# Lorentzian manifolds with special holonomy and parallel spinors

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**Abstract**

We investigate the holonomy group of a simply connected, indecomposable and reducible Lorentzian spin manifold and the property that it should admit parallel spinors. We give a condition for such a holonomy to be abelean, show that such manifolds with abelean holonomy have to be pp-manifolds and get as a consequence that pp-manifolds are the only Lorentzian manifolds with nontrivial abelean holonomy and parallel spinors.

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## 1 Introduction

The most effective mean to decide whether a semi-Riemannian spin manifold admits parallel spinors is the holonomy group of the manifold because the space of parallel spinors is isomorphic to the trivial subrepresentations of the spinor representation of

the holonomy group. Therefore it is first necessary to classify the holonomy groups and then to test if they could have trivial spinor subrepresentations.
The first step is done for simply connected, irreducible manifolds by M. Berger [Ber55], [Ber57] and J. Simons [Sim62]. In the Riemannian case this classification becomes complete because of the splitting theorem of G. de Rham [dR52] which asserts that a simply connected, complete Riemannian manifold can be decomposed in a product of irreducible ones. Here M. Y. Wang [Wan89] did the second step and showed that the following holonomy groups from the Berger-list are those of simply connected, irreducible, non-locally symmetric $m$ dimensional Riemannian spin manifolds admitting parallel spinors: $SU(k)$ for $m = 2k$, $Sp(k)$ for $m = 4k$, $G_2$ for $m = 7$ and $Spin(7)$ for $m = 8$.
Although there is a generalization of Wangs theorem for semi-Riemannian simply connected, irreducible, not locally symmetric spin manifolds by H. Baum and I. Kath in [BK99] theres a gap in the pseudo- Riemannian case. This is because the generalisation of de Rhams splitting theorem by H. Wu in [Wu64] only asserts the decomposition of a simply connected, complete semi-Riemannian manifold into a product of indecomposable ones. The holonomy representation of these manifolds has no invariant subspace on which the metric is nondegenerate. In the Riemannian case irreducibility and indecomposability are the same.
So one has to find first the possible holonomy representations of indecomposable, reducible pseudo-Riemannian manifolds. This is an rather open question. The only investigations are from A. Ikemakhen and L. Berard Bergery in the Lorentzian case ([Ike90], [BI93] and [Ike96]), in the case of index 2 in [Ike99] and for index $(r, r)$ in [BI97].
In this paper we will deal with the Lorentzian case and — after some introductory sections — draw some conclusions from the results of [BI93] and [Ike96] for the existence of parallel spinors on simply connected, indecomposable Lorentzian spin manifolds.
In [BI93] the possible holonomy algebras of a simply connected, indecomposable but reducible Lorentzian manifold of dimension $m$ are divided in four types of subalgebras of $(\mathbb{R} \oplus \mathfrak{so}(m-2)) \ltimes \mathbb{R}^{m-2}$. We will add the condition to the manifold to admit parallel spinors so that its holonomy algebra has to be a subalgebra of $\mathfrak{so}(m-2) \ltimes \mathbb{R}^{m-2}$ which leaves us with only two of those four types. We will then show, that the question whether these admit trivial subrepresentations of their spin representation can be reduced to the question if this is the case for their projections on the $\mathfrak{so}(m-2)$-component. Also we will show that this projection can not be abelean.
Further we investigate the simply connected, indecomposable but reducible Lorentzian spin manifolds whose holonomy is abelean, that is isomorphic to $\mathbb{R}^{m-2}$. These have parallel spinors. We will prove that these class of manifold is equal to the so called pp-manifolds, which are an generalisation of the plane-waves. As an result we obtain that pp-manifolds are the only Lorentzian manifolds which admit parallel spinors and have non- trivial abelean holonomy.

# 2 Holonomy and parallel spinors

## 2.1 Principle fibre bundles, vector bundles and holonomy

We first will recall the notion of a holonomy group of a connection in a principle fibre bundle in general.
Let $\pi : P \to M$ a principle fibre bundle with structure group $G$, $\omega : TP \to \mathfrak{g} := LA(G)$ a connection in $P$.
Then one has the horiziontal space $Th_pP := Ker\ \omega$ and the parallel displacement $\mathfrak{p}_\gamma^\omega$ along a piecewise smooth curve $\gamma$ in $M$ from $P_{\gamma(0)}$ to $P_{\gamma(1)}$.
Since $G$ acts freely and transitivly on the fibres one can define the holonomy group to $\omega$ in a point $p \in P$ with $\pi(p) = x \in M$

$$Hol_p(\omega) := \left\{ g \in G \;\middle|\; \begin{array}{c} \text{there is a curve } \gamma \text{ in } M, \gamma(0) = \gamma(1) = x, \\ R_g\left(\gamma_p^*(0)\right) = \gamma_p^*(1) \end{array} \right\} \subset G.$$

The restricted holonomy group $Hol_p^0(\omega)$ is defined by restricting to homotopically trivial curves in $M$. This group is connected and lies in the connected component of the unity in $G$. Its Lie-Algebra is denoted by $\mathfrak{hol}_p(\omega) := LA\left(Hol_p^0(\omega)\right)$.
The holonomy groups depend in the following way from the point $p \in P$: for two points $p$ and $q$ in the same fibre over $x \in M$ such that $q = R_g(p)$ for a $g \in G$ the holonomy groups are conjugated in $G$, i.e. $Hol_q(\omega) = g^{-1}Hol_p(\omega)g$; if there is a horizontal curve in $P$ from $p$ to $q$ then the holonomy groups are the same.
For the curvature $\Omega(X, Y)$ of $\omega$ and the group action holds the following

$$R_g^*\Omega = Ad\left(g^{-1}\right) \circ \Omega. \tag{1}$$

We now formulate the Ambrose-Singer-holonomy-theorem.

**Theorem 2.1** *[AS53] Let $M$ be connected. $P$ a priciple fibre bundle over $M$ with structure group $G$ and $\omega$ a connection in $P$. Then*

$$\mathfrak{hol}_p(\omega) = \left\{ \Omega_q(X, Y) \;\middle|\; \begin{array}{c} \textit{there is a horizontal curve from } p \textit{ to } q \\ X, Y \in Ker\ \omega_q \subset T_qP \end{array} \right\} \subset \mathfrak{g}. \tag{2}$$

In order to localize the connection form and the curvature one considers a local section $s : U \to P$. Then the local connection and curvature form are given by $\omega^s := s^*\omega = \omega \circ ds$ and $\Omega^s := s^*\Omega$.
Since in the terms of the right hand side of (2 ) only horizontal vectors are needed this enables us to calculate these terms via formula (1) with the help of the local curvature form:

$$\Omega_q\left(X, Y\right) = Ad\left(g^{-1}\right)\Omega_{s(x)}\left(dR_{g^{-1}}\left(X\right), dR_{g^{-1}}\left(Y\right)\right) = Ad\left(g^{-1}\right)\Omega_x^s(\hat{X}, \hat{Y}), \tag{3}$$

where $q = R_g(s(x))$ on a horizontal curve from $p$ and $dR_g\left(ds(\hat{X})\right) = X$, $dR_g\left(ds(\hat{Y})\right) = Y$ both horizontal.
Let $(V, \rho)$ a representation of $G$. We now consider the vector bundle associated to $P$ and $G$ defined by $E := (P \times V)/G$. $G$ acting on $V$ is isomorphic to a subgroup of $Gl(E_x)$ acting on the fibre $E_x$ for every fibre over $x$.

$\omega$ defines a covariant derivative by the formula $\nabla^{\omega}_{X}e = [s(x), \rho_*(\omega^s(X))(v) + dv(X)]$ where $s : U \to P$ is a local section in $P$ and $e(x) := [s(x), v(x)]$ a section in the vector bundle. The corresponding parallel displacement of $\nabla^{\omega}$ along a curve $\gamma$ from $x$ to $y$ in $M$, denoted by $\mathcal{P}^{\nabla^{\omega}}_{\gamma} : E_x \to E_y$, then satisfies for a loop $\gamma$ around $x$

$$\mathcal{P}^{\nabla^{\omega}}_{\gamma}(e) = \mathcal{P}^{\nabla^{\omega}}_{\gamma}([p,v]) = \left[\mathfrak{p}^{\omega}_{\gamma}(p), v\right] = [R_g(p), v] = \left[p, \rho(g^{-1})(v)\right] \tag{4}$$

for all $e \in E_x$. Now one can define the holonomy group in a point $x \in M$ with respect to this connection which acts on the fibre $E_x$:

$$Hol_x(E, \nabla^{\omega}) := \{\mathcal{P}^{\nabla^{\omega}}_{\gamma} | \gamma : [0,1] \to M,\ \gamma(0) = \gamma(1) = x\} \subset Gl(E_x).$$

Again one defines the restricted holonomy group $Hol^0_x(E, \nabla^{\omega})$ as a subgroup of $Hol_x(M,h)$ by restricting the possible curves to these which are homotopic to the constant curve. Because of formula (4) the identification of $G$ with a subgroup of $Gl(E_x)$ gives an identification between the holonomy groups

$$(Hol_p(\omega), V) \simeq (Hol_x(E, \nabla^{\omega}), E_x)$$

for $p \in P_x$. So the holonomy groups of two different points in $M$ can be understood as conjugated subgroups of $G$ under this identification.
We now can identify the parallel sections in $E$ with invariant vectors under the holonomy representation:

$$\begin{array}{rcl} V_{Hol_q(\omega)} := \{v \in V \,|\rho(Hol_q(\omega))(v) = v\} & \simeq & \{e \in \Gamma(E) | \nabla^{\omega} e = 0\} \\ v & \mapsto & \left[\mathfrak{p}^{\omega}_{\gamma}(p), v\right] \end{array} \tag{5}$$

Or in case that $M$ is simply connected on the Lie-algebra level

$$V_{\mathfrak{hol}_q(\omega)} := \left\{v \in V \,\middle|\rho_*(\mathfrak{hol}_q(\omega))(v) = 0\right\} \simeq \{e \in \Gamma(E) | \nabla^{\omega} e = 0\}. \tag{6}$$

### 2.2 Holonomy of semi-Riemannian manifolds

Let $(M,h)$ be a semi-Riemannian manifold of dimension $m = r + s$, index $r$ and $\nabla$ the Levi-Civita-connection.
Let $P$ the bundle of orthonormal frames over $M$ with structure group $O(r,s)$. Then one has as in the previous section $P \times_{O(r,s)} \mathbb{R}^m \simeq TM$.
The Levi-Civita connection $\nabla$ defines a connection $\omega$ in $P$ whose local connection form is given by

$$\omega^s = \sum_{1 \le i < j \le n} h\left(\nabla s_i, s_j\right) E_{ij}$$

where $s = (s_1, \dots s_m)$ is a section in $P$, that is a local orthormal frame-field and $E_{ij}$ the standard basis in $\mathfrak{o}(r,s)$. Then is $\nabla = \nabla^{\omega}$. One defines the holonomy groups

$$\begin{array}{rcccl} (Hol_x(M,h), T_xM) & := & (Hol_x(TM,\nabla), T_xM) & \simeq & (Hol_p(\omega), \mathbb{R}^m) \\ \left(Hol^0_x(M,h), T_xM\right) & := & \left(Hol^0_x(TM,\nabla), T_xM\right) & \simeq & \left(Hol^0_p(\omega), \mathbb{R}^m\right) \end{array}.$$

The relation between parallel vector fileds and invariant subspaces of the holonomy representation is given by (see for example [Bes87])

**Theorem 2.2** *Let $(M, h)$ be a semi-Riemannian manifold. Then holds the following*

1. *For $1 \leq k < m$ these propositions are equivalent:*

   *(i) There is a $k$-dimensional distribution, which is invariant under parallel displacement.*

   *(ii) The holonomy representation leaves invariant a $k$-dimensional subspace.*

   *This distribution has to be involutive.*

2. *There is a parallel vectorfield on $M$ if and only if the holonomy representation has a trivial subrepresentation, that means it leaves fixed a vector.*

The properties of the representation are used to characterize the manifold.

**Definition 2.1** *A semi-Riemannian manifold $(M^{r,s}, h)$ is called*

1. *(strictly) irreducibel if the (reduced) holonomy representation has no invariant subspace,*

2. *(strictly) indecomposable if the (reduced) holonomy representation has no invariant subspace on which $h$ is non-degenerate,*

3. *reducibel/decomposable if it is not irreducibel/indecomposable.*

Irreducibility entails indecomposeability and decomposeability reducibility. For Riemannian manifolds both notions are the same. The tangential space of a semi-Riemannian manifold can be decomposed in a orthogonal sum of invariant subspaces which can be (in case of nontrivial signature of $h$) reducible with a degenerate, invariant subspace. For given semi-Riemannian manifolds $(M_1^{r,s}, h_1)$ and $(M_2^{p,q}, h_2)$ the product manifold $(M_1^{r,s} \times M_2^{p,q}, h_1 \oplus h_2)$ has the tangential space $T_{(x_1,x_2)}(M_1 \times M_2) = T_{x_1}M_1 \oplus T_{x_2}M_2$. Now the holonomy representation of the product is the product of the holonomy representations (see again [Bes87])

$$
\begin{aligned}
Hol_{(x_1,x_2)}(M_1 \times M_2, h_1 \oplus h_2) &= Hol_{x_1}(M_1, h_1) \times Hol_{x_2}(M_2, h_2) \\
&= (Hol_{x_1}(M_1, h_1) \times 1) \oplus (1 \times Hol_{x_2}(M_2, h_2)) .
\end{aligned}
$$

The de-Rham decomposition theorem asserts that locally the converse is true.

