Computer Algebra Systems and Cultural Coherence in Mathematics Classrooms

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Even today using Computer Algebra Systems (CAS) extensively in mathematics classrooms is still very often seen as a threat to the preservation of the mathematical culture. On the other hand, denying new technologies in the teaching of mathematics endangers the cultural coherence necessary for the technological development of mathematics and of society in general. After explaining briefly the concept of cultural coherence, this paper presents a position which argues that a mediating role between these two positions should be given to new technologies, and especially to CAS.

Cultural coherence

Cultural coherence is one of the seven tasks which H. W. Heymann considers to be essential for (mathematics) classrooms focusing on general education (Heymann 1996, pp. 65-79; 154-183). Here H. W. Heymann proceeds from a broad, sociological *concept of culture*.

This concept of culture is primarily descriptive and not normative. It covers everyday social manners and skills of any defined group as well as the standards of moral behaviour and the achievements of art and science. This includes any errors which may have become established. In contrast to the common usage of this concept, such an understanding of it does not include denying the culture of a particular group or making one culture stand out against another.

For H. W. Heymann, a special culture always means making a specific selection from the whole range of human possibilities; and this choice (which at the same time is the dividing line from other cultures) gives a *society* its specific *identity*.

Only from the view of those who feel they belong to a particular cultural group does the concept of culture also take on a normative character. A special value is assigned to the specific cultural features established in one's own cultural group and the individual, as well as society, has to orient themselves to this value. This gives *cultural identity* to the individual. A *reflected cultural identity* should show itself in the appreciation of, and openness to, other cultures and cultural identities without denying the particularities of one's own culture.

In the view of H. W. Heymann, *cultural coherence* covers both a disparate and a synchronous aspect. With the disparate aspect the time component of a culture's development is assigned i.e. cultural continuity. It is concerned with the preservation and passing on of cultural achievements and particularities from one generation to the next, with the acquisition and the advancement of established culture by younger people, and with the creation of a relationship between former times and today.

The synchronous aspect focuses on considering different cultures and especially on establishing compatibility between different (sub-)cultures, both within their own societies and own cultures.

In creating cultural coherence we are aiming to fulfil both of these aspects, with emphasis being given above all to

- *maintaining everyday (mathematical) culture* in order to get by in daily situations in private life and in day-to-day work. In the case of mathematics (at school) it concerns the preservation of that standard of mathematical abilities which are regularly needed in everyday private and working life (including its careful adaptation to newer developments in society and technology).
- *communicating between the generations* in the sense of having a common basis for being able to communicate about standards, values, views, ideologies, etc. (whereby we do not mean harmonising or even solving the generation conflict). The introduction of set theory at the end of the 1960s, for example, seriously disrupted this form of communication in the field of (school-)mathematics.
- *developing one's reflected cultural identity* in the sense of seeing oneself as a part of the culture, of recognising linking components in one's culture and of being able to accept the different elements of other cultures as being equal. In the case of (school-)mathematics it concerns, for example, the task of making it possible to experience the relevance of the special characters and universality of mathematics (based on abstract objects, symbolic representations, rules without contradictions) for the whole culture. The point therefore is the appropriate understanding of the ways of thinking and working in mathematics. Fundamental ideas play the central role in this as relations overlapping different mathematical concepts from which connections, as well as differences, between mathematical and non-mathematical cultures can be clearly recognised.

These demands include the passing on, the acquisition of and the reflection upon, the delivered knowledge as well as its critical consideration and advancement.

CAS as mediators between the traditional teaching of mathematics and technological innovations

For the majority of people, becoming familiar with mathematics corresponds closely to "doing calculations" of various types and of increasing levels of difficulty. One can offer various more or less convincing reasons for this dominance of operational activities (transformations rule-based). Mathematics educators, however, criticise this dominance of operational activities as "blind calculating" (H. Winter, as quoted in Fischer & Malle 1985, p. 221) or even as "conditioning and training something not understood" (M. Wagenschein, as quoted in Vollrath 1987, p. 376). On behalf of a lot of didactics experts, let me quote H.-J. Vollrath, who has, in my opinion, expressed some of the essential didactical doubts clearly and concisely:

> The great emphasis on exercises has upset the didactic balance. The introductory phases have been reduced; reasoning has been given a subordinate role; understanding has been pushed to the background; routine counts for more than intuition; being able to do something is worth more than knowing how to do it; skills are more important than understanding. (Vollrath 1987, p. 376)

