumentation claims for validity are formulated in a discursive form. Data, warrant and conclusion are expressed verbally. Also, relations observed in the figure are negotiated and justified exclusively on a linguistic level. The individual argumentation steps are brought out sequentially and concatenated with one another. The overall argumentation develops as a structure or chain of individual statements. One negotiates which steps of the argumentation are developed and made explicit.

The leitmotif of this type of proving discourse, which was illustrated by the Pascal prototype, is assertion. Data, conclusions and warrants in the argumentations of this proving discourse are asserted discursively and publicly. They are expressed verbally more completely and explicitly as well as being recorded in writing on the board. In this way claims for validity and their grounds are publicly documented and fixed. This leitmotif is clear in the following example (P3 <123-131>).

123 Thierry : Angles DCH and BCG.
124 Teacher: # are
125 Thierry: DCH are complementary.
126 Teacher: Yes. (9 seconds) And then, are complementary, *have you written that*? Yes? So?
127 Stephanie: We write that C is, so the angle C is equal to 180 minus …
128 Teacher: You have to say first that HC, why 180 ?
129 Stephanie: 180, because it's straight.
130 Teacher: Ah. One must say it nevertheless, eh? *We did not say that yet. We said it but did not write it.*
131 *In a proof you must write everything that you said,* so, the next line is, there, HCG equals 180 degrees.

In this type of discourse the writing of argumentations reinforces the discursive character of the proving processes. Reasoning that is developed by individual steps in an argumentation chain has to be made explicit publicly. A conclusion is reached based on concepts and relations of concepts that are asserted and accepted as common knowledge. Writing reinforces this publication of knowledge, as well as making clear that the structure of the whole proof is brought out sequentially by individual argumentation steps which are concatenated with one another. Argumentations in this type of discourse are far from being formal, but they are rigorous in the sense that each step and justification in the chain of arguments has to be made explicit and should be based on concepts that are shared in the community of the class.

Reasoning in this type of discourse can therefore be described as reductionistic, i.e. coming to a conclusion and an understanding of a complex theorem, e.g. the Pythagorean Theorem, through single steps. The idea is here not *'to recognize the invisible in the visible'*, but to divide the whole into sound steps in a chain of reasoning. Further, reasoning is taught through publication of argumentations which have been worked out by the students but reach a public status in the class, reinforced by writing them down on the blackboard. The students have to come to a common understanding of mathematical theorems and their proofs in class instead of diverging individual interpretations.

Discussion

Comparing these different kinds of argumentations and proving discourses in class indicate that, when proving, different classroom cultures exist. As the Pythagorean Theorem is the subject of these different lessons and an arithmetical proof is worked out in both cases discussed here, it is perhaps surprising that these different types of proving processes were found. If we propose the hypothesis that these differences are due to different cultural traditions of reasoning, many of the characteristics of the discourses described above might become less surprising, and more comprehensible. By referring to a hermeneutic tradition in Germany I give a possible cultural interpretation of what I have observed in class. This perspective on classroom proving situations requires more work and is worth exploring in further research. Historical and philosophical research on the traditions of mathematics education might give new insights into the proving processes which we observe in class. This research could also help us to become aware of thinking traditions which foster our descriptions of proving processes and perhaps to overcome some of our cultural biases.

Cross cultural comparisons, as used in the research presented in this paper, turn out to be fruitful for questioning proving practices from a perspective that has to take into account the different contexts in which these processes happen. They can help us to become aware of the cultural choices we tend not to question, either in our own research, or in our teaching practices. This paper leaves many questions open. The question of why differences in proving processes in class exist has been raised, but much more work needs to be done before it can be answered. How different cultural traditions in reasoning are transmitted, so that they become manifest in classroom cultures, is another important and as yet unanswered question. Both of these questions are of major importance for teacher education and for any proposed reforms and changes in mathematics education.

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Bibliography

Blum, W. & Kirsch, A. (1991). Preformal Proving: Examples and Reflections. Educational Studies in Mathematics 22(2): 183-203.

Hanna, G. (1989). Proofs that Prove and Proofs that Explain. In: G. Vergnaud, J. Rogalski und M. Artigue (Eds.), Proceedings of the 13th Conference of the International Group for the Psychology of Mathematics Education. Paris: Université de Paris, 2, 45-51.

Healy, L. & Hoyles, C. (1998). Justifying and proving in school mathematics. Technical Report on the Nationwide Survey. London: Institute of Education, University of London.

Herbst, P. G. (2002). Engaging Students in Proving: A Double Bind on the Teacher. In: Journal for Research in Mathematics Education, 33(3), 176-203.

Herbst, P. G. (1998). What works as proof in the mathematics class. Athens, GA.: The University of Georgia, Unpublished doctoral thesis.

Jahnke, H. N. (1990) Mathematik und Bildung in der Humboldtschen Reform. Vandenhoeck & Ruprecht, Göttingen.

Kautschitsch, H. & Metzler, W. (Eds.) (1989) Anschauliches Beweisen. Wien: Verlag Hölder-Pichler-Tempsky

Knipping, C. (2003) Argumentation structures in classroom proving situations. Paper contributed to working Group 4: Argumentation and Proof, Third Conference of the European Society for Research in Mathematics Education (CERME 3), Bellaria 2003.

Knipping, C. (2002a) Beweisprozesse in der Unterrichtspraxis – Vergleichende Analysen von Mathematikunterricht in Deutschland und Frankreich. Unpublished doctoral thesis, University of Hamburg, 2002.

Knipping, C. (2002b) Die Innenwelt des Beweisens im Mathematikunterricht. Vergleiche von französischen und deutschen Mathematikstunden In: Zentralblatt für Didaktik der Mathematik**,** 34 (6), 258-266.

Krummheuer, G. (1995). The ethnography of argumentation. In: P. Cobb & H. Bauersfeld (Ed.), The Emergence of Mathematical Meaning: Interaction in Classroom Cultures. Hillsdale, NJ.: Lawrence Erlbaum Associates, 229-269.

Krummheuer, G. & Brandt, B. (2001). Paraphrase und Traduktion. Partizipationstheoretische Elemente einer Interaktionstheorie des Mathematiklernens in der Grundschule. Weinheim, Basel: Beltz - Deutscher Studienverlag.

Lassmann, Peter (Ed.) (1989) Max Weber's "Science as a vocation". London: Unwin Hyman.

Pedemonte, B. (2002). Relation between argumentation and proof in mathematics: cognitive unity or break? In: Proceedings of the 2nd Conference of the European Society for Research in Mathematics Education, 2, Marienbad, 2001.

Reid, D. (1999). Needing to explain: The mathematical emotional orientation. In: O. Zaslavsky (Eds.), Proceedings of the 23rd Annual Conference of the International Group for the Psychology of Mathematics Education. Haifa: Department of Education in Technology and Science, Vol. 4, 105-112.

Reid, D. (1995). The need to prove. Unpublished doctoral thesis, Department of Secondary Education, University of Alberta.

Reiss, K. et al. (2002). Reasoning and Proof in Geometry: Prerequisites of knowledge acquisition in secondary school students. In: A.D. Cockburn, & E. Nardi (Eds.), Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education. Norwich: University of East Anglia, 4, 113-120.

Toulmin, S. E. (1958). The uses of argument. Cambridge: Cambridge University Press.

Wittmann, E. C. & Müller, G. (1988). Wann ist ein Beweis ein Beweis? In: P. Bender (Eds.), Mathematikdidaktik: Theorie und Praxis. Berlin: Cornelsen, 237-257.