**Theorem 2.3** *[dR52] [Wu64] Every simply-connected, complete semi-Riemannian manifold is isometric to a product of simply-connected, complete manifolds, of which one can be flat and all other are indecomposable.*

That means locally it is sufficient to know the indecomposable manifolds. For the subclass of irreducible ones there is the classification of Berger and Simons.

**Theorem 2.4** *[Ber55], [Sim62], [Ale68], [BG72] and [Bry87] Let $(M^{r,s}, h)$ be a simply-connected, irreducible, non local-symmetric semi-Riemannian manifold of dimension*

*$m=r+s$ and index $r$. Then the holonomy representation on $\mathbb{R}^m$ is one of the following (modulo conjugation in $O(r,s)$):*

$$
\begin{array}{lcllcl}
m = r+s \geq 2 & : & SO(r,s) & & & \\
m = 2p+2q \geq 4 & : & U(p,q) & & \subset & SO(2p,2q) \\
 & oder & SU(p,q) & & \subset & SO(2p,2q) \\
m = 4p+4q \geq 8 & : & Sp(p,q) & & \subset & SO(4p,4q) \\
 & oder & Sp(p,q)\cdot Sp(1) & & \subset & SO(4p,4q) \\
m = r+r \geq 4 & : & SO(r,\mathbb{C}) & & \subset & SO(r,r) \\
m = 2p+2p \geq 8 & & Sp(p,\mathbb{R})\cdot Sl(2,\mathbb{R}) & & \subset & SO(2p,2p) \\
m = 4p+4p \geq 16 & & Sp(p,\mathbb{C})\cdot Sl(2,\mathbb{C}) & & \subset & SO(4p,4p) \\
m = 7 = 0+7 & : & G_2 & & \subset & SO(7) \\
m = 7 = 4+3 & : & G^*_{2(2)} & & \subset & SO(4,3) \\
m = 14 = 7+7 & : & G_2^{\mathbb{C}} & & \subset & SO(7,7) \\
m = 8 = 0+8 & : & Spin(7) & & \subset & SO(8) \\
m = 8 = 4+4 & : & Spin(4,3) & & \subset & SO(4,4) \\
m = 16 = 8+8 & : & Spin(7)^{\mathbb{C}} & & \subset & SO(8,8)
\end{array}
$$

In case of symmetric spaces there is a classification of Berger [Ber57], which gives the following corollary for Lorentzian manifolds.

**Corollary 2.1** *A simply-connected, complete and irreducible Lorentzian manifold has the trivial holonomy group or the full $SO_0(1,m-1)$.*

### 2.3 Spinor representations and the spinor bundle

We now want to define the spinor bundle of a m=(r+s)-dimensional semi- Riemannian manifold. (See [LM89], [Bau81] also [Bau94] for details.) Let $\mathcal{C}l_{r,s}$ be the Clifford-algebra of $(\mathbb{R}^m, \langle .,. \rangle_{r,s})$ where $\langle .,. \rangle_{r,s}$ is the standard scalarproduct of index $r$ on $\mathbb{R}^m$ We define the following groups in $\mathcal{C}l_{r,s}$ ($\langle X,X\rangle_{r,s} =: \|X\|_{r,s}$):

$$
\begin{array}{rcl}
Spin(r,s) & := & \{X_1\cdot\ldots\cdot X_{2k} \mid \|X_i\|_{r,s} = \pm 1,\ k\geq 0\} \\
Spin_0(r,s) & = & \{X_1\cdot\ldots\cdot X_{2k}\cdot Y_1\cdot\ldots\cdot Y_{2l} \mid \|X_i\|_{r,s} = 1, \|Y_i\|_{r,s} = -1,\ k,l\geq 0\}
\end{array}
$$

$$
K = \left\{ X_1\cdot\ldots\cdot X_{2k}\cdot Y_1\cdot\ldots\cdot Y_{2l} \;\middle|\; \begin{array}{l} \|X_i\|_{r,s} = 1, \|Y_i\|_{r,s} = -1, \quad k,l\geq 0 \\ X_i \in span(e_1,\ldots,e_r),\ Y_j\in span(e_{r+1},\ldots,e_m) \end{array} \right\}
$$

where $(e_1,\ldots e_m)$ is a basis with $\langle e_i,e_j\rangle = \delta_{ij}\kappa_i$, $\kappa_1 = \ldots \kappa_r = -1$ and $\kappa_{r+1} = \ldots \kappa_m = 1$.

$Spin_0(r,s)$ is the connected component of the unit and $K$ its maximal compact subgroup.

Now let $\lambda : Spin(r,s) \to SO(r,s)$ be the twofold covering of $SO(r,s)[SO_0(r,s)]$ by $Spin(r,s)[Spin_0(r,s)]$. Now $\lambda_*$ is an algebra isomorphism between $\mathfrak{spin}(r,s) := LA(Spin(r,s)) \subset \mathcal{C}l_{r,s}$ and $\mathfrak{so}(r,s)$. Then

$$\mathfrak{spin}(r,s) = span\{e_i\cdot e_j | 1\leq i<j\leq m\}$$

and $\lambda_*(e_i \cdot e_j) = E_{ij}$ for matrices

$$(E_{ij})_{kl} = \begin{cases} -\kappa_j & \text{for} \quad (k,l) = (i,j) \\ \kappa_i & \text{for} \quad (k,l) = (j,i) \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

the standard basis in $\mathfrak{so}(r,s)$.
We will now give an isomorphism between the complexification of the Clifford algebra $\mathbb{C}l_{(r,s)} := \mathcal{C}l_{r,s} \otimes \mathbb{C}$ and endomorphism algebras of complex vector spaces which yields to complex representations of the spin group. First we consider the $\mathbb{C}^2$ with a basis $\left\{ u(\varepsilon) := \frac{1}{\sqrt{2}} \left( \begin{smallmatrix} 1 \\ -\varepsilon\, i \end{smallmatrix} \right) | \varepsilon = \pm 1 \right\}$ and the isomorphisms of $\mathbb{C}^2$:

$$E := Id \ , \ T := \begin{pmatrix} o & -i \\ i & 0 \end{pmatrix} \ , \ U := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ , \ V := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then we have $T^2 = -V^2 = -U^2 = E$ , $UT = -iV$ , $VT = iU$ , $UV = -iT$ and $Tu(\varepsilon) = -\varepsilon u(\varepsilon)$ , $Uu(\varepsilon) = iu(-\varepsilon)$ , $Vu(\varepsilon) = \varepsilon u(-\varepsilon)$. We define the isomorphisms as follows

1. In case $m$ is even $\Phi_{(r,s)} : \mathbb{C}l_{(r,s)} \to \mathbb{C}\left(2^{\frac{m}{2}}\right)$ is defined by

$$\begin{aligned} \Phi_{(r,s)}(e_{2k-1}) &:= \tau_{2k-1} E \otimes \ldots \otimes E \otimes U \otimes \underbrace{T \otimes \ldots T}_{(k-1)\text{-times}} \\ \Phi_{(r,s)}(e_{2k}) &:= \tau_{2k} E \otimes \ldots \otimes E \otimes V \otimes \underbrace{T \otimes \ldots T}_{(k-1)\text{-times}} \end{aligned}$$

   with $\tau_1 = \ldots = \tau_r = i$ and $\tau_{r+1} = \ldots = \tau_m = 1$ and $k = 1 \ldots \frac{n}{2}$.

2. In case $m$ is odd $\Phi_{(r,s)} : \mathbb{C}l_{(r,s)} \to \mathbb{C}\left(2^{\frac{m-1}{2}}\right) \oplus \mathbb{C}\left(2^{\frac{m-1}{2}}\right)$ is defined by

$$\begin{aligned} \Phi_{(r,s)}(e_k) &= \left(\Phi_{(m-2,1)}(e_k), \Phi_{(m-2,1)}(e_k)\right) \ , \ k = 1 \ldots m-1 \\ \Phi_{(r,s)}(e_m) &= (iT \otimes \ldots \otimes T, -iT \otimes \ldots \otimes T) \, . \end{aligned}$$

This yields to representations of the spin group and algebra in case $m$ even by restriction and in case $m$ odd by restriction and projection onto the first component. The representation space $\Delta_{r,s} = \mathbb{C}^{2^{[\frac{m}{2}]}}$ is called spinor module. We write $A \cdot v := \Phi_{(r,s)}(A)(v)$ for $A \in \mathbb{C}l_{r,s}$ and $v \in \Delta_{r,s}$. A useful basis in $\Delta_{r,s}$ is the following: $(u\,(\varepsilon_k, \ldots, \varepsilon_1) := u(\varepsilon_k) \otimes \ldots \otimes u(\varepsilon_1) | \varepsilon_i = \pm 1)$.
In case $m$ is even the representation space $\Delta_{r,s}$ splits into two irreducible subspaces $\Delta^{\pm}_{r,s} := span(u\,(\varepsilon_k, \ldots, \varepsilon_1)\,| \varepsilon_k \cdot \ldots \cdot \varepsilon_1 = \pm 1)$.
On $\mathbb{C}^{2^{[\frac{m}{2}]}}$ are given two scalar products, the standard scalar product $(.,.)$, which is positiv definit and invariant under the action of the maximal compact subgroup $K$ of $Spin_0(r,s)$. The second is defined as follows

$$\langle u, v \rangle := i^{\frac{r(r-1)}{2}} (e_1 \cdot \ldots \cdot e_r \cdot u, v)$$

and it is indefinit and invariant under $Spin_0(r,s)$. The basis given by $u\,(\varepsilon_k,\ldots,\varepsilon_1)$ with $k := [\frac{m}{2}]$ is an ortonormal basis for $(.,.)$.
We now consider a semi-Riemannian manifold which is assumed to be spin. Let $(Q,f)$ be the $\lambda$-reduction for the orthonormal frame bundle $P$. Then we have $TM = P \times_{SO_0(r,s)} \mathbb{R}^m = Q \times_{Spin_0(r,s)} \mathbb{R}^m$. The vector bundle

$$S := Q \times_{Spin_0(r,s)} \Delta_{r,s}$$

is called spinor bundle. The Clifford multiplication is defined as follows

$$\begin{array}{rcl} TM \otimes S = \left(Q \times_{Spin_0(r,s)} \mathbb{R}^m\right) \otimes \left(Q \times_{Spin_0(r,s)} \Delta_{r,s}\right) & \to & S \\ X \otimes \varphi = [q,x] \otimes [q,v] & \mapsto & [q, X \cdot v] =: X \cdot \varphi \end{array}$$

Since the indefinit scalar product $\langle .,. \rangle$ is invariant under $Spin_0(r,s)$ it defines a scalar product on $S$. But one can also define the positiv definit scalar product invariantly by fixing a timelike (negativ definit) subbundle $\xi \subset TM$. Then there is a $K \subset Spin_0(r,s)$-reduction of $Q$ to a $K$-principle bundle $Q_\xi$ for which

$$S = Q \times_{Spin_0(r,s)} \Delta_{r,s} = Q_\xi \times_K \Delta_{r,s}$$

so that the $K$ invariance of $(.,.)$ suffices to define positiv definite scalarproduct $(.,.)_\xi$ on $S$.
The covariant derivative defined by the Levi-Civita connection is given by

$$\nabla^S_X \varphi = X(\varphi) + \frac{1}{2} \sum_{i<j} \kappa_i\ \kappa_j\ g(\nabla^{LC}_{s_i} X, s_j) s_i \cdot s_j \cdot \varphi \tag{8}$$

$\nabla^S$ is metrical with respect to $\langle .,. \rangle$ and distributive in the following way

$$\nabla^S (X \cdot \varphi) = (\nabla X) \cdot \varphi + X \cdot \nabla^S \varphi$$

For its holonomy group we have the important relation to those of $\nabla$

$$\lambda \left( Hol_x(S, \nabla^S \right) = Hol_x(M,h)$$

so that $\lambda_*$ identifies the Lie algebras

$$\mathfrak{hol}_x(S, \nabla^S) \stackrel{\lambda_*}{\simeq} \mathfrak{hol}_x(M,h).$$

## 2.4 Parallel spinors on product manifolds

First let $(M,h)$ be a semi-Riemannian spin manifold of dimension $m$. We define

**Definition 2.2** *A non-trivial spinor field $\varphi \in \Gamma(S)$ is called parallel spinor if $\nabla^S \varphi = 0$.*

If we set $\tilde{H} := \lambda^{-1}(Hol_x(M,h))$ and $\tilde{\mathfrak{h}} := \lambda_*^{-1}(\mathfrak{h})$ then we have from the previous section

$$\{\text{parallel spinors}\} \simeq V_{\tilde{H}} := \left\{ v \in \Delta_{r,s} \,\middle|\, (\Phi_{(r,s)})(\tilde{H})(v) = v \right\}$$

or in the simply connectaed case

$$\{\text{parallel spinors}\} \simeq V_{\tilde{\mathfrak{h}}} := \left\{ v \in \Delta_{r,s} \,\middle|\, (\Phi_{(r,s)})_*(\tilde{\mathfrak{h}})(v) = 0 \right\}.$$

We now want to find properties of manifolds admitting parallel spinors. On has the following

**Lemma 2.1** *([Hit74] for the Riemannian case, [Bau81] for the general) The existence of a parallel spinor, entails an totally isotropic Ricci endomorphism, i.e.* $h(Ric(X), Ric(Y)) = 0$ *for all* $X, Y \in TM$.

