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Abstract. This paper provides the exact analytical solution for the standard model of endogenous growth when consumers have present-biased preferences and make time-inconsistent savings plans, which they revise continuously. It is shown that long-run growth is not necessarily lower under present-biased preferences. In fact, an equivalence result holds. If hyperbolical discounting provides the same present value of a constant infinite income stream as standard exponential discounting, then the equilibrium rate of economic growth is also the same under both discounting methods. In this sense present-bias and the entailed time-inconsistency of savings plans are harmless for economic growth. The result is robust to the introduction of non-homothetic utility and a variable elasticity of intertemporal substitution in consumption.

Keywords: hyperbolic discounting, time-inconsistency, endogenous growth, adjustment dynamics.
JEL: D91; E21; 040.

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1. Introduction

Most research in economic growth theory confines itself to exponential discounting of future utility (sometimes also called geometric discounting). Exponential discounting is analytical convenient since it implies a constant time preference rate and, usually, the time-consistency of the obtained solution. The conventional approach, however, is potentially problematic since research in psychology and behavioral economics suggests that time preference rates are declining over time, presumably in a hyperbolic fashion (see Frederick et al., 2002, and DellaVigna, 2009, for surveys). Without further assumptions a time-variant discount rate implies the time-inconsistency of intertemporal plans (Strotz, 1956, Pollak, 1968). The solution then depends on whether and to what degree the optimizing agent can commit to an intertemporal plan and usually it deviates from the solution obtained under exponential discounting. In particular, it has been argued that individuals save and invest too little when they discount the future hyperbolically and are plagued by self-control problems, i.e. when they are revising their time-inconsistent saving plans at later dates (e.g. Laibson, 1996, 1998). Naturally this raises the question whether hyperbolical discounting and time-inconsistent behavior is harmful for economic growth.

In this paper I show that the rate of economic growth implied by the standard model of endogenous growth is invariant to the introduction of hyperbolical discounting given that a plausible restriction holds. The standard model of endogenous growth is the Romer (1986) model or any other endogenous growth model that leads in reduced form to a linear (Ak-type) growth model (see Rebelo, 1991; Chapter 4 in Barro and Sala-i-Martin, 2004). The restriction is that a constant infinite stream (of e.g. income) should provide the same present value under hyperbolic and exponential discounting. The restriction removes the arbitrariness of results and leads to a kind of "controlled theoretical experiment". It helps to focus on the issue of declining time preference and time-inconsistency. The result means that the method of hyperbolic discounting per se and the formulation of time-inconsistent saving plans per se leaves economic growth unaffected.[1]

The paper extends a small literature of similar equivalence results. Barro (1999) shows that the solution of the neoclassical growth model is of the same structure (though possibly not identical) under exponential and quasi-hyperbolic discounting when individuals continuously revise their time-inconsistent consumption plans.[2] Krusell et al. (2002) follow a different approach by focussing on time-consistent (Markovian) consumption strategies. They solve the neoclassical growth model in discrete time and for quasi-geometric discounting when individuals calculate their current consumption as a function of the current capital stock. They show that the solution is structurally similar to the one of the standard neoclassical growth model with exponential discounting if individuals confine themselves to linear strategies.[3] Findley and Caliendo (2013) show that exponential and hyperbolic discounting leads to structurally similar (though possibly not identical) solutions of the neoclassical growth model when the planning horizon is fixed and short. The present paper thus deviates from the existing literature in the consideration of time-inconsistent savings plans in the context of endogenous growth and in the finding that economic outcomes are not only structurally similar but identical under a plausible restriction. There is also a larger literature on generally time-variant discounting in dynamic macroeconomics to which the present paper contributes (see e.g. Epstein and Hynes, 1983; Obstfeld, 1990; Drugeon, 1996; Becker and Mulligan, 1997; Palivos et. al., 1997; Das, 2003; Schumacher, 2009; Strulik, 2012).