This remarkable resistance of the teaching of mathematics to the demands and requirements of mathematics education can nonetheless be explained by the desire for cultural continuity. As *teachers* we transmit mathematics in a manner such as we ourselves experienced it – at school or also at university. With reference to schools this means very often the treatment of "plantations of exercises" in which virtually nothing more is required than the transformation, by rules, of mathematical facts represented by symbols. Compared with mathematics in schools, mathematics instruction at university focuses more on knowledge and understanding than on skills but its goal, however, is educating mathematical experts. Much importance is also attached to operative knowledge and skills (nonetheless by the way of deductions, proofs). In this way, conserving and passing on the "traditional" (i.e. existing for at least one generation – cf. Heymann 1996, p. 72-73) image of mathematics and consequently the traditional contents and the traditional focus of mathematics at schools, is very often the factor which determines teachers' decisions when choosing teaching content and points for special emphasis.

Parents who (want to) take a more intensive interest in the school affairs of their children and especially in their mathematics instruction expect and demand that their children's mathematics lessons should only have developed slightly in comparison to their own education in mathematics. They do not want any radical break with the familiar image of mathematics from their own school days and their daily life. Any radical changes would mean shutting out the parents from their children's mathematical education, thus affecting the existing points of reference that join the older generation with the younger one and thereby changing the use of mathematics in everyday life which is familiar to the older generation. That could make communication between the generations more difficult, place obstacles in its way or even break it off completely.

Parents are also afraid that any radical changes in the taught image of mathematics will mean their children will no longer be learning "real mathematics" or will no longer be mastering the "mathematical standards". They are worried that in some fields differences will arise and cause a hiatus with the mathematical culture of everyday life. It could therefore become (more) difficult for young people to manage some professional situations in an appropriate way thus hindering them in their future careers or, at the very least, making them encounter problems in their university studies.

General education, however, should not be restricted only to the conserving of traditional knowledge and skills and to their continuity. Rather, it should endeavour to link familiar elements with new and unknown ones and so create *cultural coherence*. With regard to cultural coherence, new technologies, and *Computer Algebra Systems* (CAS) in particular, are playing a special, but ambivalent role in mathematics classrooms. CAS are able to operate with symbolically-presented mathematical objects in a manner equivalent to rule-based operations in mathematics. This is why they have been developed, and in this sense they are a (modern) materialisation of operative knowledge and skills. For the moment they can be understood as the latest step in trivialising and outsourcing the operations. Up to now CAS have been unable to master all the operations known and needed in mathematics, yet they do indeed master almost all the operations dealt with in schools. This materialised, operative knowledge and the corresponding skills are currently available to anyone at a cost of approximately €180.

Through the permanent availability of CAS, rule-based paper and pencil calculations (operations) have become less and less required as mathematical tools. The mastering of operative activities (rule-based transformations) "manually" has become an obsolete, unnecessary skill. Thus, using CAS means a radical break with regard to the image of mathematics predominantly delivered in society. What has been seen up to now as representative of the mathematics culture and as its central activity by the majority of people, loses its relevance and its necessity. The use of CAS therefore is hard to reconcile with the demands of cultural continuity and of preserving particular cultural achievements.

Turning the argument round, abandoning CAS means denying the technological development of mathematics and of society in general. Using new technology actively became one of modern society's cultural techniques a long time ago, and is today an established part of our culture. Ignoring this development in the teaching of mathematics would be a conscious fading-out of the cultural achievement of modern times and thus a radical break with our culture today. There is no convincing argument why young people today should obtain the qualities they will need tomorrow using equipment that was old yesterday – cf. Peschek 1999a, p. 265.

This ambivalence is almost always present and strongly felt in the didactical discussions about using CAS in the teaching of mathematics and it leads to different reactions. For example, some didactics experts and many teachers try to meet this threat to traditional content and teaching areas in schools mathematics by only allowing their students the use of CAS after they have proven, with paper and pencil, that they have obtained the required operative knowledge and skills. This is an attempt to try and bridge the break with (school-mathematical) traditions by developing that operative knowledge and skill which becomes obsolete by the (later) use of CAS (cf. for example the "white box/black box principle" in Heugl et al 1996, pp. 158). Others attempt to avoid this break by using CAS to simulate hand calculations (for example solving equations or systems of equations), or even to practice (drill) operating "manually".