For product manifolds holds the following

**Theorem 2.5** *On a semi-Riemannian manifold which is given as a product*

$$(M,h) \simeq (M_1,h_1) \times (M_2,h_2)$$

*of two semi-Riemannian manifolds exist parallel spinors if and only if on both* $(M_i,h_i)$ *exist parallel spinors.*

**Proof:**
Let $(M_i,h_i)$ be two semi-Riemannian manifolds of dimension $m_i$ and signature $(r_i,s_i)$ and $(M,h)$ the product of both of signature $(r = r_1 + r_2, s = s_1 + s_2)$. Its holonomy is given by the product

$$H := Hol_{(x_1,x_2)}(M,h) = Hol_{x_1}(M_1,h_1) \times Hol_{x_2}(M_2,h_2) =: H_1 \times H_2$$

We denote the corresponding lifts in the spin group via the twofold covering $\lambda$ by $\tilde{H}, \tilde{H}_1$ and $\tilde{H}_2$. Then we have the following commuting diagramm

$$\begin{array}{ccccc} H = H_1 \times H_2 & \subset & O(r_1,s_1) \times O(r_2,s_2) & \hookrightarrow & O(r,s) \\ & & \uparrow \lambda \times \lambda & & \uparrow \lambda \\ \tilde{H} = \left(\tilde{H}_1 \times \tilde{H}_2\right)_{/\mathbb{Z}_2} & \subset & \left(Spin(r_1,s_1) \times Spin(r_2,s_2)\right)_{/\mathbb{Z}_2} & \hookrightarrow & Spin(r,s) \end{array}$$

where the imbeddings of the product groups into the $O(r,s)$ resp. $Spin(r,s)$ come from the canonical identification of $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ and $\mathbb{R}^m$ sending $e_i \in \mathbb{R}^{m_1}$ onto $e_i \in \mathbb{R}^m$ and $e_i \in \mathbb{R}^{m_2}$ onto $e_{m_1+i} \in \mathbb{R}^m$.
So we have two representations of $\left(Spin(r_1,s_1) \times Spin(r_2,s_2)\right)_{/\mathbb{Z}_2}$: The first is given as the representation tensor product:

$$\begin{array}{cccl} \rho & : & \left(Spin(r_1,s_1) \times Spin(r_2,s_2)\right)_{/\mathbb{Z}_2} & \to & End(\Delta_{r_1,s_1} \otimes \Delta_{r_2,s_2}) \\ & & [(A,B)] & \mapsto & \Phi_{(r_1,s_1)}(A) \otimes \Phi_{(r_2,s_2)}(B) \end{array}$$

This definition is correct since $\rho([(A,B)]) = \rho([(-A,-B)])$.
The second results from the restriction of $\Phi_{(r,s)}$ on $Spin(r_1,s_1) \times Spin(r_2,s_2) \hookrightarrow Spin(r,s)$:

$$\Phi := \left(\Phi_{(r,s)}\right)_{|Spin(r_1,s_1) \times Spin(r_2,s_2)} \quad : \quad Spin(r_1,s_1) \times Spin(r_2,s_2) \quad \to \quad End(\Delta_{r,s}).$$

If $m$ is even one has the two irreducible subrepresentations $\Phi^{\pm}_{(r,s)} : Spin(r,s) \to End(\Delta^{\pm}_{r,s})$ and consequently $\Phi^{\pm}$ by restriction again. With this notations we prove the following

**Lemma 2.2** *For the defined representations of* $Spin(r_1,s_1) \times Spin(r_2,s_2)$ *holds the following*

1. *If $m_1$ or $m_2$ is even then we have $\Phi \sim \rho$.*

2. *For both $m_1$ and $m_2$ odd we have $\Phi^{\pm} \sim \rho$.*

**Proof: 1.)** Let $m_1 = 2\ k_1$ be even and $m = 2k$ in case $m_2$ beeing even too and $m = 2k+1$ in case $m_2$ beeing odd. The isomorphism between the representation spaces $\Delta_{r,s}$ and $\Delta_{r_1,s_1} \otimes \Delta_{r_2,s_2}$ is given by the identity in both cases, denoted by:

$$\begin{array}{rrcl} \Psi & : \qquad \Delta_{r,s} & \to & \Delta_{r_1,s_1} \otimes \Delta_{r_2,s_2} \\ & u(\varepsilon_k, \ldots, \varepsilon_1) & \mapsto & u(\varepsilon_k, \ldots, \varepsilon_{k_1+1}) \otimes u(\varepsilon_{k_1}, \ldots, \varepsilon_1) \end{array}$$

For the Clifford-multiplication, that is for $\Phi_{(r,s)}$ as the representation of the Clifford algebra we have for $X \in \mathbb{R}^{m_1}$ and the corresponding $\hat{X} \in \mathbb{R}^m$:

$$\begin{aligned} \Psi\left(\hat{X} \cdot u(\varepsilon_k, \ldots, \varepsilon_1)\right) &= \Psi\left(\Phi_{(r,s)}\left(\hat{X}\right)(u(\varepsilon_k, \ldots, \varepsilon_1))\right) \\ &= u(\varepsilon_k, \ldots, \varepsilon_{k_1+1}) \otimes \Phi_{(r_1,s_1)}(X)\,(u(\varepsilon_{k_1}, \ldots, \varepsilon_1)) \\ &= u(\varepsilon_k, \ldots, \varepsilon_{k_1+1}) \otimes X \cdot u(\varepsilon_{k_1}, \ldots, \varepsilon_1) \end{aligned}$$

and for $Y \in \mathbb{R}^{m_2}$ and the corresponding $\hat{Y} \in \mathbb{R}^m$:

$$\begin{aligned} &\Psi\left(\hat{Y} \cdot u(\varepsilon_k, \ldots, \varepsilon_1)\right) \\ &= \Psi\left(\Phi_{(r,s)}\left(\hat{Y}\right)(u(\varepsilon_k, \ldots, \varepsilon_1))\right) \\ &= (-1)^{k_1} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{k_1}\ \Phi_{(r_2,s_2)}(Y)\,(u(\varepsilon_k, \ldots, \varepsilon_{k_1+1})) \otimes u(\varepsilon_{k_1}, \ldots, \varepsilon_1) \\ &= (-1)^{k_1} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{k_1}\ Y \cdot u(\varepsilon_k, \ldots, \varepsilon_{k_1+1}) \otimes u(\varepsilon_{k_1}, \ldots, \varepsilon_1). \end{aligned}$$

Therefore we have for $g = X_1 \cdot \ldots \cdot X_{2i} \in Spin(r_1, s_1)$ with $X_l \in \mathbb{R}^{m_1}$ and $h = Y_1 \cdot \ldots \cdot Y_{2j} \in Spin(r_1, s_1)$ with $Y_l \in \mathbb{R}^{m_2}$

$$\begin{aligned} (\rho(g,h) \circ \Psi)(u(\varepsilon_k, \ldots, \varepsilon_1)) &= \Phi_{(r_2,s_2)}(g) u(\varepsilon_k, \ldots, \varepsilon_{k_1+1}) \otimes \Phi_{(r_1,s_1)}(h) u(\varepsilon_{k_1}, \ldots, \varepsilon_1) \\ &= \Psi\left(\Phi_{(r,s)}(g \cdot h)\,(u(\varepsilon_k, \ldots, \varepsilon_1))\right) \\ &= \Psi \circ \Phi(g,h)\,(u(\varepsilon_k, \ldots, \varepsilon_1)) \end{aligned}$$

which shows that both representations are equivalent.
**2.)** It is now $m_i = 2\ k_i + 1$ such that $m = m_1 + m_2 = 2\ (k+1)$ with $k = k_1 + k_2$. For technical reasons we consider a $\iota \in O(r,s)$ which is the identity on $span(e_1, \ldots, e_{m_1})$ and further defined by

$$\iota(e_{m_1+i}) = e_{m_1+1+i} \text{ for } 1 \le i < m_1 \quad \text{and} \quad \iota(e_m) = e_{m_1+1}$$

The resulting representation of the Clifford algebra $\Phi_\iota := \Phi_{(r,s)} \circ \iota$ is equivalent to $\Phi_{(r,s)}$. This holds because the definition of the covering $\lambda$ allows to write $\iota = \lambda(g)$ for $g \in Spin(r,s)$ and therefore

$$\Phi_\iota(X) = \Phi_{(r,s)}(\lambda(g)(X)) = \Phi_{(r,s)}(g \cdot X \cdot g^{-1}) = g \cdot \Phi_{(r,s)}(X) \cdot g^{-1}$$

which verifies equivalence.

We will now show that the corresponding irreducible representation $\Phi_\iota^\pm$ are equivalent to $\rho$.
The representation space of $\Phi_\iota^\pm$ is $\mathbb{C}^{2^k} \simeq \Delta_{r,s}^\pm \subset \Delta_{r,s} \simeq \mathbb{C}^{2^{k+1}}$ and of $\rho$ is $\Delta_{r_1,s_1} \otimes \Delta_{r_2,s_2} \simeq \mathbb{C}^{2^k}$. The isomorphism is the following

$$\begin{array}{rcl} \Psi^\pm : \Delta_{r,s}^\pm & \to & \Delta_{r_1,s_1} \otimes \Delta_{r_2,s_2} \\ u(\varepsilon_{k+1}, \ldots, \varepsilon_1) & \mapsto & a_{\varepsilon_{k+1},\ldots,\varepsilon_{k_1+2}} u(\varepsilon_{k+1}, \ldots, \varepsilon_{k_1+2}) \otimes u(\varepsilon_{k_1}, \ldots, \varepsilon_1) \\ & & \text{with } a_{\varepsilon_{k+1},\ldots,\varepsilon_{k_1+2}} = \left\{ \begin{array}{rcl} 1 & : & \prod_{l=1}^{k_2} \varepsilon_{k_1+1+l} = 1 \\ i\ (-1)^{k_2+1} & : & \prod_{l=1}^{k_2} \varepsilon_{k_1+1+l} = -1 \end{array} \right. \end{array}$$

One has to verify that $\Psi^\pm \circ \Phi_\iota^\pm(g,h) = \rho(g,h) \circ \Psi^\pm$. We first show this relation for $g$ and $h$ of the form $e_i \cdot e_j$ with $e_i, e_j \in \mathbb{R}^{m_2}$ resp. $\mathbb{R}^{m_1}$.
For $i = j$ this relation holds clearly since $e_i^2$ is a multiplication with a scalar.
We distinguish the following cases:
1. $e_i, e_j \in \mathbb{R}^{m_1}$ and

- both different from $e_{m_1}$. Then it is clear that

$$\begin{array}{l} \Psi^\pm \left(\iota(e_i) \cdot \iota(e_j) \cdot u(\varepsilon_{k+1}, \ldots, \varepsilon_1)\right) \\ \quad = \quad \Psi^\pm \left(u(\varepsilon_{k+1}, \ldots, \varepsilon_{k_1+1}) \otimes e_i \cdot e_j \cdot u(\varepsilon_{k_1}, \ldots, \varepsilon_1)\right) \\ \quad = \quad \rho(id, e_i \cdot e_j) \circ \Psi^\pm \left(u(\varepsilon_k, \ldots, \varepsilon_1)\right). \end{array}$$

- one of them equals to $e_{m_1}$. W.l.o.g. should be $j = m_1$. Then one has

$$\begin{array}{l} \Psi^\pm \left(\iota(e_i) \cdot \iota(e_{m_1}) \cdot u(\varepsilon_{k+1}, \ldots, \varepsilon_1)\right) \\ \quad = \quad \Psi^\pm \Big(\iota(e_i) \cdot (-1)^{k_1} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{k_1}\ i\ u(\varepsilon_{k+1}, \ldots, \varepsilon_{k_1+2}, -\varepsilon_{k_1+1}, \varepsilon_{k_1}, \ldots, \varepsilon_1)\Big) \\ \quad = \quad \Psi^\pm \Big((-1)^{k_1} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{k_1}\ i\ u(\varepsilon_{k+1}, \ldots, \varepsilon_{k_1+2}, -\varepsilon_{k_1+1}) \otimes e_i \cdot u(\varepsilon_{k_1}, \ldots, \varepsilon_1)\Big) \\ \quad = \quad \Psi^\pm \left(u(\varepsilon_{k+1}, \ldots, -\varepsilon_{k_1+1}) \otimes e_i \cdot e_{m_1} \cdot u(\varepsilon_{k_1}, \ldots, \varepsilon_1)\right) \\ \quad = \quad \rho(id, e_i \cdot e_{m_1}) \circ \Psi^\pm(u(\varepsilon_k, \ldots, \varepsilon_1)). \end{array}$$