[1]In an *uncontrolled* theoretical experiment it may well be that hyperbolic discounting leads to a lower or higher rate of economic growth (see below). The reason is not the method of discounting as such or the entailed time (in-)consistency of plans. The result is "only" an expression of the side-effect that one or the other method puts in general more or less weight on future events.

[2]The proof of equality of the solutions could not be supplied because the problem has no exact solution and because Barro did not control for equal present value under both discounting regimes.

[3]A point not made by Krusell et al., but which is easily inferred from their Proposition 2, is that the solutions actually coincide in a "controlled experiment" as in the present paper, i.e. if both discounting methods provide the same present value. One problem with the solution method (acknowledged by Krusell et al., 2002), is that the linear strategy is just one of infinitely many consumption strategies, see also Krusell and Smith (2003).

The next two sections derive the main results for a logarithmic utility function. Section 4 shows that the main results are robust to an extension towards Stone-Geary preferences, i.e. a non-homothetic utility function according to which the elasticity of intertemporal substitution varies with the level of consumption. I cannot provide a closed form solution for other types of utility function. The extension, however, suggests that the result is not an artifact of a constant elasticity of intertemporal substitution.

2. The Model

2.1. **Households.** The economy is populated by a large number of identical households of measure one who supply one unit of labor and maximize lifetime utility

$$\int_{t_0}^{\infty} \log\left[c(t)\right] \cdot D(t, t_0) \ \mathrm{d}t \tag{1}$$

subject to the budget constraint

$$\dot{k}(t) = rk(t) + w(t) - c(t). \tag{2}$$

Here $c(t)$ and $k(t)$ denote consumption and capital and $w(t)$ and r denote the wage rate and the interest rate. The interest rate will turn out to be constant over time. Aside from discounting (1) and (2) constitute the standard consumer problem in growth theory. For the main part of the text we consider hyperbolical discounting such that the discount factor in (1) is given by

$$D(t, t_0) = \frac{1}{\left[1 + \rho_0 \beta (t - t_0)\right]^{\frac{1}{\beta}}}, \tag{3}$$

in which β controls the present bias and ρ_0 controls the instantaneous discount rate of the the next instant in time. Generally, the discount rate at time t_0 of time t is defined as

$$\rho(t, t_0) = -\frac{\dot{D}(t, t_0)}{D(t, t_0)} = \frac{\rho_0}{1 + \beta \rho_0 (t - t_0)}. \tag{4}$$

We assume $\beta < 1$ in order for the utility integral to be bounded (see below). The literature has suggested also alternative methods of (quasi-) hyperbolic discounting (see e.g. Frederick et al., 2002). The here proposed method has the advantage of containing exponential discounting as a limiting case. It is thus the only method for which results under hyperbolic and exponential discounting can be directly compared with each other.[4]

The associated Hamiltonian reads $H(t,t_0) = \log\left[c(t)\right]\cdot D(t,t_0) + \lambda(t)\left[rk(t) + w(t) - c(t)\right]$. The first order condition for a maximum and the co-state equation are

$$\frac{1}{c(t)D(t,t_0)} - \lambda(t) = 0, \tag{5}$$

$$\lambda(t)r = -\dot{\lambda}(t). \tag{6}$$

Furthermore the usual transversality condition is assumed to hold: $\lim_{t\to\infty}\lambda(t)k(t) = 0$. From solving (6) we get $\lambda(t) = \lambda(t_0)\mathrm{e}^{-r(t-t_0)}$ and therewith (5) becomes

$$c(t) = \frac{D(t,t_0)}{\lambda(t_0)}\ \mathrm{e}^{r(t-t_0)}. \tag{7}$$

Inserting $c(t)$ into the budget constraint (2) and solving the differential equation for the capital stock at time T provides

$$k(T) = k(t_0)\mathrm{e}^{r(T-t_0)} + \int_{t_0}^{T} w(\tau)e^{r(T-\tau)}\mathrm{d}\tau - \frac{1}{\lambda(t_0)}\int_{t_0}^{T} D(\tau,t_0)\mathrm{e}^{r(T-t_0)}\mathrm{d}\tau.$$