Others orient themselves to a large extent on the technological development of mathematics – led by a fascination with the technological possibilities. Above all they attempt to contribute to creating cultural coherence by extensively and exhaustively using the possibilities offered by technology – without regard to other traditions. In their lessons they give preference to teaching content which could hardly be done without computers (such as extensive simulations, three-dimensional representations, various numerical procedures requiring extensive calculations – cf. e. g. Böhm & Pröpper 1999, Lehmann 1999, Waits & Demana 1997).

My view of this matter differs from all the positions mentioned here. I see CAS as a mediator between innovation and tradition. In the field of schools mathematics they act, on the one hand, as a mediator between the didactical and pedagogical requirements for significant reduction of operational activities in the teaching of mathematics. On the other hand, they produce the fear that the potential of mathematics for problem solving could be lost by the learner. Through using CAS, operative knowledge and skills are being brought into the mathematics classroom, and consequently into the discussion and treatment of mathematical problems. Operative skills and knowledge are to a great extent available to the students under particular preconditions, without them cognitively having to develop the knowledge and the skills themselves. In any CAS-supported learning environment `solve(a*x^2+b*x+c=0,x)` is the solution of the equation $ax^2 + bx + c = 0$ – similar to applying the formula for solving quadratic equations in any "traditional" (no use of CAS) learning environment. In addition `d(x^n,x)` is the derivative of the general power function $f: x \to x^n$ in the same way as applying the differentiation rule for power functions, etc. The operative activities (skills) being demanded by the students in any CAS-supported learning environment concentrate on the appropriate transformations and on the input of a fact given in mathematical notation in CAS. In this sense, *the ability to use CAS in any adequate (appropriate) way* could be considered *as the modern form of operative mathematical knowledge and skills*. So CAS offer a promising possibility for mediating between the different expectations and demands on the teaching of mathematics, such as mediating between maintaining traditions and outsourcing operations, between traditional teaching of mathematics and the use of technological innovation. Thereby, cultural coherence can be reflected upon, promoted, and actively encouraged, thus advancing a new mathematics culture in schools.

Outsourcing as characteristic of science and society

With the *principle of outsourcing* W. Peschek demands the unrestricted outsourcing of operative knowledge and skills wherever this outsourcing seems to make sense didactically (cf. Peschek 1999b, p. 407, Peschek & Schneider 2000, 2001, Schneider 2001).

The outsourcing of knowledge is a fixed part of our everyday life and of science also; thus the emancipated handling of such knowledge is of great social importance. This is particularly, and in a very specific fashion, valid in mathematics.

One fundamental mathematical way of working and thinking is representing (materialising) a non-mathematical situation by mathematical symbols (as is done in the example shown in Figure 1(1) and (2)).

(1) A youth hostel has 58 beds in 21 two and four-bed rooms. How many two and how many four-bed rooms does the youth hostel have?

(2)

$$\begin{aligned} x + y &= 21 \\ 2x + 4y &= 58 \end{aligned}$$

(3)

$$\begin{aligned} -2x - 2y &= -42 \\ \underline{2x + 4y} &\underline{= 58} \\ 2y &= 16 \\ y &= 8 \qquad x = 13 \end{aligned}$$

(4) The youth hostel has 13 two-bed and 8 four-bed rooms.

Figure 1. An example for outsourcing

In this way, it becomes possible to carry out operations at a syntactical level without having any correspondence to the reference context and without being bound to it (therefore, in a certain sense, “without understanding” – cf. Figure 1(3)). The results calculated within the formal system can be interpreted in the original context and result in the solution of the investigated problem (cf. Figure 1(4)). Such an approach is based on the outsourcing of a non-mathematical problem to the formal-operative system of mathematics. Doing so is highly economic for thinking – it reduces the complexity of the problem and it allows for solutions and methods of solving which otherwise, without the possibility of outsourcing in the formal system, would either not be found or not be so simple to find.

In mathematics, however, we constantly work with the method of outsourcing. This occurs not only in elementary procedures such as division algorithms, but also in the more complex notions and procedures up to and including proofs. One need not know why the division algorithm being applied works in order to get the correct answer when dividing. One need not bother with the logical reasoning behind an equivalence transformation when using such a transformation to solve an equation and one needs not recognize the basics of set theory or the concept of function in order to succeed in calculus when finding a derivative.

One uses many of these mathematical concepts and procedures as comprised bits of knowledge (modules) within mathematics and one needs to know their effects and "interfaces" with external agencies very well in order to be able to apply them correctly. It is not necessary, however, to know their internal workings.