So one has for $e_i, e_j \in \mathbb{R}^{m_1}$

$$\Psi^\pm \circ \Phi_\iota^\pm(id, e_i \cdot e_j) = \rho(id, e_i \cdot e_j) \circ \Psi^\pm$$

and therefore for all $h \in Spin(r_1, s_1)$

$$\Psi^\pm \circ \Phi_\iota^\pm(id, h) = \rho(id, h) \circ \Psi^\pm.$$

2. $e_i, e_j \in \mathbb{R}^{m_2}$ and

- both not equal to $e_{m_2}$. Here we have again

$$\begin{array}{l} \Psi^\pm \left(\iota(e_i) \cdot \iota(e_j) \cdot u(\varepsilon_{k+1}, \ldots, \varepsilon_1)\right) \\ \quad = \quad \Psi^\pm \left(e_{m_1+1+i} \cdot e_{m_1+1+j} \cdot u(\varepsilon_{k+1}, \ldots, \varepsilon_1)\right) \\ \quad = \quad (-1)^{2(k_1+1)} \Psi^\pm \left((e_i \cdot e_j \cdot u(\varepsilon_{k+1}, \ldots, \varepsilon_{k_1+2})) \otimes \cdot u(\varepsilon_{k_1+1}, \ldots, \varepsilon_1)\right) \\ \quad = \quad \rho(e_i \cdot e_j, id) \circ \Psi^\pm u(\varepsilon_k, \ldots, \varepsilon_1). \end{array}$$

- $e_j = e_{m_2}$. Then

$$\begin{aligned}
&\Psi^{\pm}\left(\iota(e_i)\cdot\iota(e_{m_2})\cdot u(\varepsilon_{k+1},\ldots,\varepsilon_1)\right)\\
&= \Psi^{\pm}\left(e_{m_1+1+i}\cdot e_{m_1+1}\cdot u(\varepsilon_{k+1},\ldots,\varepsilon_1)\right)\\
&= \Psi^{\pm}\left((e_i\cdot u(\varepsilon_{k+1},\ldots,\varepsilon_{k_1+2}))\otimes u(-\varepsilon_{k_1+1},\varepsilon_{k_1},\ldots,\varepsilon_1)\right)\\
&= a_{\varepsilon_{k+1},\ldots,\varepsilon_{k_1+2}}\left(e_{m_2}\cdot e_i\cdot u(\varepsilon_{k+1},\ldots,\varepsilon_{k_1+2})\right)\otimes u(\varepsilon_{k_1},\ldots,\varepsilon_1)\\
&= \rho(e_{m_2}\cdot e_i, id)\circ\Psi^{\pm}\left(u(\varepsilon_{k+1},\ldots,\varepsilon_1)\right)
\end{aligned}$$

So one has for all $g \in Spin(r_2, s_2)$ that

$$\Psi^{\pm}\circ\Phi^{\pm}_{\iota}(g, id) = \rho(g, id)\circ\Psi^{\pm}$$

and therefore the conclusion of the lemma. □

Finally we have two equivalent representations of $Spin(r_1, s_1) \times Spin(r_2, s_2)$

$$\Phi \sim \Phi_{(r_1,s_1)}\otimes\Phi_{(r_2,s_2)} \text{ or } \Phi^{\pm}\sim\Phi_{(r_1,s_1)}\otimes\Phi_{(r_2,s_2)}.$$

which are by restriction two equivalent representations of $H = H_1 \times H_2$. For this situation we prove the following

**Lemma 2.3** *Let $\rho$ be an representation of a group $G := G_1\times G_2$ and $\rho_i$ representations of $G_i$ such that $\rho \sim \rho_1\otimes\rho_2$. The $\rho$ has a trivial subrepresentation iff both $\rho_i$ have an trivial subrepresentation.*

**Proof:** One direction is clear. If $\Psi : V \to V_1 \otimes V_2$ is the isomorphism between the representation spaces and $v_i$ fixed vectors under $\rho_i$ $(i = 1, 2)$ then $v := \Psi^{-1}(v_1 \otimes v_2)$ is fixed under $\rho$.
If otherwise $v$ is a fixed vector under $\rho$. Then $\Psi(v)$ is a fixed vector under $\rho_1 \otimes \rho_2$, i. e. it defines a trivial representation of $G_1 \times G_2$. Then a general theorem in representation theory (see for example [Tit67]) says that every irreducible representation of a group product $G_1 \times G_2$ splits into a tensor product of two irreducible representations of $G_1$ and $G_2$. Thats why $\Psi(v)$ has the form $v_1 \otimes v_2$ and one gets

$$v_1\otimes v_2 = \Psi\left(\rho(g_1,g_2)v\right) = \rho_1(g_1)\otimes\rho_2(g_2)\left(v_1\otimes v_2\right) = \rho_1(g_1)v_1\otimes\rho_2(g_2)v_2.$$

But this gives $\rho_i(g_i)v_i = v_i$ for all $g_i \in G_i$, $i = 1, 2$. □

Now a parallel spinor on $(M, h)$ corresponds to a trivial subrepresentation of $H = H_1 \times H_2 \subset Spin(r_1, s_1) \times Spin(r_2, s_2)$. Due to the lemmata it corresponds to a pair of trivial subrepresentations of $H_1$ resp. $H_2$ which correspond to a pair of parallel spinors on $(M_1, h_1)$ resp. $(M_2, h_2)$. This was the assertion of the theorem. □

## 2.5 Decomposition of a Lorentzian manifold with parallel spinors

From now on let $(M, h)$ be a Lorentzian spin manifold of dimension $m$. We asign to every spinor field a vector field.

**Definition 2.3** *Let $\varphi \in \Gamma(S)$ The vector field $V_\varphi \in \Gamma(TM)$ defined by $h(V_\varphi, X) = -\langle X \cdot \varphi, \varphi \rangle$ for all $X \in TM$ is called $\varphi$-associated vector field or Dirac-current.*

For the Dirac-current the following properties hold.

**Lemma 2.4** *Let $(M, h)$ a Lorentzian spin manifold and $\varphi \in \Gamma(S)$.*

1. *It is $h(V_\varphi, V_\varphi) \leq 0$ and $V_\varphi(x) = 0$ if and only if $\varphi(x) = 0$.*
2. *$\varphi$ parallel entails $V_\varphi$ parallel.*

**Proof:** 1.) Since $(M, h)$ is spin we have $\xi$ the timelike global defined unit vectorfield. Then we have

$$h(V_\varphi, \xi) = -\langle \xi \cdot \varphi, \varphi \rangle := -(\xi \cdot \xi \cdot \varphi, \varphi)_\xi = -(\varphi, \varphi)_\xi \neq 0 \quad \text{if } \varphi \neq 0.$$

Therefore one can write $V_\varphi = (\varphi, \varphi)_\xi \, \xi + \eta$ with a spacelike vector field $\eta$.
We now consider an aritrary $x \in M$. If $\eta(x) = 0$ then $V_\varphi(x)$ would have negative length.
Let therefore $\eta(x) \neq 0$. Then on sets

$$s_1 := \xi \quad \text{and} \quad s_2 := \frac{1}{\sqrt{h(\eta, \eta)}} \eta$$

and completes this locally with vector fields $s_3, \ldots, s_m$ of lenght 1 to a local orthonormal basis. Since $h(V_\varphi, X) = -\langle X \cdot \varphi, \varphi \rangle$ we have in this basis

$$V_\varphi = -\sum_{i=1}^{n} \kappa_i \langle s_i \cdot \varphi, \varphi \rangle s_i = (\varphi, \varphi)_\xi \; s_1 - (s_1 \cdot s_2 \cdot \varphi, \varphi)_\xi \, s_2$$

and for the lenght $h(V_\varphi, V_\varphi) = -(\varphi, \varphi)_\xi^2 + (s_1 \cdot s_2 \cdot \varphi, \varphi)_\xi^2$.
The action of $e_1 \cdot e_2$ on the basis of the spinor module ($k := [\frac{m}{2}]$) is

$$e_1 \cdot e_2 \cdot u(\varepsilon_k, \ldots, \varepsilon_1) = -\varepsilon_1 u(\varepsilon_k, \ldots, \varepsilon_1).$$

So for a $\varphi \in S$ with $\varphi(x) = [\hat{s}(x), v(x)]$ where $v(x) \in \mathbb{C}^k$ we have $(\varphi(x), \varphi(x))_\xi = (v(x), v(x))$ and $(s_1 \cdot s_2 \cdot \varphi(x), \varphi(x))_\xi = (e_1 \cdot e_2 \cdot v(x), v(x))$ and we decompose $v(x)$ as follows

$$\begin{aligned} v(x) &= \sum_{\varepsilon_1, \ldots \varepsilon_k \pm 1}^{k} v_{(\varepsilon_k, \ldots, \varepsilon_1)} u(\varepsilon_k, \ldots, \varepsilon_1) \\ &= \sum_{\varepsilon_2, \ldots \varepsilon_k \pm 1}^{k} v_{(\varepsilon_k, \ldots, \varepsilon_2, 1)} u(\varepsilon_k, \ldots, \varepsilon_2, 1) + \sum_{\varepsilon_2, \ldots \varepsilon_k \pm 1}^{k} v_{(\varepsilon_k, \ldots, \varepsilon_2, -1)} u(\varepsilon_k, \ldots, \varepsilon_2, -1) \\ &=: v_+ + v_-. \end{aligned}$$

So we have $e_1 \cdot e_2 \cdot v(x) = -v_+ + v_-$ and for the lenght

$$\begin{aligned} (e_1 \cdot e_2 \cdot v(x), v(x)) &= (-v_+ + v_-, v_+ + v_-) \\ &= -(v_+, v_+) + (v_-, v_-) - \underbrace{(v_+, v_-)}_{=0} + \underbrace{(v_-, v_+)}_{=0} \\ &= -(v_+, v_+) + (v_-, v_-). \end{aligned}$$

$$
\begin{aligned}
\text{Finally: } h\left(V_{\varphi}, V_{\varphi}\right) &= -(\varphi, \varphi)^{2}+\left(s_{1} \cdot s_{2} \cdot \varphi, \varphi\right)^{2} \\
&\overset{x}{=} -\Big(\left(v_{+}, v_{+}\right)^{2}+2\ \left(v_{+}, v_{+}\right)\left(v_{-}, v_{-}\right)+\left(v_{-}, v_{-}\right)^{2}\Big) \\
&\quad +\Big(\left(v_{+}, v_{+}\right)^{2}-2\ \left(v_{+}, v_{+}\right)\left(v_{-}, v_{-}\right)+\left(v_{-}, v_{-}\right)^{2}\Big) \\
&= -4\ \left(v_{+}, v_{+}\right)\left(v_{-}, v_{-}\right) \leq 0 .
\end{aligned}
$$

From the proof also follows that $V_{\varphi}(x)=0$ if and only if $\varphi(x)=0$.
2.) From the relations between the covariant derivatives, the Clifford multiplication and the scalar product it is for all $X, Y \in TM$

$$
\begin{aligned}
h\left(\nabla_{Y} V_{\varphi}, X\right) &= Y\left(h\left(V_{\varphi}, X\right)\right)-h\left(V_{\varphi}, \nabla_{Y} X\right) \\
&= -Y(\langle X \cdot \varphi, \varphi\rangle)+\left\langle\nabla_{Y} X \cdot \varphi, \varphi\right\rangle \\
&= -\left\langle\nabla_{Y} X \cdot \varphi, \varphi\right\rangle-\left\langle X \cdot \nabla_{Y}^{S} \varphi, \varphi\right\rangle-\left\langle X \cdot \varphi, \nabla_{Y}^{S} \varphi\right\rangle+\left\langle\nabla_{Y} X \cdot \varphi, \varphi\right\rangle \\
&= 0 \text{ if } \varphi \text{ parallel.}
\end{aligned}
$$

This gives the conclusion. □

From the second assertion of the lemma follows that a non-flat Lorentzian manifold with parallel spinors can not be irreducible as we have seen in corollary 2.1.
That means that we have for a 1-connected Lorentzian manifold with parallel spinor $\varphi$ two cases:
**1.)** $h\left(V_{\varphi}, V_{\varphi}\right)<0$, i.e. $V_{\varphi}$ timelike. Since $V_{\varphi}$ is parallel the holonomy group acts trivial on $\mathbb{R}\ V_{\varphi}(x) \subset T_{x} M$, and so the manifold decomposes due to theorem 2.3 as follows

$$(M, h) \simeq(\mathbb{R},-d t) \times(N, g)$$

with $(N, g)$ a Riemannian manifold of dimension $m-1$ which again can be decomposed in flat or irreduzible Riemannian manifolds. Since irreducible local symmetric Riemannian manifolds with parallel spinors has to be flat (as a conclusion of the Ricci-flatness, see [Bes87]) one obtains from the result in the Riemannian case (see introduction) the following

**Theorem 2.6** *$(M, h)$ is a simply connected, complete Lorentzian manifold with parallel spinor whose associated vector field is timelike if and only if $(M, h)$ is isometric to a product of $(\mathbb{R},-d t^{2})$ and 1-connected, irreducible Riemannian manifolds with one of the following holonomy groups $SU(k)$, $Sp(k)$, $G_{2}$, $Spin(7)$ and eventually a flat factor.*

**2.)** Let $V_{\varphi}$ lightlike. In this case $(M, h)$ decomposes in a product of irreducible Riemannian manifolds with parallel spinors and an indecomposable Lorentzian manifold which is reducible (both are 1- connected, because of the parallel vector field coming from the parallel spinor. That means one has to investigate under which conditions such a manifold admits a parallel spinor field. This will be the aim of the following section.