Next divide by $\mathrm{e}^{r(T-t_0)}$, take the limit $T\to\infty$ and insert the transversality condition to obtain

$$0 = k(t_0) + \int_{t_0}^{\infty} w(\tau)e^{-r(\tau-t_0)}\mathrm{d}\tau - \frac{1}{\lambda(t_0)}\int_{t_0}^{\infty} D(\tau,t_0)\mathrm{d}\tau.$$

Solving for $\lambda(t_0)$ and plugging the result into (7) provides consumption at time t:

$$c(t) = \frac{k(t_0) + \int_{t_0}^{\infty} w(\tau)e^{-r(\tau-t_0)}\mathrm{d}\tau}{\int_{t_0}^{\infty} D(\tau,t_0)\mathrm{d}\tau}\ D(t,t_0)\ \mathrm{e}^{r(t-t_0)} \Rightarrow c(t) = \frac{k(t) + \int_{t}^{\infty} w(\tau)e^{-r(\tau-t)}\mathrm{d}\tau}{\int_{t}^{\infty} D(\tau,t)\mathrm{d}\tau}. \tag{8}$$

The second equality follows from the time-inconsistency of the plan made at t_0 (see Pollak, 1968; Findley and Caliendo, 2013). At any instant the (naive) consumer believes that her future selves stick to the consumption plan but in fact she sticks to the plan only at the instant of time when the plan is made. As time proceeds the plan is continuously revised

[4] In order to see this set $n = 1/\beta$ and take the limit, $\lim_{n\to\infty}\left(1+\frac{\rho\tau}{n}\right)^{-n} = \mathrm{e}^{-\rho\tau}$.

such that at any time t the plan will be set up anew with starting point t_0. Setting $t_0 = t$ provides the second equality in (8).

2.2. **Firms.** This part of the model follows strictly Romer (1986). The economy is populated by a large number of firms of measure one. For convenience the time index is suppressed. Any firm i uses capital input k_i and labor input $\ell(i)$ to produce output $y(i) = \tilde{A}(i)k(i)^\alpha \ell(i)^{1-\alpha}$. Firms operate under perfect competition such that factor prices are given by $r = \alpha\tilde{A}(i)k(i)^{\alpha-1}\ell(i)^{1-\lambda}$ and $w = (1-\alpha)\tilde{A}(i)k(i)^\alpha \ell(i)^{-\alpha}$. As in Romer (1986) there is learning-by-doing such that the technology available to any firm is a positive function of aggregate capital $\tilde{A}(i) = A\left[\int_0^1 k(j)\mathrm{d}j\right]^{1-\alpha} = Ak^{1-\alpha}$. Using the normalization of firms ($k(i) = k$) this provides the aggregate production function $y = Ak$, wages $w = (1-\alpha)Ak$ and the interest rate $r = \alpha A$.

2.3. **Economic Growth.** Inserting wages and the discount factor (3) into consumption (8) the equation simplifies to

$$c(t) = \rho_0(1-\beta)\left[k(t) + (1-\alpha)A\int_t^\infty k(\tau)e^{-r(\tau-t)}\right].$$

Suppose capital grows at a constant rate $g_k < r$ at all times τ (verified below). Then consumption simplifies further to a linear function of the capital stock.

$$c(t) = a \cdot k(t), \quad a \equiv \rho_0(1-\beta)\left[1 + \frac{(1-\alpha)A}{\alpha A - g_k}\right], \tag{9}$$

and the equation of motion (2) can be rewritten as $\dot{k}/k = g_k = A - a$. From this and (9) we solve for $a = (1-\alpha)A + \rho_0(1-\beta)$ to obtain the solution

$$c(t) = \left[(1-\alpha)A + \rho_0(1-\beta)\right]k(t), \quad g_k = \alpha A - \rho_0(1-\beta). \tag{10}$$