Outsourcing is something *genuine for mathematics*; it is one of the characteristics of mathematics and it is an essential basis for its efficiency.

One can immediately establish analogies between these scientific-theoretical considerations and considerations on the *socio-philosophical level*. Using comprised bits of knowledge has long become self-evident and an indisputable necessity in our society: "If only those people drove a car who completely understand the mechanics and electronics of their car, we would not have traffic problems. If only those people stopped at a traffic light who understand the functioning of the traffic light from the algorithm to the programming and further to the microprocessors' operating, we would also have no more traffic problems." (Peschek 1999a, p. 268).

Here mathematics is taking on a special role with increasing social relevance:

> Mathematics is relatively secure, socially accepted, codified knowledge which, notably, allows for a separation between understanding and doing ... (it) owes its high social relevance to the fact that, in utilizing outsourcing, it even works when the user has no idea anymore as to why. (Peschek 1999b, p. 406)

(A more detailed discussion on this matter can be found, for example, in Fischer 1991, Maaß & Schlöglmann 1988, Peschek 1999a, 1999b, Winkelmann 1992.)

An elaborate image of the subculture that is mathematics, of its characteristics, its ways of thinking and working and its socio-cultural relevance will include outsourcing as an important (scientific-theoretical and socio-philosophical) aspect. To develop a reflected cultural identity it is relevant to experience outsourcing as a fundamental characteristic of mathematics and to be able to understand it as a constitutive aspect of the relevance of mathematics for society. Reflected handling of outsourcing and of its scientific-theoretical and socio-philosophical relevance in the teaching of mathematics can be considered an important contribution to creating cultural coherence. CAS are for the moment just the last step in the development of creating perfect outsourcing. Hence they can serve as fully demonstrative and obvious examples and models for the process of outsourcing.

CAS and communication with experts

H. W. Heymann perceives the "problem of communication between experts and lay-persons" (Heymann 1996, p. 113) to be one of the key problems of a highly differentiated and structured democracy based upon the division of labour. The functionality of the society is based on appropriate and freely available contact with highly specialised experts' knowledge. As mature, responsible citizens we are permanently confronted with statements made by experts which we then must assess and judge in order to be able to make (our own) decisions. Normally, we will rely on the professional correctness of these experts' statements. Yet we still need to judge their importance for ourselves and for the community. Because we are ourselves experts only in a few fields we must be able to ask the experts the right questions, to assess their answers and to draw our own conclusions in all those fields in which we are lay persons.

R. Fischer considers a task of those persons who have attended institutes of higher learning (high schools and vocational high schools), "the more highly educated" (cf. Fischer n.d., p. 3), to be that of mediating between experts and the "general public". In particular more highly educated persons should be able to explain the experts' statements in an understandable fashion and judge their importance.

Being able to communicate with experts and with the general public requires the development of competences other than those which are necessary for being an expert oneself. R. Fischer identifies the following three fields of competence as those which are to be acquired:

- basic knowledge (notions, concepts, forms of representation)
- operative knowledge and skills (in order to solve problems or to generate new knowledge)
- reflection (possibilities, limits and meanings of concepts and methods).

With regard to the ability to communicate with experts (and the general public) R. Fischer considers the fields of basic knowledge and of reflection to be particularly important for lay persons who have received a general education. Basic knowledge "is a prerequisite for communicating with experts", reflection "is necessary for judging expertise" (Fischer n.d., p. 5).

However, in addition to competence in the field of basic knowledge, the activities and tasks of experts require above all profound competence in the field of operative knowledge and skills. R. Fischer points out that this classification should not be taken as an absolute. Neither should the experts be relieved of their responsibility of viewing what they are doing in a self-critical manner, nor should doing operations be completely removed from the framework of mathematics instruction for lay persons. The focus and profiles for experts and lay persons do, however, clearly differ.

Communication with experts includes outsourcing considerable operative knowledge and skills to the mathematical expert. Interaction with CAS can be perceived and seen as a specific model for such a communication. As in using CAS, in the communication between human being and machine, elements can be seen that are also quite significant for communication between lay persons and human experts (cf. also Pesckek & Schneider, in press).

A successful and profitable interaction with CAS requires

- *broad basic knowledge of mathematics (especially knowledge about important mathematical forms of representation)*

  For example, for the correct input of (symbolic) CAS-representations it is necessary to be able to recognise the structure of the given formula and to be familiar with the hierarchy of calculations since the input of (arithmetic, algebraic) formulas must be carried out sequentially on most CAS.