# 3 Parallel spinors on indecomposable, reducible Lorentzian manifolds

## 3.1 Four types of indecomposable, reducible Lorentzian manifolds

We will now cite some results of [BI93] and [Ike96] about holonomy groups of indecomposable but reducible Lorentzian manifolds.
Let $(M, h)$ be a simply connected and complete Lorentzian manifold of dimension $\geq 3$, which should be indecomposable and reducible, not necessary with a fixed lightlike vector under holonomy representation but only with an invariant degenerate subspace $E \subset T_xM$. Then $E^\perp$ is invariant and one defines $F := E \cap E^\perp \neq \{0\}$ an invariant, isotropic, one-dimensional subspace.
One now chooses a basis $(t_-, t_+, e_1, \ldots, e_n)$ $(n = m - 2)$ of $T_xM$ for which $F = \mathbb{R}t_-$ and the metric corresponds to

$$\eta := \begin{pmatrix} 0 & 1 & 0^t \\ 1 & 0 & 0^t \\ 0 & 0 & Id \end{pmatrix}. \tag{9}$$

(I.e. $t_-$, $t_+$ null vectors, $e_i$ orthonormal.)
Now one characterizes the subgroup of $SO_0(\eta) \sim SO(1, m-1)$ which leaves $F$ invariant.

**Proposition 3.1** *Let $F := \mathbb{R}t_-$ and $G := \{\mathcal{A} \in SO_0(\eta) | \mathcal{A}F \subset F\}$ the connected subgroup of $SO_0(\eta)$ which leaves $F$ invariant. For the Lie algebra $\mathfrak{g}$ of $G$ holds*

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} a & 0 & X^t \\ 0 & -a & 0 \\ 0 & -X & A \end{pmatrix} \right| a \in \mathbb{R}, X \in \mathbb{R}^n, A \in \mathfrak{so}(n) \right\}. \tag{10}$$

In order to find those subgroups which doesn't leave any non degenerate subspace invariant we fix some notations:

1. $\mathfrak{g}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}^n \oplus \mathfrak{so}(n)$ as vector space and with the commutator:

$$[(a, X, A), (b, Y, B)] = \Big(0, (A + a\ Id)\, Y - (B + b\ Id)\, X, [A, B]_{\mathfrak{so}(n)}\Big). \tag{11}$$

2. $\mathfrak{a} := \{(a, 0, 0) \,| a \in \mathbb{R}\} \subset \mathfrak{g}$ an abelean subalgebra of $\mathfrak{g}$.

3. $\mathfrak{n} := \{(0, X, 0) \,| X \in \mathbb{R}^n\} \subset \mathfrak{g}$ an abelean ideal in $\mathfrak{g}$ isomorphic to $\mathbb{R}^n$.

4. $\mathfrak{k} := \{(0, 0, A) \,| a \in \mathfrak{so}(n)\} \subset \mathfrak{g}$ a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{so}(n)$.

5. Further it is $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{n}$ and $[\mathfrak{g}, \mathfrak{k}] \subset \mathfrak{k} \ltimes \mathfrak{n}$ as well as $[\mathfrak{k}, \mathfrak{a}] = \{0\}$ and $[\mathfrak{a} \oplus \mathfrak{k}, \mathfrak{n}]$. The latter means that
$$\mathfrak{g} = (\mathfrak{a} \oplus \mathfrak{k}) \ltimes \mathfrak{n}.$$

Let now $\mathfrak{h} := \mathfrak{hol}_x(M,h) \subset \mathfrak{g}$ the holonomy algebra of an indecomposable but reducible Lorentzian manifold and set

$$\begin{array}{rcl} \mathfrak{b} & := & pr_{\mathfrak{k}}\mathfrak{h} \subset \mathfrak{k} \simeq \mathfrak{so}(n) \\ \mathfrak{t} & := & \mathfrak{z}(\mathfrak{b}) \text{ the center of } \mathfrak{b}\text{, an abelean subalgebra of } \mathfrak{k} \\ \mathfrak{d} & = & [\mathfrak{b},\mathfrak{b}] \text{ a subalgebra of } \mathfrak{k} \end{array}$$

From a general theorem (see for example [KN63, Appendix 5]) which asserts that subalgebras of $\mathfrak{so}(n)$ are reductive the following decomposition follows

$$\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{d} \text{ and } \mathfrak{d} \text{ is semisimple.} \tag{12}$$

In [BI93] now is proved another important fact about $\mathfrak{b}$ which is an analogon to the complete reducibility in the Riemannian case.

**Theorem 3.1** *Let $\mathfrak{b} := pr_{\mathfrak{k}}(\mathfrak{hol}_x(M,h))$ the projection of the holonomy algebra of an indecomposable, reducible Lorentzian manifold onto the $\mathfrak{so}(n)$-component. Then $\mathfrak{b}$ is completely reducible. I.e there existist decompositions of $\mathbb{R}^n$ in orthogonal subspaces and of $\mathfrak{b}$ in commuting ideals*

$$\mathbb{R}^n = E_0 \oplus E_1 \oplus \ldots \oplus E_r \quad \textit{and} \quad \mathfrak{b} = \mathfrak{b}_1 \oplus \ldots \oplus \mathfrak{b}_r$$

*where $\mathfrak{b}$ acts trivial on $E_0$, $\mathfrak{b}_i$ acts irreducible on $E_i$ and $\mathfrak{b}_i(E_j) = \{0\}$ for $i \neq j$.*

For the following we have to cite another general fact about subgroups of $SO(n)$

**Theorem 3.2** *[KN63, Appendix 5] Let $G \subset SO(n)$ connected.*

1. *If $G$ is semisimple so it is closed in $SO(n)$ and therefore compact.*
2. *If $G$ acts irreducible on $\mathbb{R}^n$ then the dimension of the center of $G$ is smaller than two, i.e. $Z(G)$ is discrete or isomorphic to $S^1$.*

*Together with (12) one concludes that an irreducible acting subgroup of $SO(n)$ is closed in $SO(n)$ and therefore compact.*

In [BI93] is now proved the following distinction for holonomy groups of indecomposable reducible Lorentzian manifolds. One distincts four exclusive cases (The corresponding Lie groups to the algebras are denoted by the corresponding latin capitals):

1. $\mathfrak{n} \subset \mathfrak{h}$

    **Type 1:** $\mathfrak{a} \subset \mathfrak{h}$. Therefore $\mathfrak{h} = (\mathfrak{a} \oplus \mathfrak{b}) \ltimes \mathfrak{n}$ and so $H = (A \times B) \ltimes N$. $A \simeq \mathbb{R}^+$ and $N \simeq \mathbb{R}^n$ are closed and by theorem 3.2 also $B$ is closed so that $H$ is closed.

    **Type 2:** $\mathfrak{h} = \mathfrak{b} \ltimes \mathfrak{n}$, i.e. $H = B \ltimes N$ closed.

**Type 3:** Neither $\mathfrak{a} \subset \mathfrak{h}$ nor $\mathfrak{h} \subset \mathfrak{k} \ltimes \mathfrak{n}$.

In that case exists a surjective homomorphism $\varphi : \mathfrak{t} \to \mathfrak{a}$, such that

$$\mathfrak{h} = (\mathfrak{l} \oplus \mathfrak{d}) \ltimes \mathfrak{n}$$

where $\mathfrak{l} := graph\ \varphi = \{(\varphi(T), T) | T \in \mathfrak{t}\} \subset \mathfrak{a} \oplus \mathfrak{t}$. That means $H = (L \times D) \ltimes N$. $D$ and $N$ are closed. Now one shows that $L$ and therefore $H$ are closed if and only if $Ker\ \varphi$ generates a compact subgroup of $T$.

2. **Type 4:** $\mathfrak{h}$ does not contain $\mathfrak{n}$. Then exists

   (a) a non-trivial decomposition $\mathfrak{n} = \mathfrak{p} \oplus \mathfrak{q} \simeq \mathbb{R}^p \oplus \mathbb{R}^q$, $0 < p, q < n$,

   (b) a surjective homomorphism $\varphi : \mathfrak{t} \to \mathfrak{q}$

   such that $\mathfrak{b} \subset \mathfrak{so}(p)$ and $\mathfrak{h} = (\mathfrak{d} \oplus \mathfrak{l}) \ltimes \mathfrak{p} \subset \mathfrak{g}$ where $\mathfrak{l} := \{(\varphi(A), A)\, | A \in \mathfrak{t}\} = graph\ \varphi \subset \mathfrak{q} \oplus \mathfrak{t}$. Or written as matrices

$$\mathfrak{h} = \left\{ \left( \begin{array}{cccc} 0 & 0 & X^t & \varphi(A)^t \\ 0 & 0 & 0 & 0 \\ 0 & -X & A+B & 0 \\ 0 & -\varphi(A) & 0 & 0 \end{array} \right) \middle| A \in \mathfrak{t}, B \in \mathfrak{d}, X \in \mathbb{R}^p \right\}$$

   So $H = (L \times D) \ltimes P$ with $P$ isomorphic to $\mathbb{R}^p$ and again $H$ and $L$ are closed if and only if $Ker\ \varphi$ generates a compact subgroup of the torus $T$.

   Further one observes that the non-triviality of the decomposition of $\mathfrak{n}$ forces the center of $\mathfrak{b}$ to non-trivial.

**Remark:**

1. In [BI93] are given examples of metrics for all types and especially with non-closed holonomy groups of type 3 and 4.

2. From the commutator relation (11) it is clear that the only abelean indecomposable, reducible holonomy algebra has to be of type 1 or 2, both with $pr_{\mathfrak{k}}\mathfrak{h} = 0$.

One separates the four types as follows in two classes. One defines

**Definition 3.1** *A Lorentzian manifold with a lightlike parallel vector field is called Brinkmann- wave (See [Bri25], also [Eis38]).*

**Corollary 3.1** *An indecomposable, reducible Lorentzian manifold is of type 2 or 4 if and only if it is a Brinkmann-wave.*

The proof is clear since the condition for a Brinkmann-wave is equivalent to the existence of a one dimensional trivial subrepresentation of $\mathfrak{h}$ but this occurs only for the types 2 and 4.

## 3.2 Conclusions for the existence of parallel spinors

We will now prove some first, purely algebraic properties for the components of $\mathfrak{h} := \mathfrak{hol}_x(M,h)$ in relation to the existence of parallel spinors.
With respect to the four types from the previous section the existence of parallel spinors gives an obvious restriction.

**Proposition 3.2** *Let $(M,h)$ be an indecomposable Lorentzian manifold with lightlike parallel spinor. Then $(M,h)$ is reducible of type 2 or 4.*
*If further its holonomy algebra is abelean then it is of type 2, namely it equals to $\mathfrak{n}$.*

The proof is clear because indecomposable manifolds with parallel spinors are Brinkmann-waves.
We consider the Lie-algebras of both types

$$\mathfrak{h}_1 = \mathfrak{b}_1 \ltimes \mathfrak{n} \quad \text{and} \quad \mathfrak{h}_2 = (\mathfrak{d} \oplus \mathfrak{l}) \ltimes \mathfrak{p}$$

with $\mathfrak{b}_1 \subset \mathfrak{k} \simeq \mathfrak{so}(n)$ for $n+2=m$ and $\mathfrak{d}$ a semisimple and $\mathfrak{t}$ an abelean subalgebra of $\mathfrak{so}(p)$ which commute whith each other, $\mathfrak{n} = \mathfrak{p} \oplus \mathfrak{q} \simeq \mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ for $0<p<n$ and $\mathfrak{l} := graph(\phi) \subset \mathfrak{t} \oplus \mathfrak{q}$ for $\phi : \mathfrak{t} \to \mathfrak{q}$ surjektiv. We ask under which condition is

$$V_{\tilde{\mathfrak{h}}_i} := \left\{ v \in \Delta_{1,m-1} \,\middle|\, \lambda_*^{-1}(\mathfrak{h}_i)v = 0 \right\}$$

non trivial.
First we can reduce the problem in the following way.