The solution verifies the initial supposition that a constant growth rate $g_k < r$ exists. The rate of economic growth is declining in the initial time preference rate ρ_0 and increasing in the speed of declining impatience β. The fact that growth is constant and that consumption is a constant fraction of capital makes the model observationally equivalent to the standard endogenous growth model with exponential discounting (see e.g. its treatment in Barro and Sala-i.Martin, 2004, Chapter 4). In this sense we confirm for the endogenous growth case the similarity result found by Barro (1999) in the context of neoclassical growth and quasi-hyperbolic discounting.

3. Comparing Growth under Hyperbolical and Exponential Discounting

This section shows that we can actually go farther than stating "only" structural similarity under hyperbolical and exponential discounting. Given a plausible disciplining restriction on the choice of the discounting parameters the two models provide the *identical* solution.

For exponential discounting the discount factor is given by $D(t, t_0) = \exp(-\bar{\rho}t)$, irrespective of t_0 and it is well known that the first order conditions (5) and (6) then provide the time-consistent consumption plan according to the Ramsey rule $\dot{c}/c = r - \bar{\rho}$. In the framework of endogenous growth this further simplifies to the constant growth rate $\dot{c}/c = \alpha A - \bar{\rho}$. For the solution suppose consumption is a constant share of capital, $c = bk$, such that consumption and capital grow at equal rates. From (2), which simplifies to $g_k = A - b$, we then obtain the solution $b = (1 - \alpha)A + \bar{\rho}$, that is

$$c(t) = [(1 - \alpha)A + \bar{\rho}]\, k(t), \quad g_k = \alpha A - \bar{\rho}. \tag{11}$$

Comparing (10) and (11) confirms that the solutions are structurally similar. We can furthermore compare the implied growth rates.

Proposition 1. *Long run growth is higher [lower] under exponential discounting than under hyperbolical discounting for* $\bar{\rho} < (1 - \beta)\rho_0$ *[for* $\bar{\rho} > (1 - \beta)\rho_0$*].*

The proof follows from inspection of (10) and (11).

3.1. A Strict Equivalence Result.

The result in Proposition 1 is intuitive but not very informative without further parameter restrictions. In particular it would be interesting to know whether hyperbolical discounting *per se* leads to lower growth. In other words we would want to figure out whether the feature of non-constant time preference as such and the formation of time-inconsistent plans as such affect growth.

In order to make this assessment we impose the condition that a constant infinite flow provides the same present value irrespective of whether it is discounted with exponential or hyperbolic preferences.

Assumption 1 (Equivalent Present Value). *Discounting parameters are such that a discounted infinite stream provides the same present value. i.e.*

$$\int_0^\infty \exp(-\bar{\rho}t)\mathrm{d}t \overset{!}{=} \int_0^\infty (1+\rho_0\beta t)^{-1/\beta}\mathrm{d}t. \tag{12}$$

If growth would differ under the restriction we could immediately conclude that the mode of preference formation as such affects economic growth.

Proposition 2 (Equivalent Growth). *Given equivalent present value (12) the equilibrium rate of economic growth is the same for hyperbolic and exponential discounting.*