  Working with CAS often requires an increased use of functions. Hence the possibility offered by CAS to store formulas as modules and to operate with these modules is based on the interpretation of these formulas as formulas of a function and requires the appropriate knowledge in dealing with functions (also in several variables).

  The interpretation of incorrect graphic representations (such as in points of discontinuity) requires significant knowledge and understanding of the given fact and of the intended aims of the representation.

  To use adequately each of the (symbolic, graphical, and tabular) representations offered particularly by CAS, requires knowledge about the potentials and problems of

the different forms of representation in addition to understanding the given situation and the goals intended with these representations.
This list could be continued by numerous other examples referring to required basic knowledge.

- *very precise conceptions of the fundamental possibilities and limits as well as estimations of the local abilities of CAS*
  The prerequisites, effects, applicability, conditions of use, and limits of the modules offered by the expert CAS must be known and familiar to the user in order to guarantee understanding and an efficient use of “CAS-knowledge”. (These features can sometimes be determined “experimentally” in the interaction with CAS).

- *the willingness and ability to ask the “right” questions, to be precise when formulating one’s own questions and considerations and to present them in a form which can be interpreted by CAS*

  Communication with CAS assumes that the problem is formulated precisely and with a syntax adequate to the CAS. To achieve this, a degree of exactness is required which exceeds even the exactness needed for communication with human experts. CAS react only and in the most rigid manner to the questions being asked and to the form in which these are formulated. Inputs that have not been exactly understood by CAS inevitably lead to a “refusal to answer” in the form of an error message, requiring (“forcing”) the user to be more precise. A negotiation process, such as occurs with human experts in order to clarify any imprecision, is only possible in a very limited form with CAS, either by means of error messages from the systems or of questions not being answered adequately.

- *a verification as well as an appropriate interpretation and assessment of the answers given by CAS*
  Even the first steps in using CAS require interpretation abilities (a common mathematical notation of a represented formula must be interpreted and recognised with regard to its structure in order to be able to transfer the input into a workable form for CAS). These abilities are also particularly required for the screen outputs produced by CAS. The solutions offered by CAS must not only be recognised as the solution of an equation, the integral of a function, the graph of a function, etc., in order to be able to interpret them in context. What is also required is an (inner-mathematical) verification, interpretation and evaluation of the solutions offered:
  - why does this equation not have a real solution?
  - how can it be explained that the integral has a negative value?
  - are all of the essential points of the graph of a function visible on the screen?

  etc.
  If different CAS representations are used, additional interpretation abilities as well as extensive “qualifications of translation” (between the different forms of representation) will be required.

Whenever CAS users (students) are working in specific forms of interaction, other aspects can be observed. These may be the transmission of the answers provided by CAS to other lay persons and the discussion of these answers among the lay persons. There may also be negotiation processes for the interpretation, the relevance and the justification of these answers as well as of any further questions to the expert. All in all, these are essential components of what R. Fischer (n.d., p. 4) describes as communication with the general public.

These considerations will neither equate CAS with a mathematical expert nor the communication between experts and lay persons with the communication between human beings and machines. CAS cannot become a substitute for human experts (and particularly not for teachers). They are too limited and rigid in their communication with users; their basic knowledge of mathematics, their abilities of representation and of interpretation are too insufficient. CAS can sometimes even disappoint us in the operative field. However one can find correspondences in both directions for some important aspects (those mentioned above) or at least similarities. For these reasons I plead for discussion of the use of CAS as a simple model of communication between mathematical experts and lay persons and for reflections upon it.

Concluding remarks

CAS make an important contribution to creating cultural coherence in more than one way in mathematics classrooms. Cultural continuity is ensured by the opportunities that CAS offer for providing access to operative knowledge in mathematics classrooms. This knowledge is available to the students without them cognitively having to develop it themselves. At the same time the advancement of delivered knowledge is supported by its adaptation to newer developments in technology and society.

The relevance of mathematics for the whole culture, an understanding of its characteristics and its efficiency can be shown and experienced by the example of outsourcing operations with the help of CAS. This is, in fact, a “materialised” and highly developed model of outsourcing. Hereby a considerable contribution is made to developing reflected cultural identity. In addition to this, CAS can be used as a simple model for the communication between (mathematical) experts and lay persons which is one of the key problems of our culture, and in this sense they also can support and promote the creation of cultural coherence.

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