**Proposition 3.3** *Let $(M,h)$, $x \in M$, $t_-, t_+, e_1, \dots, e_n$ a basis in $T_xM$ as in the previous section and $\big(u(\varepsilon_k, \dots, \varepsilon_0) | \varepsilon = \pm 1, k = [\frac{n}{2}]\big)$ a basis of $\Delta_{1,m-1} = \Delta_{1,n+1}$. Then holds the following*

1. $V_{\tilde{\mathfrak{n}}} = span\left(\{u(\varepsilon_k, \dots, \varepsilon_1, -1) | \varepsilon = \pm 1\}\right)$.
2. *Let $\mathfrak{b}_i = pr_{\mathfrak{k}}\mathfrak{h}_i$, $i = 1,2$. Then $dim V_{\tilde{\mathfrak{h}}_i} = \frac{1}{2} dim V_{\tilde{\mathfrak{b}}_i}$.*

**Proof:** 1.) We consider the following orthonormal frame in $T_xM$:

$$\begin{aligned} e_- &:= \frac{1}{\sqrt{2}}(-t_- + t_+) \quad \text{timelike unit vector} \\ e_+ &:= \frac{1}{\sqrt{2}}(t_- + t_+) \end{aligned}$$

and the $e_1, \dots, e_n$. In this basis the isotropy subalgebra $\mathfrak{g}$ of $t_-$ in $\mathfrak{so}(1,m-1) \supset \mathfrak{g} \supset \mathfrak{h}_i$ has the following shape

$$\begin{pmatrix} 0 & 0 & -X^t \\ 0 & 0 & X^t \\ -X & -X & A \end{pmatrix}.$$

For the standard basis in $\mathfrak{so}(1,m-1)$ (see (7)) it is $\mathfrak{n} = span\,(E_{-\,i} - E_{+\,i} | i = 1, \dots, n)$ and $\tilde{\mathfrak{n}} := \lambda_*^{-1}(\mathfrak{n}) = span\,(e_- \cdot e_i - e_+ \cdot e_i = e_i \cdot t_- | i = 1, \dots, n)$. Since the $e_i$'s are isomorphisms it is $V_{\tilde{\mathfrak{n}}} = V_{\mathbb{R}e_i \cdot t_-} = V_{\mathbb{R}t_-}$ and from

$$t_- \cdot u(\varepsilon_k, \dots, \varepsilon_0) = -(\varepsilon_0 + 1)\, u(\varepsilon_k, \dots, \varepsilon_1, -\varepsilon_0)$$

follows the conclusion.
2.) Since $\tilde{\mathfrak{b}}_i \subset \mathfrak{spin}(n) = span\,(e_i \cdot e_j | 1 \leq i < j \leq n)$ does not act on $u(\varepsilon_0)$ in $u(\varepsilon_k, \ldots, \varepsilon_0)$ it is clear that $V_{\tilde{\mathfrak{b}}_i}$ has a basis of the form $\left(v_1^+, v_1^-, \ldots v_l^+, v_l^-\right)$ with $v_i^\pm \in span\left(\{u\left(\varepsilon_k, \ldots, \varepsilon_1, \pm 1\right) | \varepsilon_i = \pm 1\}\right)$.
Then the $v_k^-$ lie in $V_{\tilde{\mathfrak{h}}_i}$ for both $i$. We have to show that they are a basis of both.
For $i = 1$ this is clear because of $V_{\tilde{\mathfrak{h}}_1} = V_{\tilde{\mathfrak{n}}} \cap V_{\tilde{\mathfrak{b}}}$.
For $i = 2$ one takes a $v \in V_{\tilde{\mathfrak{h}}_2}$. Since it has to be in $V_{\tilde{\mathfrak{p}}}$ it must lie in $span\left(\{u\left(\varepsilon_k, \ldots, \varepsilon_1, -1\right) | \varepsilon_i = \pm 1\}\right)$ and therefore atomatically in $V_{\tilde{\mathfrak{q}}}$. This entails $v \in V_{\tilde{\mathfrak{l}}}$. The assumption to $v$ gives also $v \in V_{\tilde{\mathfrak{d}}}$ and therefore finally $v \in V_{\tilde{\mathfrak{b}}}$. Thats why it is $V_{\tilde{\mathfrak{h}}_2} = V_{\tilde{\mathfrak{b}}} \cap V_{\tilde{\mathfrak{n}}}$ which is spanned by the $v_i^-$. □

As a corollary we obtain a sufficient condition for the existence of parallel spinors

**Corollary 3.2** *Let $(M, h)$ a simply connected, indecomposable, reducible Lorentzian manifold with $\mathfrak{hol}_x(M, h) = \mathfrak{n}$. Then $(M, h)$ admits parallel spinors with light like Dirac current.*

**Remark:**

1. Since the indecomposable Lorentzian symmetric spaces have holonomy $\mathfrak{n}$ (see [CW70], also [BI93] for the result) by this corollary is given a first group of examples of indecomposable Lorentzian manifold with parallel spinors. In section 4.2 we give more examples and describe these spaces further.

2. The parallel spinors resulting from the basis of $V_{\tilde{\mathfrak{n}}}$ namely $\varphi_{(\varepsilon_k, \ldots, \varepsilon_1)} := [q, u\left(\varepsilon_k, \ldots, \varepsilon_1, -1\right)]$ are pure. That means the complex dimension of the space $\{Z \in TM^{\mathbb{C}} | Z \cdot \varphi_{(\varepsilon_k, \ldots, \varepsilon_1)} = 0\}$ is maximal, that is equal to $[\frac{n+2}{2}]$ (see [Kat98]). A basis of this space is given by $\left(t_-, e_1 - i\varepsilon_1 e_2, \ldots, e_{n-1} - i\varepsilon_{[\frac{n}{2}]} e_n\right)$.

3. The second point of the proposition reduces the problem to find subalgebras $\mathfrak{h}_i \subset \mathfrak{so}(1, m-1)$ with $V_{\tilde{\mathfrak{h}}_i} \neq 0$ to the problem to find subalgebras $\mathfrak{b} \subset \mathfrak{so}(n)$ with $V_{\tilde{\mathfrak{b}}} \neq 0$ where here $\lambda_*^{-1}$ can be understood as the differential of the twofold covering $\lambda : Spin(n) \to SO(n)$ since the representations are equivalent $\Delta_{1,m-1} \simeq \Delta_{1,1} \otimes \Delta_n$.

From theorem 3.1 one obtains the following for $\mathfrak{b}$.

**Proposition 3.4** *Let $\mathfrak{b} \subset \mathfrak{k} \simeq \mathfrak{so}(n)$.*

1. *Let $V_{\tilde{\mathfrak{b}}} \neq 0$, $\mathbb{R}^n = E_0 \oplus E_1 \oplus \ldots \oplus E_r$ and $\mathfrak{b} = \mathfrak{b}_1 \oplus \ldots \oplus \mathfrak{b}_r$ the decomposition in irreducible subrepresentations due to theorem 3.1. Then for $i = 1 \ldots k$ it has to be dim $E_i = 4k_i$ or $\mathfrak{z}(\mathfrak{b}_i) = 0$.*

2. *For $\mathfrak{b} = pr_{\mathfrak{k}}\mathfrak{h}$ and $\mathfrak{h}$ the holonomy algebra of an indecomposable Lorentzian manifold with a parallel spinor. If $\mathfrak{b}$ is abelean, then it is trivial.*

**Proof:**
1.) From theorem 3.2 we have for irreducible acting $\mathfrak{b}_i$ that the center of $\mathfrak{b}_i$ is trivial or one-dimensional, i.e. $dim\ E_i = 2k_i$ and $\mathfrak{t}_i = \mathfrak{z}(\mathfrak{b}_i) = \mathbb{R}J$ with $J^2 = -Id$. $J$ is of the form

$$J = \begin{pmatrix} j & 0 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & j \end{pmatrix} \quad \text{with } j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now it is $\tilde{J} := \lambda_*^{-1}(J) = \sum_{l=1}^{k_i} e_{2l-1} \cdot e_{2l}$ and therefore

$$\tilde{J}u\left(\varepsilon_k, \ldots, \varepsilon_1\right) = -\left(\varepsilon_{k_i} + \ldots + \varepsilon_1\right) u\left(\varepsilon_{k_i}, \ldots, \varepsilon_1\right).$$

In case that $V_{\tilde{\mathfrak{t}}_i} \neq 0$ the $k_i$'s must be even.
2.) Let $\mathfrak{b}$ given as in the assumptions. From theorem 3.1 we have $\mathfrak{b} = \mathfrak{b}_1 \oplus \ldots \oplus \mathfrak{b}_r$ and $\mathbb{R}^n = E_1 \oplus \ldots \oplus E_r \oplus E_0$ such that $\mathfrak{b}$ acts trivial on $E_0$ and $(\mathfrak{b}_i, E_i), i = 1 \ldots r$ are irreducible representations. Since $\mathfrak{b}$ abelean, so are the $\mathfrak{b}_i$'s, but this entails $\mathfrak{b}_i \simeq \mathfrak{so}(2)$ and $E_i = \mathbb{R}^2$, $i = 1, \ldots r$ or $\mathfrak{b}_i$ trivial. Now the second proposition follows from the first. $\square$

# 4 Brinkmann-waves with special holonomy and parallel spinors

## 4.1 The local form of the metric of an indecomposable, reducible Lorentzian manifold

In [Ike96] is proved the following theorem about the local form of the metric. See also [Bri25] and [Eis38] for the second part.

**Proposition 4.1** *[Ike96] Let $(M, h)$ be an indecomposable, reducible Lorentzian manifold. Then exist coordinates $(U, \xi := (x_-, x_+, x_1, ..., x_n))$ such that the metric $h$ has the following form*

$$h = 2\ dx_- dx_+ + \sum_{i=1}^{n} u_i dx_i dx_+ + f dx_+^2 + \sum_{i,j=1}^{n} g_{ij} dx_i\ dx_j$$

*where $f, u_i, g_{ij} \in C^\infty(U)$ with $\frac{\partial}{\partial x_-} g_{ij} = \frac{\partial}{\partial x_-} u_i = 0$.*
*Is $(M, h)$ of type 2 or 4 that is a Brinkmann-wave then one has in addition $\frac{\partial}{\partial x_-} f = 0$.*

It is clear that in case of a Brinkmann-wave the vector field $\partial_- := \frac{\partial}{\partial x_-}$ corresponds to the parallel lightlike vector field.
If one considers small $n$-dimensional submanifolds in $U$ through $x = (x_-, x_+, x_1, \ldots, x_n)$ defined by

$$W_{(x_-, x_+)} := \{(x_-, x_+, y_1, \ldots y_n) | (y_1, \ldots y_n) \in \mathbb{R}^n \cap \xi(U)\}$$

then one can understand the $g_{ij}$ as coefficients of a family of Riemannian metrics $g_{x_+}$ on $W_{(x_-, x_+)}$ which depends from a parameter $x_+$.
In this sense [Ike96] proves the following (in a weaker version)

**Proposition 4.2** *Let* $x = (x_-, x_+, x_1, \dots, x_n) \in W_{(x_-,x_+)} \subset U$ *and let*

$$G_x := \left\{ \mathcal{P}_\delta^{\nabla^h} \,|\delta \text{ a closed curve around } x \text{ with fixed } x_-, x_+ \text{ coordinates} \right\} \subset Hol_x(M,h).$$

*Then follows* $Hol_x(W_{(x_-,x_+)}, g_{x_+}) = G_x$.
*In particular it is* $Hol_x(W_{(x_-,x_+)}, g_{x_+}) \subset pr_K\left(Hol_x(M,h)\right)$.

**Remark:**

1. In general there is no equality. In [Ike96] is given an example of a 7- dimensional Lorentzian manifold for which $g_{x_+}$ is flat for all $x_+$, $\mathfrak{b}$ is 3-dimensional and acts irreducible on $\mathbb{R}^5$. But due to the Berger classification $\mathfrak{b}$ cannot be the holonomy algebra of a Riemannian manifold which has to be $\mathfrak{so}(5)$ in case of irreducible action. That means that there is no coordinate transformation such that the holonomy of the transformed $g$ corresponds to $\mathfrak{b}$.

2. If the holonomy groups of the $g_{x_+}$ are trivial then it is $\left(g_{x_+}\right)_{ij} = c^2(x_+)\ \delta_{ij}$ with a conformal factor. With a coordinate transformation $\tilde{x_i} := c(x_+)x_i,\ i = 1, \dots, n$ one reaches $\tilde{c} \equiv 1$ so that in case of trivial holonomy one can assume that $c$ is independent of $x_+$.

3. It is known (see [Eis38], [Sch74]) that every Brinkmann-wave can be transformed into the following shape

$$h = 2\ dx_- dx_+ + \sum_{i,j=1}^{n} g_{ij} dx_i\ dx_j \tag{13}$$

   where $g_{ij} \in C^\infty(U)$ with $\frac{\partial}{\partial x_-} g_{ij} = 0$. One can say that this makes the dependence of $g_{ij}$ from $x_+$ more complicated. We will use this fact later on.

From proposition 3.3 one obtains the following corollary for the existence of parallel spinors.

**Proposition 4.3** *Let* $(M,h)$ *be a simply connected, complete Brinkmann-wave with parallel spinor. Then all metrics* $g_{x_+}$ *from the family of metrics given by the local form of* $h$ *admit parallel spinors.*

For the Ricci-tensor one can prove the following. The proof is due to [FO99].

**Lemma 4.1** *Let* $(M,h)$ *be a Brinkmann-wave given in the local form* $h = h\,(f, u_i, g_{ij})$. *Its Ricci-endomorphism is totally isotropic if and only if* $Ric(\partial_i, .) = 0$, *i.e.* $Ric = f\ dx_+ \circ dx_+$ *for a function* $f$.