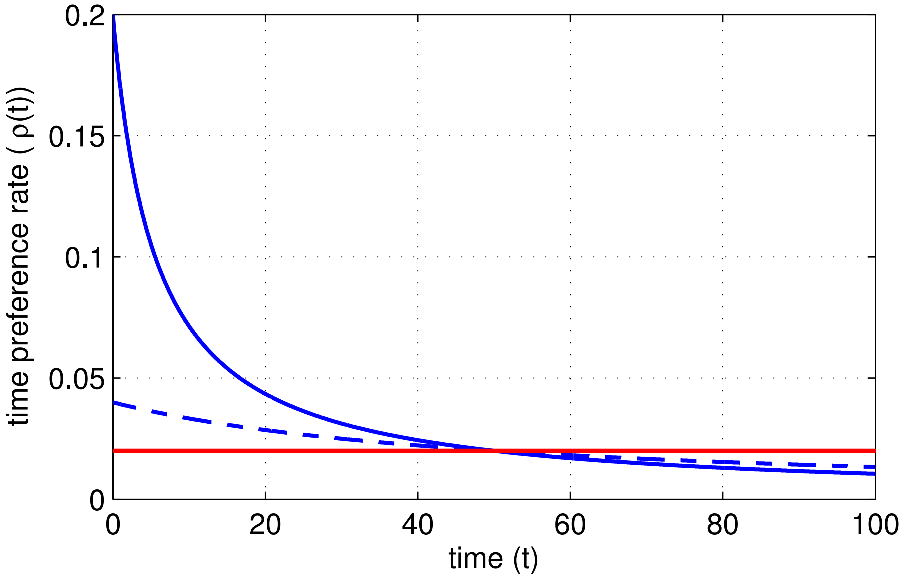
The proof first shows that (12) is solved for $\bar{\rho} = \rho_0(1-\beta)$ and then concludes from inspection of (10) and (11) that the rates of economic growth are equal under this condition. Figure 1 visualizes how time preference evolves over time for some examples fulfilling the equivalence assumption. The red line shows a constant discount rate of $\bar{\rho} = 0.02$. The solid blue line shows the hyperbolical preference rate when $\rho_0 = 0.2$ and $\beta = 0.9$. The dashed line shows another example for $\rho = 0.04$ and $\beta = 0.5$. The preference rates intersect at a unique time t^*. It is straightforward to verify that all discount rates fulfilling Assumption 1 intersect at time $t^* = 1/\bar{\rho}$, which is at $t = 50$ for the examples in Figure 1.

4. Non-Homothetic Utility

In this section we introduce non-homothetic utility by generalizing the objective function. Specifically (1) is replaced by

$$\int_{t_0}^\infty \log\left[c(t) - \bar{c}\right] \cdot D(t,t_0)\ \mathrm{d}t. \tag{13}$$

Figure 1: Equivalent Time Preference Rates



Three time preference rates providing the same present value of a constant infinite flow. Red: constant time preference rate $\bar{\rho} = 0.02$. Blue: hyperbolical discount rate with $\beta = 0.9$ and $\rho_0 = 0.2$ (solid); $\beta = 0.5$ and $\rho_0 = 0.04$ (dashed).

This form of the instantaneous utility function is often referred to as Stone-Geary type. The term $\bar{c}$ is frequently called subsistence consumption in the sense that only consumption above $\bar{c}$ provides positive utility. If consumption falls below $\bar{c}$ the above expression is not defined and we would assign a value of minus infinity to utility. We ensure by assuming a favorable initial condition, namely $k(0) > \bar{c}/(\alpha A)$, that utility will be always positive (see below).

Strictly speaking $\bar{c}$ has little to do with subsistence needs (see Dalgaard and Strulik, 2010) but it is a convenient yet plausible device to generate a non-homothetic utility function. It implies that the elasticity of substitution in consumption goes from zero to one as consumption goes from $\bar{c}$ to infinity. This way the model comprises the basic model as a limiting case and assigns a low elasticity of substitution to poor households with consumption close to subsistence. When placed into the Ak-growth context, non-homothetic utility – in contrast to the basic model – generates plausible long-run adjustment dynamics. It supports the stylized fact that both the savings rate and the rate of economic growth are gradually increasing in the course of development until they approach constant steady-state values (Steger, 2000; Strulik, 2010).