**Proof:** Let $h^{\alpha\beta}$ the inverse matrix of $h_{\alpha\beta} = h\,(\partial_\alpha, \partial_\beta)\ \alpha, \beta = -, +, 1, \dots, n$, i. e.

$$h_{\alpha\beta} = \begin{pmatrix} -f & 1 & \left(-g^{-1}u\right)^t \\ 1 & 0 & 0 \\ -g^{-1}u & 0 & g^{-1} \end{pmatrix}$$

Since $\partial_-$ is parallel we get $Ric(\partial_-,.) = 0$ and therefore

$$\begin{aligned} Ric(X) &= \sum_{\alpha,\beta} h^{\alpha\beta} Ric(X,\partial_\alpha)\partial_\beta \\ &= \left(Ric\,(X,\partial_+) - \sum_{k=1}^{n} \left(g^{-1}u\right)_k Ric\,(x,\partial_k)\,\right)\partial_- + \sum_{k,l=1}^{n} g^{kl} Ric(X,\partial_k)\partial_l \end{aligned}$$

So it is $h(Ric(X), Ric(X)) = \sum_{k,l=1}^{n} g^{kl} Ric(X,\partial_k) Ric(X,\partial_l)$. So a isotropic Ricci-tensor forces $Ric(X,\partial_k) = 0$ for all vector fields $X$. The converse is obviously true from the formula. □

### 4.2 Brinkmann-waves with holonomy $\mathfrak{n}$

Since Brinkmann-waves with holonomy $\mathfrak{n}$ admit parallel spinors we will now desribe these class of manifolds in order to see if there are more then locally symmetric spaces. Therefore we consider a Brinkmann-metric $h = h(f, u_i, g_{ij})$ where the metrics $g_{x_+}$ have trivial holonomy. From the above remark we can assume that the $g_{x_+}$ are the flat metric, that means $g_{ij} \equiv \delta_{ij}$. For the curvature of such a Brinkmann- wave one proves the following

**Lemma 4.2** *Let $h(f, u_i, \delta_{ij})$ be a Brinkmann-metric with flat Riemannian part. Then we have for its covariant derivatives*

$$\nabla^{LC}_X Y \in span(\partial_-, \partial_1, \dots \partial_n) \text{ and } \nabla_{\partial_i}\partial_j = \frac{1}{2}\left(\partial_i u_j - \partial_j u_i\right)\partial_-$$

*and for the curvature:*

$$\begin{aligned} \mathcal{R}(\partial_i,\partial_j)\partial_k &= \frac{1}{2}\left(\partial_k\partial_i u_j - \partial_k\partial_j u_i\right)\partial_- && (14) \\ \mathcal{R}(\partial_i,\partial_+)\partial_k &= \frac{1}{2}\left(\frac{1}{2}\sum_{l=1}^{n}\left(\partial_i u_l \partial_l u_k + \partial_k u_l \partial_l u_i - \partial_i u_l \partial_k u_l - \partial_l u_i \partial_l u_k\right)\right. \\ &\quad \left. + \sum_{l=1}^{n} u_l\left(\partial_i\partial_l u_k - \partial_i\partial_k u_l\right) + \partial_i\partial_k f - \partial_+\left(\partial_i u_k + \partial_k u_i\right)\right)\partial_- \\ &\quad + \frac{1}{2}\sum_{l=1}^{n}\Big(\partial_i\partial_k u_l - \partial_i\partial_l u_k\Big)\partial_l. && (15) \end{aligned}$$

**Remark:** Now for a Brinkmann-wave with flat Riemannian part admitting parallel spinors we can express the necessary condition of Ricci isotropy in terms of the functions $u_i$. The Ricci isotropy is equivalent to the followindg conditions to the $u_i$'s:

$$\sum_{k=1}^{n} \partial_k^2 u_i = \partial_i \sum_{k=1}^{n} \partial_k u_k.$$

For $\hat{u} = (u_1, \dots u_n)$ and $\Delta\hat{u} = (\Delta_0 u_1, \dots, \Delta_0 u_n)$ for $\Delta_0$ the flat Laplace operator this means

$$\Delta\hat{u} = grad\ div\ \hat{u}$$

Now for finding the conditions for the holonomy of a Brinkmann-wave beeing equal to $\mathfrak{n}$ we prove another

**Lemma 4.3** *Let $\Omega$ the curvature form of the Levi-Civita-connection $\omega$ on an indecomposable, reducible, simply-connected Lorentzian manifold. Let $\{U_\alpha\}_{\alpha\in I}$ be a cover of open sets with corresponding local sections in the bundle of orthonormal frames $s^\alpha \in \Gamma(U_\alpha, P)$. For all local curvature forms holds $pr_{\mathfrak{k}}\left(\Omega^{s^\alpha}(X,Y)\right) = 0$(that is $\left(\Omega^{s^\alpha}(X,Y)\right) \in \mathfrak{n}$) for all $X, Y \in TM$ if and only if $pr_{\mathfrak{k}}\left(\mathfrak{hol}_x(M,h)\right) = 0$ (that is $\mathfrak{hol}_x(M,h) \in \mathfrak{n}$).*

**Proof:** Let $x \in M$ and $p \in P_x$. The fact that the holonomy in $x$ of $(M,h)$ is contained in $\mathfrak{n}$ is by the Ambrose-Singer-holonomy theorem equivalent to

$$pr_{\mathfrak{k}}\left(\Omega_q\left(U,V\right)\right) = 0 \text{ for all } q \text{ on horizontal curves starting in } p \text{ and all } U,V \in Th_qP. \tag{16}$$

By formula (3) and $T_{\pi(q)}M \simeq Th_qP$ this is equivalent to

$$pr_{\mathfrak{k}}\left(Ad\left(g^{-1}\right)\Omega^s_{\pi(q)}(X,Y)\right) = 0 \;\text{ for all } q \text{ as above and all } X,Y \in T_{\pi(q)}M \tag{17}$$

where $g \in SO_0(1,m-1)$ is determined by $q = R_g\left(s\left(\pi(q)\right)\right)$. This would give the proposed equivalence from the lemma if we could prove the following relation for all $A \in Hol_x(M,h) \subset \mathfrak{k} \ltimes \mathfrak{n}$:

$$pr_{\mathfrak{k}}\left(Ad\left(g^{-1}\right)(A)\right) = 0 \text{ iff } pr_{\mathfrak{k}}\left(A\right) = 0. \tag{18}$$

We will prove this. Since $\mathfrak{n}$ is an ideal and therefore $N$ an normal subgroup it is

$$\begin{aligned}
&pr_{\mathfrak{k}}\left(Ad\left(g^{-1}\right)(A)\right)\\
&\quad= pr_{\mathfrak{k}}\left(Ad\left(g^{-1}\right)\left(pr_{\mathfrak{k}}(A) + pr_{\mathfrak{n}}(A)\right)\right)\\
&\quad= pr_{\mathfrak{k}}\left(\frac{d}{dt}\left[g\left(exp\left(t\ pr_{\mathfrak{k}}(A)\right)\right)g^{-1}\right]_{t=0}\right) + pr_{\mathfrak{k}}\left(\frac{d}{dt}[\underbrace{g\left(exp\ t\ pr_{\mathfrak{n}}(A)\right)g^{-1}}_{\in N}]_{t=0}\right)\\
&\quad= pr_{\mathfrak{k}}\left(Ad(g^{-1})\left(pr_{\mathfrak{k}}(A)\right)\right).
\end{aligned}$$

If $pr_{\mathfrak{k}}(A) = 0$ this term vanishes which gives one direction of (18). On the other hand the vanishing of this projection gives that

$$N \ni exp\left[tAd(g^{-1})(pr_{\mathfrak{k}}(A))\right] = g^{-1}\left(exp\left[t\ pr_{\mathfrak{k}}(A)\right]\right)g \;\text{ for all } t$$

With the following notations

$$exp\left[t\ pr_{\mathfrak{k}}(A)\right] := \begin{pmatrix} 1 & 0 & 0^t \\ 0 & 1 & 0^t \\ 0 & 0 & (a_{ij}(t)) \end{pmatrix},$$

$$g := \left(g_{\alpha\beta}\right)_{\alpha,\beta=+,-,1,\dots,n}, \text{ and } g^{-1} := \left(g^{\alpha\beta}\right)_{\alpha,\beta=+,-,1,\dots,n},$$

$$
\begin{aligned}
\text{this entails } 1 &\equiv pr_K\left(g^{-1}\left(exp\left[t\; pr_{\mathfrak{k}}(A)\right]\right) g\right) \\
&= \left(g_{i-}g^{-j} + g_{i+}g^{+j} + \sum_{k,l=1}^{n} g_{ik}a_{kl}(t)g^{lj}\right)_{i,j=1}^{n}.
\end{aligned}
$$

But that means $a_{ij} = const$, therefore $exp\left[t\; pr_{\mathfrak{k}}(A)\right] \equiv 1$ and finally $pr_{\mathfrak{k}}(A) = 0$. □

With both lemmata one proves the following

**Proposition 4.4** *Let $(M,h)$ be a simply-connected Brinkmann-wave with $h = h(f, u_i, g_{ij})$ locally. Then it is $Hol_x(M,h) \subset \mathfrak{n}$ if and only if $g_{ij} \equiv \delta_{ij}$ and for the functions $u_1, \ldots, u_n$ holds*

$$
\partial_k\left(\partial_i u_j - \partial_j u_i\right) = 0. \tag{19}
$$

**Proof:** Because of $Hol_x(M,h) \subset \mathfrak{n}$ it is $g_{ij} = \delta_{ij}$. For an arbitrary $y \in M$ then $h$ is given in its local form $h(f, u_i, \delta_{ij})$ in a neighborhood. We define the following orthonormal frame for $h$:

$$
\begin{aligned}
s_- &= \frac{1}{\sqrt{2}}\left(\left(1+\frac{f}{2}\right)\partial_- - \partial_+\right) \quad \text{timelike unit vector field} \\
s_+ &= \frac{1}{\sqrt{2}}\left(\left(1-\frac{f}{2}\right)\partial_- + \partial_+\right) \\
s_i &= \partial_i - u_i\partial_-
\end{aligned}
\tag{20}
$$

We will now calculate the conditions for $\Omega^s(X,Y) \in \mathfrak{n}$ for $s = (s_-, s_+, s_1, \ldots s_n)$ the section in $P$. The local connection form is given by

$$
\omega^s = \sum_{\alpha<\beta=-,+,1}^{n} h\left(\nabla s_\alpha, s_\beta\right) E_{\alpha\beta} \quad \text{for } E_{\alpha\beta} \text{ a basis of } \mathfrak{so}(n+2,1).
$$

The local curvature form is then given by

$$
\begin{aligned}
\Omega^s(\partial_\alpha, \partial_\beta) &= \partial_\alpha\left(\omega^s\left(\partial_\beta\right)\right) - \partial_\beta\left(\omega^s\left(\partial_\alpha\right)\right) \\
&= \sum_{\gamma<\delta=-,+,1..n} \left[\partial_\alpha\left(h\left(\nabla_{\partial_\beta}s_\gamma, s_\delta\right)\right) - \partial_\beta\left(h\left(\nabla_{\partial_\alpha}s_\gamma, s_\delta\right)\right)\right] E_{\gamma\delta}
\end{aligned}
$$

Now it is

$$
\begin{aligned}
& pr_{\mathfrak{k}}\left(\Omega^s(\partial_\alpha, \partial_\beta)\right) \\
&= \sum_{1\le k<l\le n} \Big[h\left(\mathcal{R}\left(\partial_\alpha, \partial_\beta\right) s_k, s_l\right) + h\left(\nabla_{\partial_\beta}s_k, \nabla_{\partial_\alpha}s_l\right) + h\left(\nabla_{\partial_\alpha}s_k, \nabla_{\partial_\beta}s_l\right)\Big] E_{kl} \\
&= \sum_{1\le k<l\le n} \Big[h\left(\mathcal{R}\left(\partial_\alpha, \partial_\beta\right) \partial_k, \partial_l\right) + h\left(\nabla_{\partial_\beta}\partial_k, \nabla_{\partial_\alpha}\partial_l\right) + h\left(\nabla_{\partial_\alpha}\partial_k, \nabla_{\partial_\beta}\partial_l\right) \\
&\qquad - \underbrace{h\left(\partial_\alpha u_l \nabla_{\partial_\beta}\partial_k + \partial_\alpha u_k \nabla_{\partial_\beta}\partial_l + \partial_\beta u_l \nabla_{\partial_\alpha}\partial_k + \partial_\beta u_k \nabla_{\partial_\alpha}\partial_l, \partial_-\right)}_{=0}\Big] E_{kl} \\
&= \begin{cases} 0 & \text{for} \quad \alpha, \beta = i, j \text{ oder } \alpha = - \\ \sum_{1\le k<l\le n} \partial_i\left(\partial_k u_l - \partial_l u_k\right) E_{kl} & \text{for} \quad \alpha = i, \beta = + \end{cases}
\end{aligned}
$$

This gives the proposition. □

**Remark:**
Brinkmann-waves with holonomy $\mathfrak{n}$ are Ricci-isotropic (this follows by the existence of parallel spinors as well as by direct calculations) and for the only non vanishing Ricci-component holds

$$\begin{aligned} Ric(\partial_+, \partial_+) &= \frac{1}{4} \sum_{i,l=1}^{n} (\partial_i u_l)^2 + (\partial_l u_i)^2 - 2\ \partial_i u_l \partial_l u_i \\ &\quad + \frac{1}{2} \Delta_0 f - \partial_+ \left( \sum_{i=1}^{n} \partial_i u_i \right). \end{aligned} \tag{21}$$

We will now describe Brinkmann-waves with holonomy $\mathfrak{n}$ as so called pp-manifolds which are generalisations of plane waves. The latter are generalisations of locally symmetric Brinkmann-waves. (See [Sch74], [Tak62].)