The easiest way to solve the problem of maximizing (13) with respect to (2) is to introduce a transformation of variables. By setting $\tilde{c} \equiv c - \bar{c}$ and $\tilde{k} = k - \bar{c}/r$ it can be rewritten as:

$$\int_{t_0}^{\infty} \log\left[\tilde{c}(t)\right] \cdot D(t,t_0)\ \mathrm{d}t \quad s.t. \qquad \dot{\tilde{k}}(t) = r\tilde{k}(t) + w(t) - \tilde{c}(t). \tag{14}$$

Notice that this is – aside from the name of variables – exactly the original problem from Section 2. In the case of hyperbolic time preferences the solution can be read off from (10): $\bar{c}(t) = [\rho_0(1-\beta) + (1-\alpha)A]\,\tilde{k}$. Re-transformation of variables provides the solution we are interest in:

$$c(t) = [(1-\alpha)A + \rho_0(1-\beta)]\,k(t) + \bar{c}\left[1 - \frac{\rho_0(1-\beta)}{\alpha A}\right] \tag{15}$$

$$g_k = \alpha A - \rho_0(1-\beta) - \frac{\bar{c}}{k}\left[1 - \frac{\rho_0(1-\beta)}{\alpha A}\right]. \tag{16}$$

For $\bar{c} = 0$ the solution collapses to (10). Otherwise the model predicts that the rate of economic growth is gradually increasing with economic development (g_k is gradually increasing in k). The presence of plausible adjustment dynamics is thus no obstacle in getting a closed form solution for the endogenous growth model under hyperbolic time preference.

Proceeding in the same way for the case of exponential discounting provides a set of equations isomorph to (15) and (16) in which $\bar{\rho}$ replaces the term $\rho_0(1-\beta)$. This confirms that the equivalence result of Proposition 2 is robust against the extension towards non-homothetic utility and the introduction of plausible long-run adjustment dynamics.

5. Discussion

This paper has introduced hyperbolic discounting into endogenous growth theory. The result from the neoclassical growth model that growth behavior is structurally identical with that under exponential discounting has been confirmed for the endogenous growth case. Moreover I found an exact analytical solution for savings behavior and equilibrium growth. This feature has been exploited in order to assess the quantitative implications of hyperbolic discounting. It has been shown that hyperbolic discounting supports the same rate of long-run growth as the standard exponential discounting method, given that both methods provide the same present value of a constant infinite stream.

This equivalence result suggest that the possibility of present bias preferences and time-inconsistent savings decisions, which are then continuously discarded, is no threat to the standard model of endogenous growth. Against the background that many seemingly solid insights in economics appear to be non-robust to hyperbolic discounting, it is perhaps reassuring to know that the basic workhorse of modern economic growth theory is robust.

The equivalence result has been derived first for the case of log-utility. In principle it would be desirable to generalize it to other types of utility functions as well. The problem, however, gets quite quickly very complicated. For example, I have not even managed to obtain a closed form solution for the general iso-elastic utility case. I found, however, a solution and confirmed the main result for the Stone-Geary utility case, in which the elasticity of intertemporal substitution varies between zero and unity, depending on the level of consumption. From this finding we can conclude that an intertemporal elasticity of unity (i.e. the balance of income and substitution effects) is non-crucial for the main result of this paper. Moreover, recent empirical research suggest that log-utility seems to be an acceptable first approximation of reality (e.g. Chetty, 2006).

Another objection could be that the basic endogenous growth model is "too stylized". A similar quasi-linear structure, however, is the foundation of many other approaches to endogenous growth as well. In fact, Carroll et al. (2000) argue, based on Rebelo (1991), that the linear Ak production function is the ultimate structure of all endogenous growth models.

I share the belief that the robustness of the endogenous growth model most likely cannot be maintained for other types of manipulations of the utility function that lead to time-inconsistent plans. Naturally, we should expect different output from different input. In this paper I performed a kind of “controlled theoretical experiment”. The result is that, ceteris paribus, endogenous economic growth remains unaffected by present biased preferences and time-inconsistent savings behavior.

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