**Definition 4.1** *A Brinkmann-wave is called pp-manifold if for its curvature tensor holds the following trace condition*

$$tr_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R}) = 0. \tag{22}$$

Schimming [Sch74] proved that this condition is equivalent to the existence of local coordinates such that the metric $h$ has the following form

$$h = 2\ dx_- dx_+ + f dx_+^2 + \sum_{i=1}^{n} dx_i\ dx_i \quad \text{with } \partial_- f = 0.$$

We now can prove

**Theorem 4.1** *An simply connected indecomposable, reducible Lorentzian manifold has holonomy $\mathfrak{n}$ if and only if it is a pp-manifold.*

**Proof:** Because of the local form of the metric of a pp-manifold it is clear from proposition 4.4 that it has holonomy $\mathfrak{n}$.
For the other direction we will show that the trace condition for the curvature is fullfilled. From Proposition 4.4 and lemma 4.2 it follows $\mathcal{R}(X,Y)\partial_k = r_k(X,Y)\ \partial_-$ for skew $(2,0)$-tensors $r_k$. We will make the calculation with orthonormal fields (20) from the proof of proposition 4.4.

$$\begin{aligned} tr_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R})(X,Y,U,V) &= \sum_{\alpha,\beta=-,+,1\ldots n} \kappa_\alpha\ \kappa_\beta\ \mathcal{R}(X,Y,s_\alpha,s_\beta)\mathcal{R}(s_\alpha,s_\beta,U,V) \\ &= \sum_{\alpha=-,+,1\ldots n} \kappa_\alpha\ h\left(\mathcal{R}(X,Y)s_\alpha, \mathcal{R}(U,V)s_\alpha\right) \\ (\partial_- \text{ parallel}) \quad &= \sum_{k=1}^{n} h\left(\mathcal{R}(X,Y)\partial_k, \mathcal{R}(U,V)\partial_k\right) \\ &= 0 \ \text{ since } \mathcal{R}(X,Y)\partial_k = r_k(X,Y)\ \partial_-. \end{aligned}$$

$\square$

So we get that the class of Brinkmann-waves with holonomy $\mathfrak{n}$ is equal to the one of pp-manifolds, which is obviously larger then the class of symmetric Brinkmann-waves. The metric of the latter is given by

$$h = 2\ dx_- dx_+ + f dx_+^2 + \sum_{i=1}^{n} dx_i\ dx_i \quad \text{with } f = \sum_{k=1}^{n} c_k x_k^2.$$

Further we get the most general form of the metric of a pp-manifold that is the local form of a Brinkmann-metric with condition (19) from propsition 4.4 for the $u_i$'s. About such $u_i$'s and the resulting metric we remark the following facts (see also [FO99] for these results in low dimensions).

1. The most general functions $u_i$ which are fullfilling the condition $\partial_k\left(\partial_i u_j - \partial_j u_i\right) = 0$ are of the following form

$$u_i = \sum_{k=1}^{n} x_k\ f_{ik} + \partial_i \phi \qquad \text{where } \partial_j(f_{ik}) = \partial_-(f_{ik}) = 0,\ f_{ii} = 0, \partial_-(\phi) = 0.$$

   This is clear by the following computations: One considers the Taylor formula for $u_i$ around the point $(x_+, 0, \ldots, 0)$ and writes is as follows

$$u_i(x_+, x_1, \ldots, x_n) = \sum_{\substack{k=1 \\ k \neq i}}^{n} \left(\partial_k(u_i)(x_+, 0, \ldots, 0)\right)\ x_k + f_i(x_+, x_1, \ldots, x_n)$$

   for certain funktions $f_i$. Then it is $f_i = \partial_i \phi$ if and only if the integrability condition $\partial_j f_i - \partial_i f_j = 0$ is fullfilled. One gets

$$\begin{aligned}\left(\partial_j f_i - \partial_i f_j\right)(x_+, x_1, \ldots x_n) &= \partial_j u_i(x_+, x_1, \ldots, x_n) - \partial_i u_j(x_+, x_1, \ldots, x_n) \\ &\quad - \left(\partial_j u_i(x_+, 0, \ldots, 0) - \partial_i u_j(x_+, 0, \ldots, 0)\right) \\ &= 0 \ \text{ since } \partial_i u_j - \partial_j u_i \text{ independent of all } x_k.\end{aligned}$$

   So one gets the proposed form with $f_{ik} := \partial_k(u_i)(x_+, 0, \ldots, 0)$.

2. By the coordinate transformation $\tilde{x}_- := x_- - \frac{1}{2}\phi$ one can remove the term $\partial_i \phi$. Since

$$\frac{\partial}{\partial \tilde{x}_-} = \partial_-\ ,\quad \frac{\partial}{\partial \tilde{x}_+} = \partial_+ - \frac{1}{2}\partial_+\phi\ \partial_-\ ,\quad \frac{\partial}{\partial \tilde{x}_i} = \partial_i - \frac{1}{2}\partial_i \phi\ \partial_-$$

$$\text{it is } \tilde{u}_i = h\left(\frac{\partial}{\partial \tilde{x}_+}, \frac{\partial}{\partial \tilde{x}_i}\right) = \sum_{k=1}^{n} x_k\ f_{ik}.$$

So one gets the following

**Corollary 4.1** *The most general local form of the metric of a pp-manifold is given by*

$$h = 2\ dx_- dx_+ + \sum_{i,k=1}^{n} x_k f_{ik} dx_i dx_+ + f dx_+^2 + \sum_{i=1}^{n} dx_i\ dx_i$$

*with $f$ and $f_{ij}$ independent of $x_-$, $\partial_k(f_{ij}) = 0$ and $f_{ii} = 0$.*

With respect to the existence of parallel spinors we conclude the following from this and the previous section (in particular from the fact that Riemannian manifolds with parallel spinors have non abelean holonomy and that symmetric Riemannian manifolds with parallel spinors have to be flat).

**Corollary 4.2** *The pp-manifolds are the only Lorentzian manifolds with parallel spinors and nontrivial abelean holonomy.*

Finally one can show that in very small dimensions the existence of an parallel spinor with lightlike Dirac current forces the manifold to be a pp-manifold. This result is also given in [FO99].

**Corollary 4.3** *Let $(M,h)$ a simply connected, indecomposable Lorentzian manifold of dimension $m=n+2$ with $n=1,2,3$ and a parallel spinor with lightlike Dirac current. Then $(M,h)$ has to be a pp-manifold.*

**Proof:** An indecomposable Lorentzian manifold admitting a parallel spinor with light Dirac current has to be a Brinkmann-wave whose metric can locally be given in the simplified form of (13) (in an above remark). Then the Riemannian parts $g_{x_+}$ must admit parallel spinors and because of $n<4$ their holonomy must be trivial and therefore $g_{x_+} = c^2(x_+)\ \delta_{ij}$ with a conformal factor. Then the coordinate transformation $\tilde{x_i} := c(x_+)x_i,\ i=1,\dots,n$ gives $\tilde{c}\equiv 1$ and $\tilde{u}_i = -(\partial_+ c)x_i$. But these functions are fullfilling the condition (19) for having holonomy $\mathfrak{n}$ and therefore for being a pp-manifold. □

### 4.3 Further examples and problems

We will now construct a class of examples of Brinkmann-waves with parallel spinors based upon a Riemannian metric with parallel spinors. First we prove the following

**Proposition 4.5** *Let $(M,h)$ be a Brinkmann-wave locally given by $h=h(f,u_i,g_{ij})$ with the additional condition $\partial_+ g_{ij}=0$ (i. e. $g_{x_+}\equiv g$) and $u_i=0$. For $f$ sufficient general (for example $\partial_k^2 f\neq 0$) and $x=(x_-,x_+,x_1,\dots x_n)$ it is*

$$Hol_x(M,h) = Hol_x(W_{(x_-,x_+)},g)\ltimes N \qquad (N\simeq\mathbb{R}^n).$$

**Proof:** Direct calculations give

$$\nabla_{\partial_+}\partial_i = \frac{1}{2}(\partial_i f)\partial_- \quad \text{and} \quad \nabla_{\partial_i}\partial_j = \nabla^g_{\partial_i}\partial_j \tag{23}$$

$$\mathcal{R}\left(\partial_i,\partial_+\right)\partial_k = \frac{1}{2}(\partial_i\partial_k f)\partial_- \quad \text{and} \quad \mathcal{R}\left(\partial_i,\partial_j\right)\partial_k = \mathcal{R}^g\left(\partial_i,\partial_j\right)\partial_k \tag{24}$$

where $\nabla^g$ and $\mathcal{R}^g$ are connection and curvature of $g$. We define again an orthonormal basis:

$$\begin{aligned} s_- &= \frac{1}{\sqrt{2}}\left(\left(1+\frac{f}{2}\right)\partial_- - \partial_+\right) \text{ timelike} \\ s_+ &= \frac{1}{\sqrt{2}}\left(\left(1-\frac{f}{2}\right)\partial_- + \partial_+\right) \\ s_i &= \sum_{k=1}^n \xi_k^i\ \partial_k \text{ orthonormal for } g \text{ with } \xi_k^i \text{ independent of } x_+. \end{aligned}$$

Let $s := (s_-, s_+, s_1, \dots s_n) \in \Gamma(U,P)$ and $\hat{s} := (s_1, \dots s_n) \in \Gamma(U, P^g)$ a section in the bundle of $g$-orthonormal frames. From (23) and (24) one gets for the local curvature forms that $pr_{\mathfrak{k}}\left(\Omega^s(\partial_i, \partial_j)\right) = \Omega_g^{\hat{s}}(\partial_i, \partial_j)$. Because of (23) it is

$$pr_{\mathfrak{k}}\left(\Omega^s(\partial_i, \partial_+)\right) = \sum_{1 \le k < l \le n} h\left(\mathcal{R}\left(\partial_i, \partial_+\right) s_k, s_l\right) E_{kl}$$

and further

$$\begin{aligned}
h\left(\mathcal{R}\left(\partial_i, \partial_+\right) s_k, s_l\right) &= \sum_{p,q=1}^{n} \xi_p^k\ \xi_q^l\ h\left(\mathcal{R}\left(\partial_i, \partial_+\right) s_k, s_l\right) \\
&\quad + \sum_{p,q=1}^{n} \underbrace{\partial_+(\xi_p^k)}_{=0} \Big(\partial_i(\xi_q^l)\ h(\partial_p, \partial_q) + \xi_q^l\ h(\partial_p, \nabla_{\partial_i}\partial_q)\Big) \\
&\quad + \sum_{p,q=1}^{n} \xi_p^k (\partial_i(\xi_q^l)\ \underbrace{h(\nabla_{\partial_+}\partial_p, \partial_q)}_{=0} + \xi_q^l\ \underbrace{h(\nabla_{\partial_+}\partial_p, \nabla_{\partial_i}\partial_q)}_{=0}) \\
&= \sum_{p,q=1}^{n} \xi_p^k\ \xi_q^l\ h\left(\partial_i\partial_p(f)\ \partial_-, \partial_q\right) = 0
\end{aligned}$$

So it is $pr_{\mathfrak{k}}\left(\Omega^s(X,Y)\right) = \Omega_g^{\hat{s}}(X,Y)$ for all $X, Y \in TU$.
Now from the Ambrose-Singer-theorem, the formula (3) and considerations analogous to the proof of lemma 4.3 which yields to $pr_{\mathfrak{k}}\left(Ad(g)A\right) = Ad\left(pr_K(g)\right) pr_{\mathfrak{k}}(A)$ one gets the conclusion. □

This yields to another class of Brinkmann-waves with parallel spinors.

**Corollary 4.4** *Let $h = h(f, 0, g_{ij})$ be a Brinkmann-wave with $\partial_+ g_{ij} = 0$ and the metric $g$ admits a parallel spinor. Then $(M,h)$ admits a parallel spinor.*

The question remains if there are types of holonomy groups with trivial subrepesentations of its spin representation, groups which are not of the form $Hol_x(g) \ltimes N$ and whether theres a metric with this holonomy.
Bryant ([Bry99a], see also [FO99]) has recently proved that for dimensions $m \le 11$ the maximal groups admiting parallel spinors can be obtained by the method of proposition 4.5, but it is not known whether all subgroups can be obtained in this way. The example of Ikemakhen in [Ike96] which can not be constructed in this way does not admit parallel spinors.
Bryant also shows in [Bry99b] that the general form of the metric in dimension 11 having the maximal holonomy $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ has to be conformal anti-selfdual. This means that the family of the $Spin(7)$-defining 4-form $\Theta_{x+}$ satisfies $\partial_+ \Theta_{x+} = f\ \Theta + \Upsilon$ with a function $f$ and an anti-selfdual 4-form $\Upsilon$. This condition is locally trivial, i.e. for every given family of $Spin(7)$ metrics one can find diffeomorphisms such that the family of the transfomed metrics with $Spin(7)$ holonomy is conformal anti-selfdual.
Another class of metrics was obtained by Figueroa O'Farril who proved in [FO99] the following. Let $(M, h = h(f, u_i, g_{ij})$ be a Brinkmann- wave and $(\hat{M}, \hat{g})$ a Riemannian

manifold and $\sigma \in C^\infty(M)$ but only dependent from $x_+$. Then for the holonomy group of the product manifold and the warped product metric $\hat{h} = h + \sigma\hat{g}$ holds the following

$$Hol_{(x,y)}(M \times N, \hat{h}) = \Big( Hol_x(M,h) \times Hol_y(\hat{M},\hat{g}) \Big) \ltimes \mathbb{R}^n .$$

So by choosing both factors with parallel spinors — in [FO99] this is done in dimension 11 — one gets a new class of metrics with parallel spinors, but no new examples of holonomy groups admitting trivial subrepresentations which cannot be constructed by the method of proposition 4.5.

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